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FUNDAMENTAL GROUP AND TWISTED ALEXANDER POLYNOMIAL OF LINK COMPLEMENT IN 3-TORUS

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ABSTRACT. We consider a diagrammatic approach to investigate tame knots and links in three dimensional torus T^3 . We obtain a finite set of generalised Reidemeister moves for equivalent links up to ambient isotopy. We give a presentation for fundamental group of link complement in 3-torus T^3 and the first homology group. We also compute Alexander polynomial and twisted Alexander polynomials of this class of links.

Keywords: knots, links, torus, three-dimension, twisted Alexander polynomial, first homology group, Alexander-Fox matrix, general Reidemeister moves.

Dedicated to Professor Andrey Vesnin on the occasion of his 60th birthday

1. INTRODUCTION

The dawn of mathematical theory of knots dates back to eighteenth century. It was first treated mathematically in 1771 by Alexandre-Théophile Vandermonde who explicitly noted the importance of topological features when discussing the properties of knots related to the geometry of position. Then it was gradually developed till the early part of twentieth century. The second part of the 20th century was a real golden age for knot theory. The theory became a vast subject, that is ubiquitous in topology and now extending beyond its traditional root in topology to algebraic and differential geometry, number theory, mathematical physic. Dale Rolfsen wrote in his famous book "Knots and Links"[1] that the

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best thing that happened to knot theory, however, is that many more scientists are interested in it, not just topologists, and contributing in their own ways. M. Dehn, J. W. Alexander and others studied knots from the point of view of the knot group and invariants from homology theory. The Alexander polynomial is a knot invariant discovered in 1923 by J. W. Alexander [2]. In technical language, the Alexander polynomial arises from the homology of the infinitely cyclic cover of a knot complement. Any generator of a principal Alexander ideal is called an Alexander polynomial. The Alexander polynomial remained the only known knot polynomial until the Jones polynomial was discovered in 1984. V. Jones's discovery of the new polynomial inspired a "polynomial fever" rampant [1]. This breakthrough shed light to our natural world of knots, this led to the discovery of many new polynomials. The new polynomials are just the tips of icebergs, deep results and sophisticated structures are hidden itself under water such as quantum groups and Floer homology.

Classical knot theory is the study of knots and links in the 3-dimensional sphere S^3 or just 3-dimensional Euclidean space R^3 , that are the simplest 3-manifolds. For the latest years studies on knots and links have been generalised in other spaces as solid torus (see [3], [4], [5]), in projective space (see [6], [7]), in lens spaces (see, for example, [8], [9], [10]), in homology 3-sphere ([11]).

In this work we extend further the study in this direction to knots and links in other manifold, namely the three dimensional torus T^3 , that we shortly call it 3-torus. Some results about links in the space as product of a surface and a circle appear recently making use of skein module theory. R. Detcherry and M. Wolff in [13] provide an explicit spanning family for the skein modules, associated with any closed oriented surface. Combined with earlier work of Gilmer and Masbaum [12, 14], they obtain the dimension of the skein modules for product of a surface and a circle. In [15] M. K. Dabkowski and M. Mroczkowski introduce diagrams and Reidemeister moves for links in $F \times S^1$, where F is an orientable surface. Using these diagrams they compute the Kauffman Bracket Skein Modules for $D^2 \times S^1$, $A \times S^1$ and $F_{0,3} \times S^1$, where D^2 is a disk, A is an annulus and $F_{0,3}$ is a disk with two holes.

Our modest purpose is using the classical diagrammatic approach to obtain a scheme for computation the fundamental group of link complement in 3-torus and also the first homology group. By doing so we introduce a set of Reidemeister type moves for diagrams of link in 3-torus, which is similar to that of Mroczkowski and Dabkowski in [15]. We are also interested in the twisted Alexander polynomial, using Fox free differential calculus we point out how to compute it for links in 3-torus. We prove some basic properties of the twisted Alexander polynomial for some simple links and local links.

2. DIAGRAMS

A link L with n components in three-dimensional torus T^3 is an embedding of a disjoint union of n circles S^1 into three-dimensional torus. If $n = 1$ the link is called knot. Two links are considered equivalent if they are ambient isotopic, that is, if there exists a continuous deformation of T^3 which takes one link to the other.

The three-dimensional torus, or 3-torus, is defined as any topological space that is homeomorphic to the Cartesian product of three circles $T^3 = S^1 \times S^1 \times S^1$. The 3-torus is a three-dimensional compact manifold with no boundary. It can

be obtained by "gluing" the three pairs of opposite faces of a cube, where being "glued" can be intuitively understood to mean that when a particle moving in the interior of the cube reaches a point on a face, it goes through it and appears to come forth from the corresponding point on the opposite face, producing periodic boundary conditions (see Figure 1). Thus, the 3-torus is the quotient of a cube C by the equivalence relation on the boundary ∂C of the cube which identifies its opposite faces. We denote by $F : C \rightarrow T^3 = C / \sim$ the quotient map. Denote by A the bottom face, A' the top face, B' the right face, B the left face, C the front face and C' the back face of the cube C .

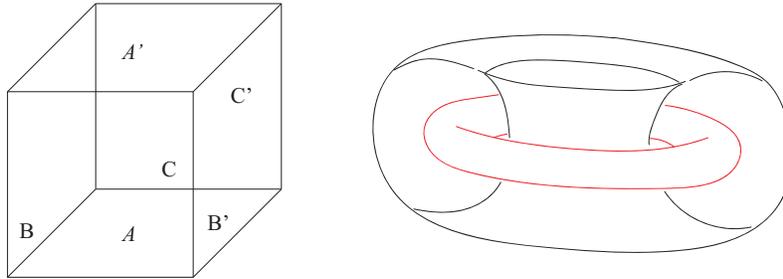


FIG. 1. 3-torus

We will define diagram for links in 3-torus analogous to that of diagram of links in lens spaces given in [8]. Let L be a link in 3-torus T^3 and consider $L' = F^{-1}(L)$. By moving via small isotopy in T^3 , we can suppose that:

- a) L' does not meet the vertices and edges of the cube C ;
- b) $L' \cap \partial C$ consists of a finite set of points;
- c) L' is not tangent to ∂C ;

Thus, L' is the disjoint union of closed curves in the interior of the cube C and arcs properly embedded in C , that is only the boundary points belong onto ∂C .

Denote by A the bottom face of the cube. Let $p : C \rightarrow A$ be the usual orthogonal projection defined by $p(x) = l(x) \cap A$, where $l(x)$ is the line, that is orthogonal to the face A and passing through x . Take L' and project it via $p|_{L'} : L' \rightarrow A$. For a point $P \in p(L')$, the preimage $p^{-1}(P)$ may contain more than one point. In this case, we say that P is a multiple point. In particular, if it contains exactly two points, we say that P is a double point. We might assume, by moving L via a small isotopy, that the projection $p|_{L'} : L' \rightarrow A$ of L is regular, that is:

- 1) the projection of L' contains no cusps;
- 2) all auto-intersections of $p(L')$ are transversal;
- 3) the set of multiple point is finite, and all of them are actually double points;
- 4) no double point is on the edges of bottom face A .

We call a double point in the projection of L' in the bottom face A a crossing. As for classical knot diagram we specify over arcs and under arcs for each crossing relative to the space inside the cube C , that is over and under crossing are defined in the context of bottom and top cube's faces. By doing such projection we forget certain information about the knot in 3-torus. Namely knots in 3-torus can wrap up through faces of the cube, the projection does not carry the information when knots wrap up through the bottom face A and its identification top face. Further we call bottom face the floor and top face the ceiling. For that reason we add vertices

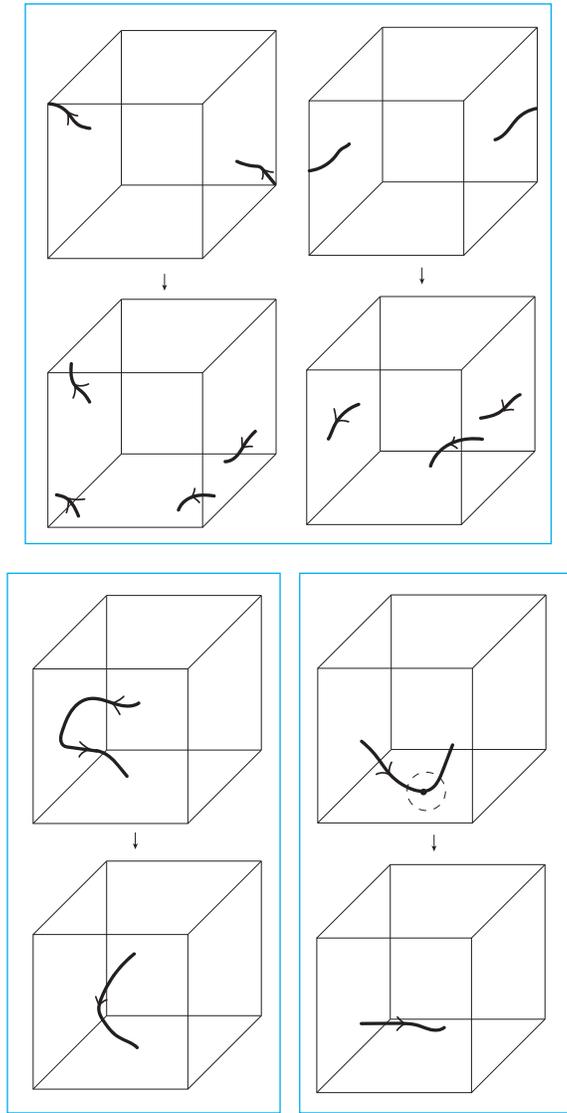


FIG. 2. Example of small isotopy to reach conditions a), b), c)

with poles (positive and negative) to the projected diagram of knot whenever an arc touches the floor or ceiling, that touched point will be projected to a vertex in knot diagram. We specify the neighbourhood of the vertex is the positive pole if it touches the ceiling and negative pole if it touches the floor (see Figure 3). Note that a vertex always has two different poles as we have supposed in the condition c) above that L' is not tangent to faces ∂C . Also an isolated vertex does not appear since every multiple point in a projection are actually double points. We add a condition of regularity for a projection

- 5) no vertex at multiple point

Now let Q be a double point, consider $p_{|L'}^{-1}(Q) = P_1, P_2$ and suppose that P_2 is closer to bottom face A than P_1 . Let U be a connected open neighbourhood of P_2 in L' such that $p(U)$ contains no other double point and does not meet edges of A . We call U underpass relative to Q . Every connected component of the complement in L' of all the underpasses (as well as its projection in face A) is called overpass.

A diagram of a link L in T^3 is a regular projection of $L' = F^{-1}(L)$ on the bottom face A , with specified overpasses and underpasses and the projections of the underpasses are not depicted in the diagram (see Figure 3). Thus we have a diagram of knot in a square, the opposite edges of which are identified.

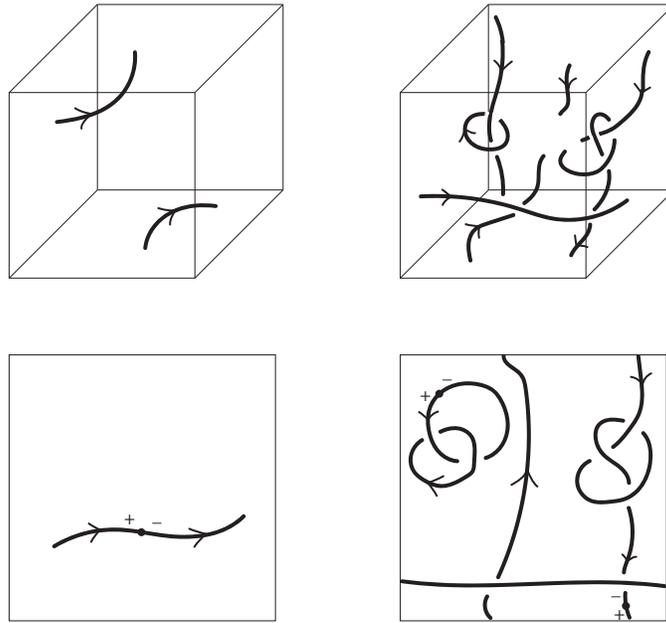


FIG. 3. Diagram of knot

3. REIDEMEISTER MOVES AND VERTEX MOVES

In this section we will have a finite set of moves connecting two different diagrams of the same link. The generalised Reidemeister moves on a diagram of a link $L \subset T^3$, are the moves R_1, R_2, R_3, R_4, R_5 (see Figure 4). The vertex moves are the moves V_1, V_2, V_3 (see Figure 5). Remark that V_4 is a forbidden move.

Theorem 1. *Two links L_0 and L_1 in 3-torus T^3 are equivalent if and only if their diagrams can be joined by a finite sequence of generalised Reidemeister moves R_1, R_2, R_3, R_4, R_5 , vertex moves V_1, V_2, V_3 and diagram isotopies.*

Proof. It is easy to see that each Reidemeister and vertex move connects equivalent links, hence a finite sequence of the moves and diagram isotopies does not change the equivalent class of the link. Reversely, if we have two equivalent links L_0 and L_1 , then there exists an isotopy of the ambient space $H : T^3 \times [0, 1] \rightarrow T^3$ such that $H_0(L_0) = L_0$ and $H_1(L_0) = L_1$. For each $t \in [0, 1]$ we have a link $L_t = H_t(L_0)$.

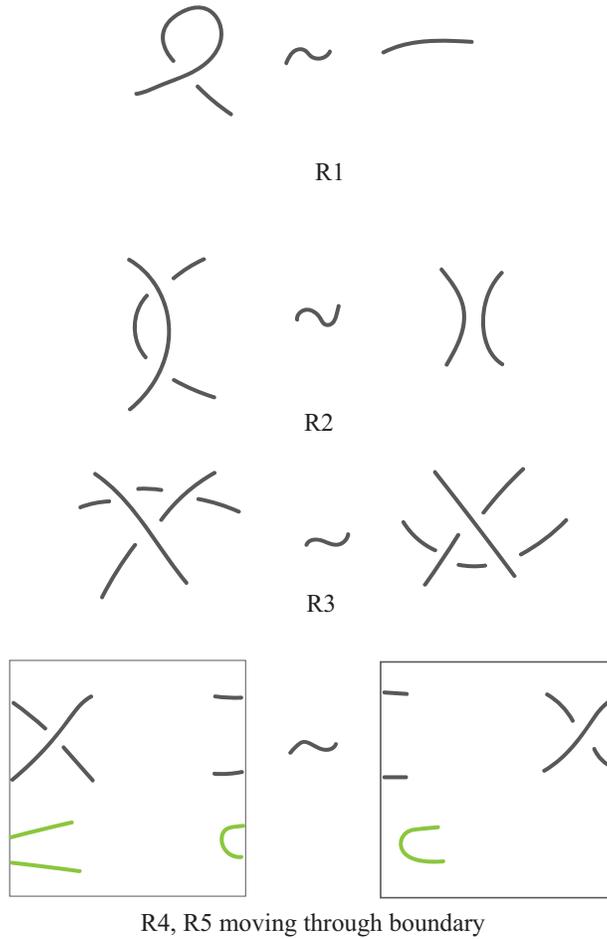


FIG. 4. Generalised Reidemeister moves

The link L_t may violate conditions a) b) c) and its projections can violate the regularity conditions 1) 2) 3) 4).

It is not hard to see that the isotopy H can be chosen in such a way that conditions b) is always satisfied. We can assume that there are a finite number of forbidden configurations using general position theory (see [16]). And for each $t \in [0, 1]$, only one forbidden configuration may occur. Figure ?? illustrates the situations when a condition is violated during the isotopy.

- conditions 1) 2) 3) generate configurations S_1, S_2, S_3 .
- condition c) generates configurations S_4 and P_1
- condition 4) generates configuration S_5
- condition 5) generates configuration P_3

From each type of forbidden configuration a transformation of the diagram appears, i.e. a generalized Reidemeister move or vertex move, as follows

- from S_1, S_2, S_3 we obtain the usual Reidemeister moves R_1, R_2, R_3 ;
- from S_4 we obtain move R_4 ;

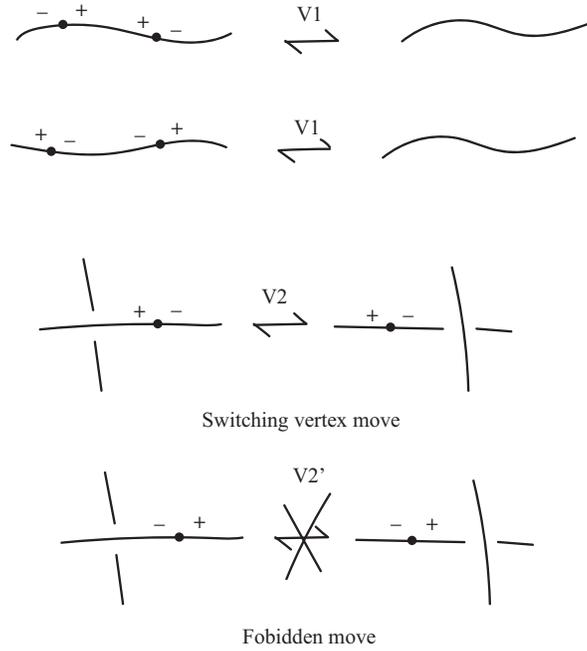


FIG. 5. Vertex moves

- from P_1 we obtain move V_1
- from S_5 we obtain move R_5 ;
- from P_2 we obtain move V_2 ;

From the vertex moves V_1, V_2 we can obtain two other transformations for a diagram to get a new diagram, that is equivalent to the old one. Namely, the first transformation is the move V_3 (see Figure 8), that describes how we can move a vertex through a double point. The Figure 8 illustrates the case when the moving vertex is in the under arc, the same can be applied when the moving vertex lays in over arc. The second is we can add vertices at intersection of diagram with a Jordan curve, so that the Jordan curve does not meet any double point and the additional vertices have same poles relative to inside and outside of the Jordan curve. Naturally, we can do the reverse process as removing vertices under the same condition.

Thus any pair of diagrams of two equivalent links can be joined by a finite sequence of generalised Reidemeister moves R_1, \dots, R_5 ; vertex moves V_1, V_2 and diagram isotopies. □

4. FUNDAMENTAL GROUP

In this section we obtain, directly from the diagram, a finite presentation for the fundamental group of the complement of links in T^3 .

Let L be a link in T^3 , and consider a diagram of L . Fix an orientation for L , which induces an orientation on both L' and $p(L')$. We call a boundary point is either a vertex or an intersection point of $p(L')$ and the square boundary of the face

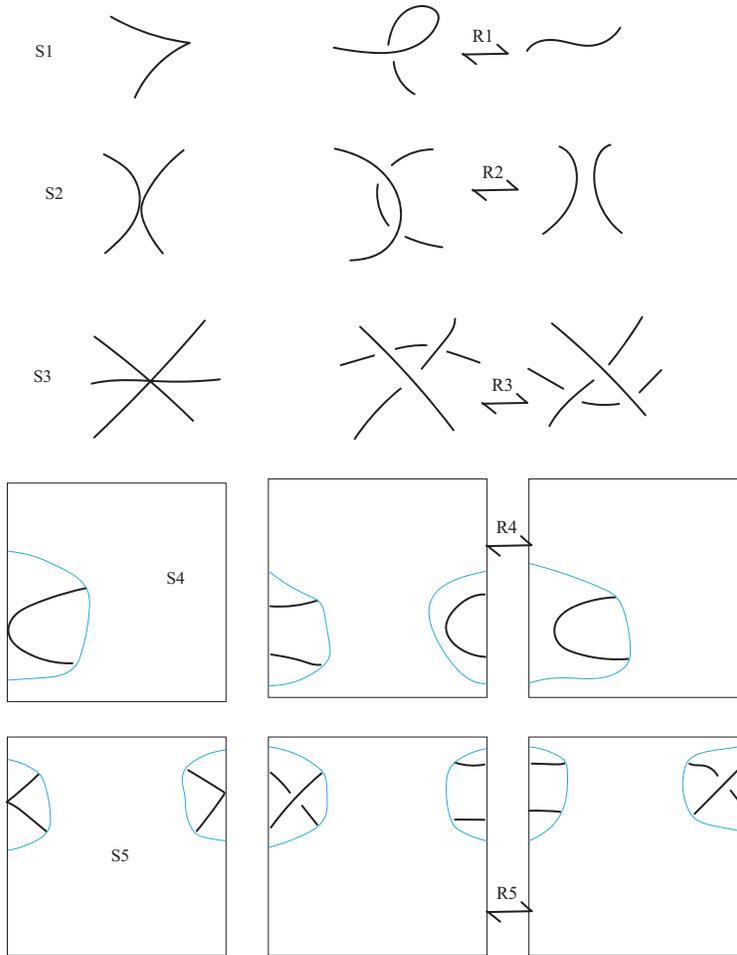


FIG. 6. Forbidden configurations and diagram moves

A. Perform an R_1 move on each overpass of the diagram having both endpoints on the same boundary point; in this way every overpass has at most one boundary point. Then label the overpasses as follows: X_1, \dots, X_n are the ones ending in the left face B ; X'_1, \dots, X'_n are the overpasses ending in the right face B' ; Y_1, \dots, Y_m are the overpasses ending in the front face C ; Y'_1, \dots, Y'_m are the overpasses ending in the back face C' ; Z_1, \dots, Z_l are the overpasses ending in the top face A' and Z'_1, \dots, Z'_l are the overpasses ending in the bottom face A . The remaining overpasses are labelled by A_1, \dots, A_r . For each $i = 1, \dots, n$, let $\epsilon_i = 1$ if, according to the link orientation, the overpass X_i starts from a point in the left side of the square; otherwise, if X_i ends in the point, let $\epsilon_i = -1$. Also for each $j = 1, \dots, m$, let $\nu_j = 1$ if, according to the link orientation, the overpass Y_j starts from a point in the bottom side of the square; if Y_j ends in the point, let $\nu_j = -1$. And for each $k = 1, \dots, l$, according to the link orientation, let $\tau_k = 1$ if the overpass Z_k starts from positive pole of a vertex and $\tau_k = -1$ if the overpass starts from negative pole.

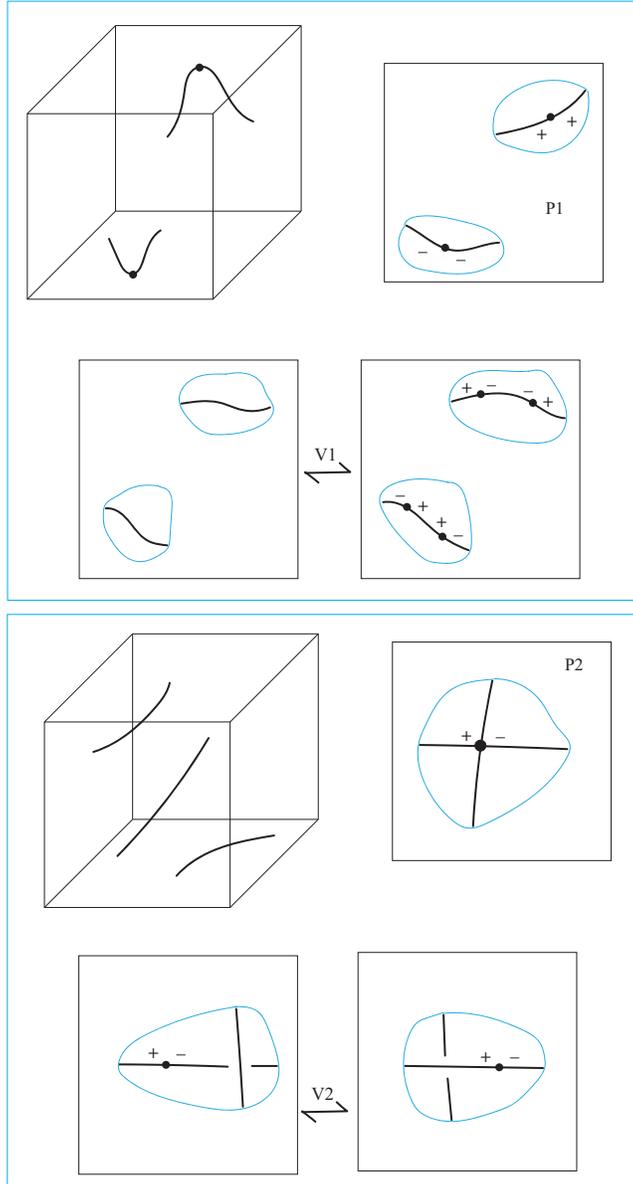


FIG. 7. Forbidden configurations and diagram moves

Associate to each overpass X_i, Y_j, Z_k, A_q a generator x_i, y_j, z_k, a_q respectively, which is a loop around the overpass as in the classical Wirtinger theorem, oriented following the left hand rule. Moreover let x, y, z be the generators of the fundamental group of the three torus T^3 depicted in Figure 10. The relations are the following:

W : w_1, \dots, w_s are the classical Wirtinger relations for each crossing, that is of the type $a_i a_j a_i^{-1} a_k^{-1} = 1$ or $a_i a_j^{-1} a_i^{-1} a_k^{-1} = 1$, see to Figure 11;

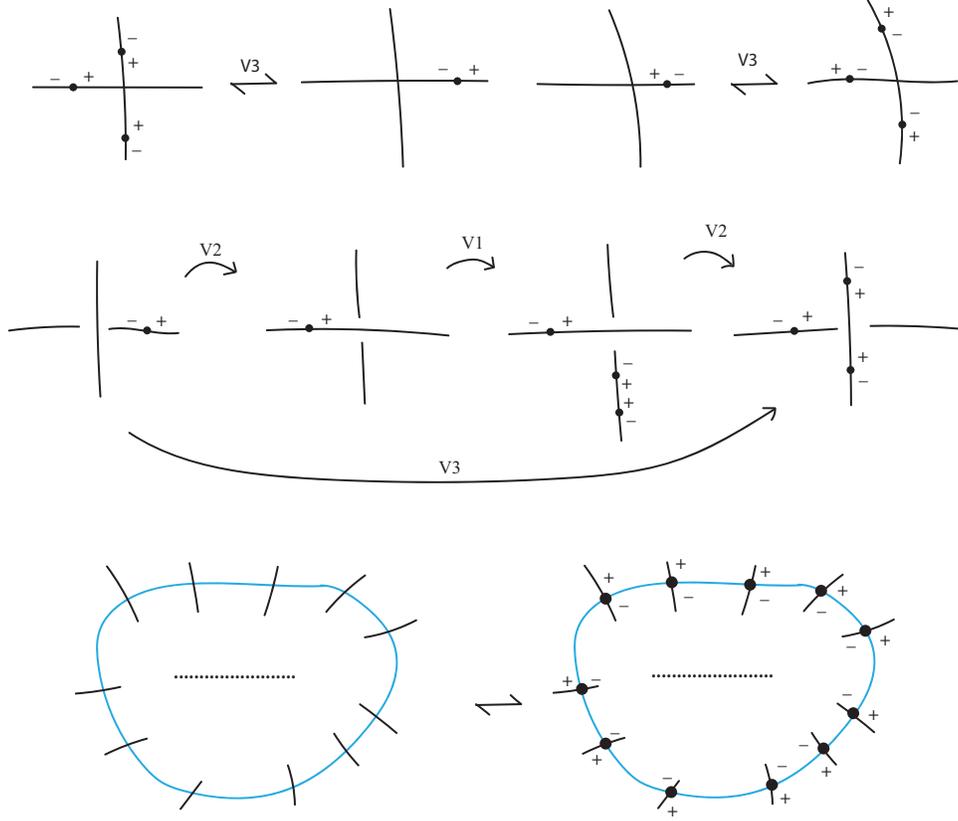


FIG. 8. Additional moves

Q: Relations between loops corresponding to overpasses with identified endpoints on the boundary $x'_i = xx_ix^{-1}$, $y'_j = yy_jy^{-1}$, $z'_k = zz_kz^{-1}$

T: Torus relations $x_1^{\epsilon_1} \dots x_n^{\epsilon_n} = zxz^{-1}x^{-1}$, $y_1^{\nu_1} \dots y_m^{\nu_m} = yzy^{-1}z^{-1}$, $z_1^{\tau_1} \dots z_l^{\tau_l} = yxy^{-1}x^{-1}$, when n, m or l is zero then the corresponding product define to be 1.

Theorem 2. Let base point be the vertex of the cube, then the group $\pi_1(T^3 \setminus L)$ of the link L in 3-torus T^3 has generators $x, x_1, \dots, x_n, x'_1, \dots, x'_n, y, y_1, \dots, y_m, y'_1, \dots, y'_m, z, z_1, \dots, z_l, z'_1, \dots, z'_l, a_1, \dots, a_r$ and relations W, T and Q .

Proof. Suppose that $L' = F^{-1}(L)$ is such that $p|_{L'} : L' \rightarrow A$ is a regular projection. Consider a cube surface C_ϵ^2 , so that the maximal distance between the surface of the sphere and the cube boundary ∂C is ϵ . The topological sphere C_ϵ^2 divides the cube C into two pieces: call I_ϵ the internal one and E_ϵ the external one. Choose ϵ small enough such that all the underpasses belong into $\text{int}(I_\epsilon)$. Let V, V_ϵ be the front bottom left vertices of C and C_ϵ^2 correspondingly (see Fig. 10). We consider $\tilde{C}_\epsilon^2 = C_\epsilon^2 \cup \overline{VV_\epsilon}$ and $\tilde{I}_\epsilon = I_\epsilon \cup \overline{VV_\epsilon}$.

We use Seifert - van Kampen theorem to compute the fundamental group of link complement $\pi_1(T^3 \setminus L, *)$ in three torus T^3 with decomposition $(T^3 \setminus L) = (F(\tilde{I}_\epsilon \setminus L) \cup (F(E_\epsilon \setminus L))$.

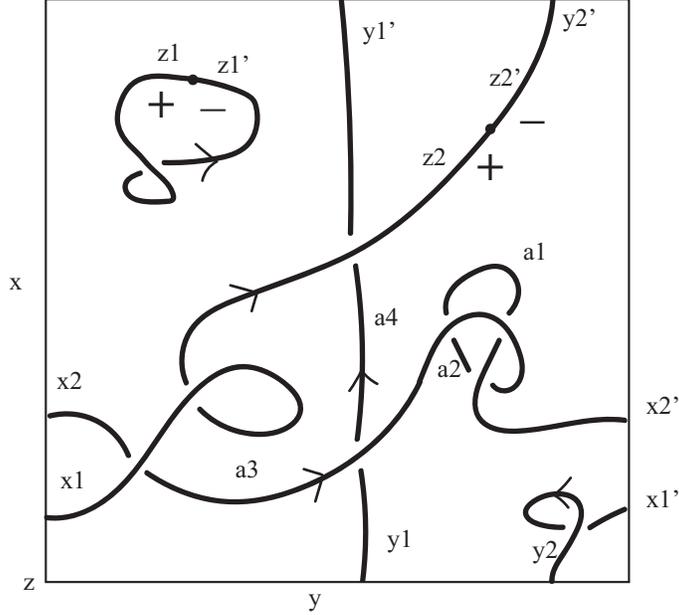


FIG. 9. Example of overpasses labelling for a link in T^3

The fundamental group of $F(\tilde{I}_\epsilon \setminus L)$ can be obtained as in the classical Wirtinger Theorem:

$$\pi_1(F(\tilde{I}_\epsilon \setminus L, *) = \langle a_1, \dots, a_r | w_1, \dots, w_s \rangle$$

For $F(E_\epsilon \setminus L)$, we proceed in the following way: first of all observe that we can retract $F(E_\epsilon \setminus L)$ to $E \setminus L$, where E is $\partial C / \sim$. Now the 2-complex E is a CW-complex consists of: one 0-cell V since all vertices of the cube are identified; three 1-cells correspond to three sets of parallel edges; and three 2-cells correspond to three pair of parallel faces. In order to obtain $\pi_1(E \setminus L)$, we need to add the loops $d_{x_1}, \dots, d_{x_n}, d_{y_1}, \dots, d_{y_m}, d_{z_1}, \dots, d_{z_l}$ around the points of L (see Fig. 12). The relations given by the 2-cell are $d_{x_1} \dots d_{x_n} = xzx^{-1}z^{-1}$, $d_{y_1} \dots d_{y_m} = yzy^{-1}z^{-1}$, $d_{z_1} \dots d_{z_k} = xyx^{-1}y^{-1}$. Hence the fundamental group of $E \setminus L$ is:

$$\pi_1(E \setminus L, *) = \langle d_{x_1}, \dots, d_{x_n}, d_{y_1}, \dots, d_{y_m}, d_{z_1}, \dots, d_{z_l} | d_{x_1} \dots d_{x_n} = xzx^{-1}z^{-1}, \\ d_{y_1} \dots d_{y_m} = yzy^{-1}z^{-1}, d_{z_1} \dots d_{z_l} = xyx^{-1}y^{-1} \rangle$$

Finally, the fundamental group of $F(\tilde{C}_\epsilon^2) \setminus L = (F(\tilde{I}_\epsilon \setminus L) \cap (F(E_\epsilon \setminus L)))$ is generated by $x_1, \dots, x_n, x'_1, \dots, x'_n, y_1, \dots, y_m, y'_1, \dots, y'_m, z_1, \dots, z_l, z'_1, \dots, z'_l$. By Seifert-Van Kampen theorem, we identify each $x_1, \dots, x_n, y_1, \dots, y_m, z_1, \dots, z_l$ with the corresponding generator $d_{x_1}, \dots, d_{x_n}, d_{y_1}, \dots, d_{y_m}, d_{z_1}, \dots, d_{z_l}$. Furthermore we need to identify $x'_1, \dots, x'_n, y'_1, \dots, y'_m, z'_1, \dots, z'_l$ with suitable loops in the CW-complex, we have the following relations $x'_i = xx_ix^{-1}$, $y'_j = yy_jy^{-1}$, $z'_l = zz_lz^{-1}$. In the Figure 12 we illustrate how we identify x'_1 (loop in red) to $xd_{x_1}x^{-1} = xx_1x^{-1}$ (loop in blue).

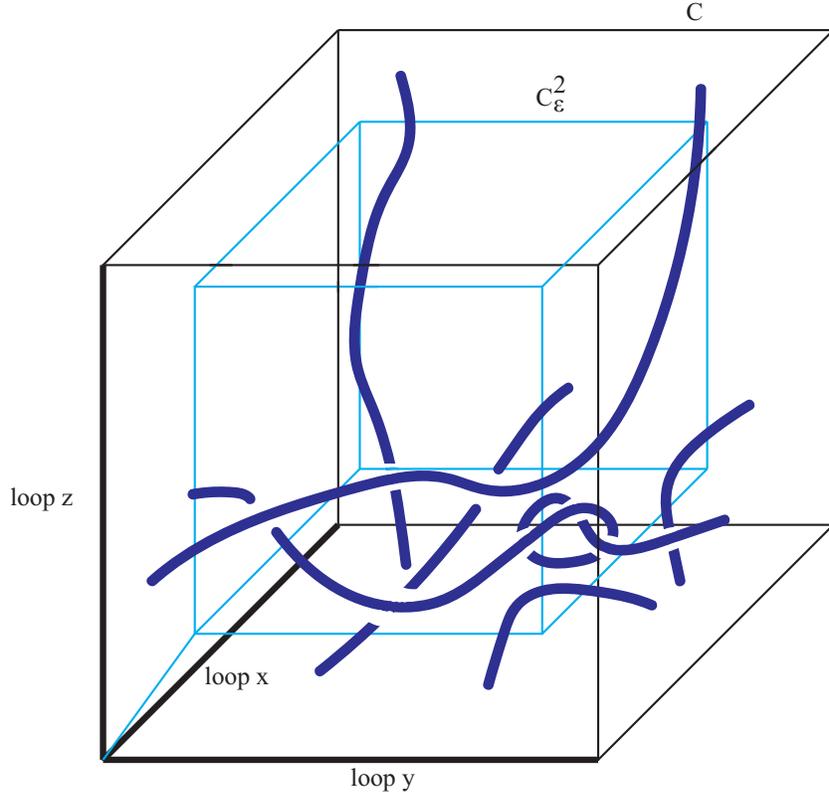


FIG. 10. A link in 3-torus

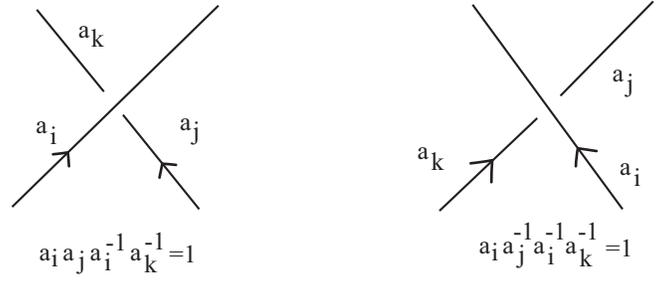


FIG. 11. Wirtinger relations

At last we remove $d_{x_1}, \dots, d_{x_n}, d_{y_1}, \dots, d_{y_m}, d_{z_1}, \dots, d_{z_l}$ from the group presentation, obtaining a presentation for the fundamental group of link complement in three torus. That has generators $x, x_1, \dots, x_n, x'_1, \dots, x'_n, y, y_1, \dots, y_m, y'_1, \dots, y'_m, z, z_1, \dots, z_l, z'_1, \dots, z'_l, a_1, \dots, a_r$ and relations W, T and Q .

□

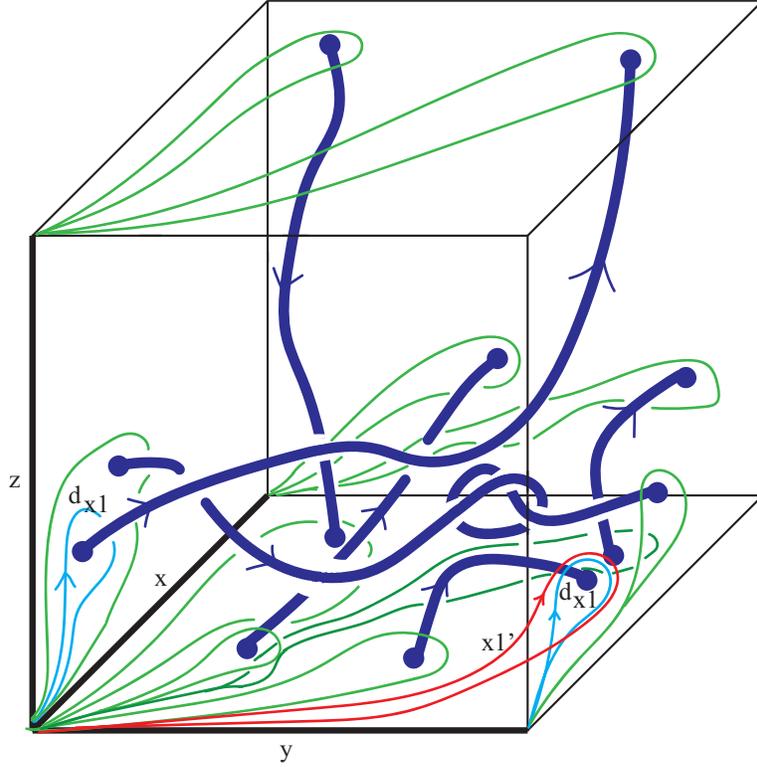


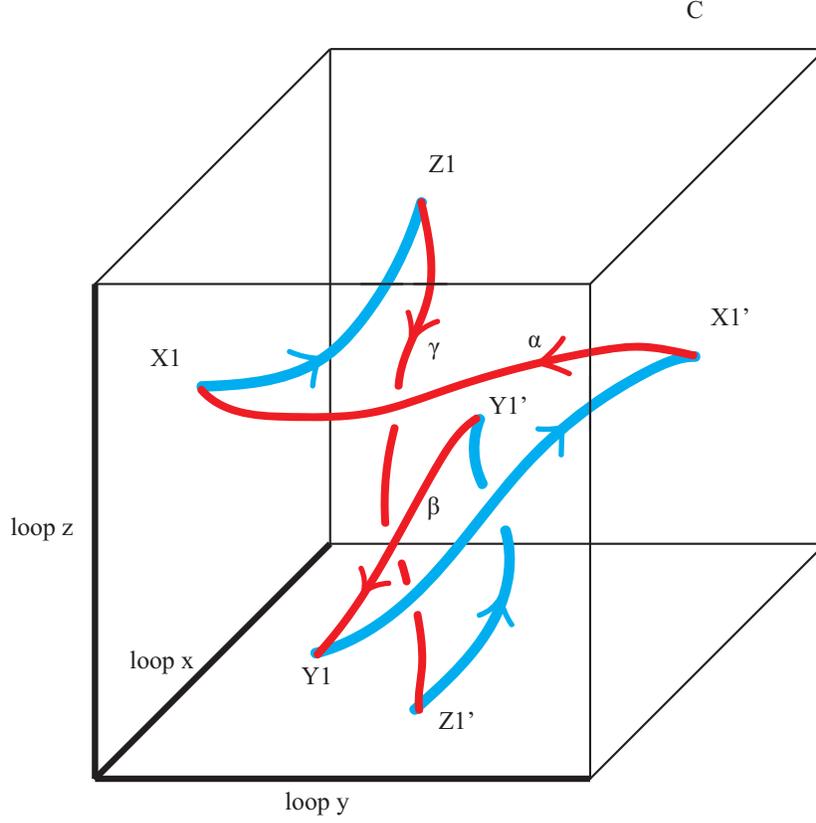
FIG. 12. Example of identifying loops for a link in 3-torus

5. THE FIRST HOMOLOGY GROUP

In this section we show how to determine from the diagram the first homology group of links in 3-torus T^3 . Consider a diagram of an oriented knot $K \subset T^3$ and let $\epsilon_i, \nu_j, \tau_k$ be as defined in the previous section. Define $\delta = \sum_{i=0}^n \epsilon_i$, $\sigma = \sum_{j=0}^m \nu_j$, $\xi = \sum_{k=0}^l -\tau_k$.

Lemma 1. *If $K \subset T^3$ is an oriented knot and $[K]$ is the homology class of K in $H_1(T^3)$, then $[K] = (\delta, \sigma, \xi)$.*

Proof. Let x, y, z be generator of $H_1(T^3) = \mathbb{Z} \oplus \mathbb{Z} \oplus \mathbb{Z}$, as shown in Figure 13. Let $K \cap (\partial C / \sim) = \{X_1, \dots, X_n, X'_1, \dots, X'_n, Y_1, \dots, Y_m, Y'_1, \dots, Y'_m, Z_1, \dots, Z_l, Z'_1, \dots, Z'_l\}$. For $i = 1, \dots, n; j = 1, \dots, m; k = 1, \dots, l$ consider the identification class $[X_i] = \{X_i, X'_i\}; [Y_j] = \{Y_j, Y'_j\}; [Z_k] = \{Z_k, Z'_k\}$. The notations $X_i, X'_i, Y_j, Y'_j, Z_k, Z'_k$ are in the manner as at the section 1 about diagrams. Denote with α_i the loop (red curve) in T^3 connecting X_i with X'_i as in Figure 13, oriented according to the orientation of the knot K as depicted when $\epsilon_i = 1$ and in the opposite direction if $\epsilon_i = -1$. Analogously we connect Y_j and Y'_j , Z_k and Z'_k by loops β_j and γ_k respectively. The homology class of these loops are $[\alpha_i] = \epsilon_i y$, $[\beta_j] = \nu_j x$, $[\gamma_k] = -\tau_k z$. The loop $K' = K \cup_{i=1}^n \alpha_i \cup_{j=1}^m \beta_j \cup_{k=1}^l \gamma_k$ is homologically trivial, so we have:


 FIG. 13. Connecting arcs for knot in T^3

$$(0, 0, 0) = [K'] = [K] + \left(\sum_{i=1}^n [\alpha_i], \sum_{j=1}^m [\beta_j], \sum_{k=1}^l [\gamma_k] \right)$$

Thus, $[K] = (\delta, \sigma, \xi)$, where $\delta = \sum_{i=0}^n \epsilon_i$, $\sigma = \sum_{j=0}^m \nu_j$, $\xi = \sum_{k=0}^l -\tau_k$. \square

Lemma 2. Let L be a link in 3-torus T^3 , with components L_1, \dots, L_ω . For each $\iota = 1, \dots, \omega$, let $(\delta_\iota, \sigma_\iota, \xi_\iota) = [L_\iota] \in \mathbb{Z}^3 = H_1(T^3)$. Then

$$H_1(T^3 \setminus L) \cong \begin{cases} \mathbb{Z}^3 \oplus \mathbb{Z}_\rho, & \text{if } \omega = 1 \\ \mathbb{Z}^3 \oplus \mathbb{Z}_\kappa \oplus \mathbb{Z}_\lambda, & \text{if } \omega = 2 \\ \mathbb{Z}^\omega \oplus \mathbb{Z}_\zeta \oplus \mathbb{Z}_\eta \oplus \mathbb{Z}_\theta, & \text{if } \omega \geq 3. \end{cases}$$

where $\rho = \gcd(\delta_1, \sigma_1, \xi_1)$; κ and λ are the invariant factor of the matrix M_1 ; ζ, η and θ are the invariant factor of the matrix M_2 .

$$M_1 = \begin{pmatrix} \delta_1 & \delta_2 \\ \sigma_1 & \sigma_2 \\ \xi_1 & \xi_2 \end{pmatrix};$$

$$M_2 = \begin{pmatrix} \delta_1 & \delta_2 & \dots & \delta_\omega \\ \sigma_1 & \sigma_2 & \dots & \sigma_\omega \\ \xi_1 & \xi_2 & \dots & \xi_\omega \end{pmatrix}.$$

Proof. We abelianize the fundamental group presentation given in previous section. Relation of type W and Q imply that generators corresponding to the same link component are homologous. So $H_1(T^3 \setminus L)$ is generated by g_1, \dots, g_ω , which are generators corresponding to the link components, and x, y, z generators corresponding to the 3-torus T^3 . Relation T become:

$$\begin{aligned} (1) \quad & \delta_1 g_1 + \dots + \delta_\omega g_\omega = 0, \\ (2) \quad & \sigma_1 g_1 + \dots + \sigma_\omega g_\omega = 0, \\ (3) \quad & \xi_1 g_1 + \dots + \xi_\omega g_\omega = 0. \end{aligned}$$

Thus we have the first homology group $H_1(T^3 \setminus L)$ of link complement in 3-torus by Hurewicz's theorem, that's generated by $g_1, \dots, g_\omega, x, y, z$ and its relators are (1)(2)(3) as above. If $\omega = 1$ then $H_1(T^3 \setminus L) \cong \mathbb{Z}^3 \oplus \mathbb{Z}_\rho$, where $\rho = \gcd(\delta_1, \sigma_1, \xi_1)$. If $\omega = 2$ then $H_1(T^3 \setminus L) \cong \mathbb{Z}^3 \oplus \mathbb{Z}_\kappa \oplus \mathbb{Z}_\lambda$, where κ and λ are the invariant factor of the matrix M_1 . If $\omega \geq 3$ then $H_1(T^3 \setminus L) \cong \mathbb{Z}^\omega \oplus \mathbb{Z}_\zeta \oplus \mathbb{Z}_\eta \oplus \mathbb{Z}_\theta$, where ζ, η and θ are the invariant factor of the matrix M_2 . □

6. THE ALEXANDER-FOX MATRIX AND TWISTED ALEXANDER POLYNOMIALS OF LINKS IN THREE DIMENSIONAL TORUS

Given a presentation of the group of a link, one may calculate its Alexander polynomial using Fox free calculus [17]. We recall the following definition of Alexander polynomials (see [18]). Let

$$P = \langle x_1, \dots, x_n \mid r_1, \dots, r_m \rangle$$

be a presentation of a group G and denote by $H = G/G'$ its abelianization. Let $F = \langle x_1, \dots, x_n \rangle$ be the corresponding free group. We apply the chain of maps

$$\mathbb{Z}F \xrightarrow{\frac{\partial}{\partial x}} \mathbb{Z}F \xrightarrow{\gamma} \mathbb{Z}G \xrightarrow{\alpha} \mathbb{Z}H,$$

where $\frac{\partial}{\partial x}$ denotes the Fox differential, γ is the quotient map by relations r_1, \dots, r_m and α is the abelianization map. The Alexander-Fox matrix of the presentation P is the matrix $A = [a_{i,j}]$, where $a_{i,j} = \alpha(\gamma(\frac{\partial r_i}{\partial x_j}))$ for $i = 1, \dots, m$ and $j = 1, \dots, n$. For $k = 1, \dots, \min\{m-1, n-1\}$, the k -th elementary ideal $E_k(P)$ is the ideal of $\mathbb{Z}H$, generated by the determinants of all the $(n-k)$ minors of A . The first elementary ideal $E_1(P)$ is the ideal of $\mathbb{Z}H$, generated by the determinants of the all the $(n-1)$ minors of A .

Definition 1. . Let $L \subset S^3$ be a link, and let $E_k(P)$ be the k -th elementary ideal, obtained from a presentation P of fundamental group $\pi_1(S^3 \setminus L, *)$. Then the k -th link polynomial $\Delta_k(L)$ is the generator of the smallest principal ideal containing $E_k(P)$. The Alexander polynomial of L , denoted by $\Delta(L)$, is the first link polynomial of L .

For a classical link L in S^3 , the abelianization of $\pi_1(S^3 \setminus L, *)$ is the free abelian group, whose generators correspond to the components of L . For a link in 3-torus T^3 , the abelianization of its link group may also contain torsion, as we know by Lemma 2. In this case, we need the notion of a twisted Alexander polynomial.

Let G be a group with a finite presentation P and abelianization $H = G/G'$ and denote $K = H/Tors(H)$. Then every homomorphism $\sigma : Tors(H) \rightarrow \mathbb{C}^* = \mathbb{C} \setminus \{0\}$ determines a twisted Alexander polynomial $\Delta^\sigma(P)$ as follows. Choosing a splitting $H = Tors(H) \times K$, σ defines a ring homomorphism $\sigma : \mathbb{Z}H \rightarrow \mathbb{C}K$ sending $(f, g) \in Tors(H) \times K$ to $\sigma(f)g$. Thus we apply the chain of maps

$$\mathbb{Z}F \xrightarrow{\frac{\partial}{\partial x}} \mathbb{Z}F \xrightarrow{\gamma} \mathbb{Z}G \xrightarrow{\alpha} \mathbb{Z}H \xrightarrow{\sigma} \mathbb{C}K$$

and obtain the σ -twisted Alexander matrix $A^\sigma = \left[\sigma(\alpha(\gamma(\frac{\partial r_i}{\partial x_j}))) \right]$. The twisted Alexander polynomial is then defined by $\Delta^\sigma(P) = \gcd(\sigma(E_1(P)))$.

Definition 2. Let $L \subset T^3$ be a link in the three dimensional torus T^3 . For any presentation P of the link group $\pi_1(T^3 \setminus L, *)$, we may define the following.

The Alexander polynomial of L , denoted by $\Delta(L)$, is the generator of the smallest principal ideal containing $E_1(P)$.

For any homomorphism $\sigma : Tors(H_1(T^3 \setminus L)) \rightarrow \mathbb{C}^*$, the σ -twisted Alexander polynomial of L is $\Delta^\sigma(L) = \gcd(\sigma(E_1(P)))$.

We know from Lemma 2 that the torsion subgroup of $H_1(T^3 \setminus L)$ is the group $\mathbb{Z}_\zeta \oplus \mathbb{Z}_\eta \oplus \mathbb{Z}_\theta$ in general. So the image of the group homomorphism $\sigma : Tors(H_1(T^3 \setminus L)) \rightarrow \mathbb{C}^*$ is contained in the cyclic group, generated by Ω , the d -root of unity, where d is $lcm(\zeta, \eta, \theta)$. The σ -twisted Alexander polynomial $\Delta^\sigma(L) \in \mathbb{Z}[\Omega][K]$ is defined up to multiplication by $\Omega^j g$, with $g \in K$.

A link is called local or affine if it is contained in a ball embedded in 3-torus T^3 . For local links the following properties hold.

Proposition 1. Let L be a local link in 3-torus T^3 . Then the Alexander polynomial $\Delta_L = 0$.

Proof. The fundamental group of L can be presented with the relation of Wirtinger type and the torus relations only. The generator for the group are torus generator x, y, z and the generators corresponding to arcs of the diagram as the one for L in S^3 . The Jacobian matrix J has the following form

$$J = \left[\begin{array}{ccc|c} y - yxy^{-1}x^{-1} & 1 - yxy^{-1} & 0 & 0 \\ z - zxz^{-1}x^{-1} & 0 & 1 - zxz^{-1} & 0 \\ 0 & 1 - yzy^{-1} & y - yzy^{-1}z^{-1} & 0 \\ \hline & 0 & & J_{\bar{L}} \end{array} \right]$$

where $J_{\bar{L}}$ is the Jacobian matrix of the link group of the link L consider as a link in S^3 .

The upper left block of the matrix A is the part corresponding to torus generator x, y, z and their relations $yxy^{-1}x^{-1}, zxz^{-1}x^{-1}, yzy^{-1}z^{-1}$. For this case the homology group does not contain torsion, so via natural projection to $\mathbb{Z}H$, sending every generators to t . We have the Alexander-Fox matrix has the form

$$A = \left[\begin{array}{ccc|c} t-1 & 1-t & 0 & \\ t-1 & 0 & 1-t & 0 \\ 0 & 1-t & t-1 & \\ \hline & 0 & & A_L \end{array} \right]$$

From this we easily see that $\Delta_L = \Delta_L \det T$, where T is the matrix corresponding to the upper left block of the matrix A . We have $\det T = 0$, this implies $\Delta_L = 0$. \square

As a consequence a knot in 3-torus with a nontrivial Alexander polynomial cannot be local.

Let $L = L_1 \# L_2$, where $\#$ denotes the connected sum and L_2 is a local link. The decomposition $(T^3, L) = (T^3, L_1) \# (S^3, L_2)$ induces monomorphism $j_1 : H_1(T^3 \setminus L_1) \rightarrow H_1(T^3 \setminus L)$ and $j_2 : H_1(T^3 \setminus L_2) \rightarrow H_1(T^3 \setminus L)$. Given $\sigma : \mathbb{Z}[H_1(T^3 \setminus L)] \rightarrow \mathbb{C}[G]$ induced by $\sigma \in \text{hom}(\text{Tor}s(H_1(T^3 \setminus L)) \rightarrow \mathbb{C}^*)$, denote with σ_1, σ_2 its restrictions to $\mathbb{Z}[j_1(H_1(T^3 \setminus L_1))]$ and $\mathbb{Z}[j_2(H_1(T^3 \setminus L_2))]$ respectively. We have the following proposition, that is analogous to the Proposition 8 in [8].

Proposition 2. *Let $L = L_1 \# L_2 \subset T^3$, where L_2 is a local link. Then $\Delta^\sigma(L) = \Delta^{\sigma_1}(L_1) \cdot \Delta^{\sigma_2}(L_2)$.*

Proof. Let $\pi_1(T^3 \setminus L_1) = \langle a_1, \dots, a_n | r_1, \dots, r_n \rangle$ is the fundamental group of complement of the link L_1 in T^3 and $\pi_1(S^3 \setminus L_2) = \langle b_1, \dots, b_m | s_1, \dots, s_m \rangle$. Then by the Van Kampen theorem we get a presentation for $\pi_1(T^3 \setminus L)$ is $\langle a_1, \dots, a_n, b_1, \dots, b_m | r_1, \dots, r_n, s_1, \dots, s_m, a_1 = b_1 \rangle$. We have the Alexander-Fox matrix of L as follow

$$A_L = \begin{bmatrix} & & A_{L_1} & & & & 0 & \\ & & 0 & & & & A_{L_2} & \\ -1 & 0 & \dots & 0 & 1 & 0 & \dots & 0 \end{bmatrix},$$

where A_{L_i} is the Alexander-Fox matrix of L_i , for $i = 1, 2$. If $d_k(A)$ denotes the greatest common division of all k -minors of a matrix A , then by straight forward computation will show that $d_{m+n-1}(A_L) = d_{n-1}(A_{L_1}) \cdot d_{m-1}(A_{L_2})$. So for the case of first elementary ideal of π , which is the ideal of $\mathbb{C}G$, that implies the equation $\Delta^\sigma(L) = \Delta^{\sigma_1}(L_1) \cdot \Delta^{\sigma_2}(L_2)$ for twisted Alexander polynomial. \square

7. EXAMPLES

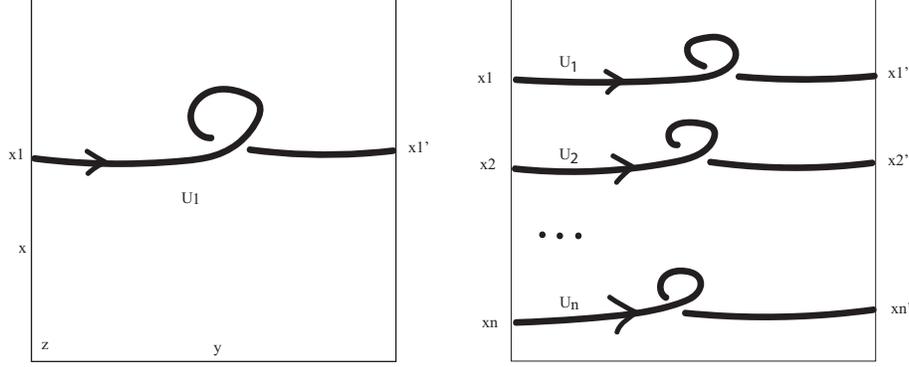
In this section we carry computation of (twisted) Alexander polynomials for some simple knots in 3-torus.

Example 1. Local trivial knot. The Alexander polynomial of it is 0.

Example 2. Global trivial knot U_1 , winding along the loop x of the 3-torus once. A diagram for it is depicted in the Fig. 14. The fundamental group $\pi_1(T^3 \setminus U_1)$ is

$$\begin{aligned} \pi_1(T^3 \setminus U_1) &= \langle x, y, z, x_1, x'_1 | x_1 = x'_1, x'_1 = x x_1 x^{-1}, z x z^{-1} x^{-1} = 1, \\ & \quad x y x^{-1} x^{-1} = 1, y z y^{-1} z^{-1} = x_1 \rangle = \langle x, y, z, x_1 | x_1 x x_1^{-1} x^{-1} = 1, \\ & \quad z x z^{-1} x^{-1} = 1, y x y^{-1} x^{-1} = 1, y z y^{-1} z^{-1} = x_1 \rangle \end{aligned}$$

Its first homology group is $\langle x, y, z, x_1 | x_1 = 1, x y = y z, x z = z x, y z = z y \rangle = \langle x, y, z | x y = y z, x z = z x, y z = z y \rangle \cong \mathbb{Z}^3$. The Jacobian matrix of $\pi_1(T^3 \setminus U_1)$ is


 FIG. 14. Diagrams of U_1 (left), L_n (right)

$$A_{U_1} = \left[\begin{array}{c|cccc} \text{relators} & x & y & z & x_1 \\ \hline x_1 x x_1^{-1} x^{-1} = 1 & x_1 - x_1 x x_1^{-1} x^{-1} & 0 & 0 & 1 - x_1 x x_1^{-1} \\ z x z^{-1} x^{-1} = 1 & z - z x z^{-1} x^{-1} & 0 & 1 - z x z^{-1} & 0 \\ y x y^{-1} x^{-1} = 1 & y - y x y^{-1} x^{-1} & 1 - y x y^{-1} & 0 & 0 \\ y z y^{-1} z^{-1} x_1^{-1} = 1 & 0 & 1 - y z y^{-1} & z - y z y^{-1} z^{-1} & -y z y^{-1} z^{-1} x_1^{-1} \end{array} \right]$$

The first homology group of $T^3 \setminus U_1$ is torsion free, so we have Alexander-Fox matrix is

$$A_{U_1} = \begin{bmatrix} 0 & 0 & 0 & 1-t \\ t-1 & 0 & 1-t & 0 \\ t-1 & 1-t & 0 & 0 \\ 0 & 1-t & t-1 & -1 \end{bmatrix},$$

and the Alexander polynomial is $\Delta_{U_1}(t) = (t-1)^2$

Example 3. Global trivial link L_n from n unlinked component U_1, \dots, U_n , all components winding along the loop x of the 3-torus once. A diagram for it is depicted in the Fig. 14. The fundamental group of $\pi_1(T^3 \setminus L_n)$ is

$$\begin{aligned} \langle x, y, z, x_1, x_1', \dots, x_n, x_n' \mid x_1 = x_1', \dots, x_n = x_n', x_1' = x x_1 x^{-1}, \dots, x_n' = x x_n x^{-1}, \\ z x z^{-1} x^{-1} = 1, y x y^{-1} x^{-1} = 1, y z y^{-1} z^{-1} = x_1 \rangle \\ = \langle x, y, z, x_1, \dots, x_n \mid x_1 x x_1^{-1} x^{-1} = 1, \dots, x_n x x_n^{-1} x^{-1} = 1, \\ z x z^{-1} x^{-1} = 1, y x y^{-1} x^{-1} = 1, y z y^{-1} z^{-1} = x_1 \rangle \end{aligned}$$

Its first homology group is commutative additive group $\langle x, y, z, x_1, \dots, x_n \mid x_1 + \dots + x_n = 0 \rangle = \langle x, y, z, x_2, \dots, x_n \rangle \cong \mathbb{Z}^{n+2}$. The Jacobian matrix A_{U_1} of $\pi_1(T^3 \setminus L_n)$ is

$$\left[\begin{array}{c|cccccccc} \text{relators} & x & y & z & x_1 & x_2 & \dots & x_n \\ \hline x_1 x x_1^{-1} x^{-1} = 1 & x_1 - x_1 x x_1^{-1} x^{-1} & 0 & 0 & 1 - x_1 x x_1^{-1} & 0 & \dots & 0 \\ x_2 x x_2^{-1} x^{-1} = 1 & x_2 - x_2 x x_2^{-1} x^{-1} & 0 & 0 & 0 & 1 - x_2 x x_2^{-1} & \dots & 0 \\ \dots & \dots \\ x_n x x_n^{-1} x^{-1} = 1 & x_n - x_n x x_n^{-1} x^{-1} & 0 & 0 & 0 & 0 & \dots & 1 - x_n x x_n^{-1} \\ z x z^{-1} x^{-1} = 1 & z - z x z^{-1} x^{-1} & 0 & 1 - z x z^{-1} & 0 & 0 & \dots & 0 \\ y x y^{-1} x^{-1} = 1 & y - y x y^{-1} x^{-1} & 1 - y x y^{-1} & 0 & 0 & 0 & \dots & 0 \\ y z y^{-1} z^{-1} x_1^{-1} = 1 & 0 & 1 - y z y^{-1} & z - y z y^{-1} z^{-1} & -y z y^{-1} z^{-1} x_1^{-1} & -y z y^{-1} z^{-1} x_1^{-1} x_2^{-1} & \dots & -y z y^{-1} z^{-1} x_1^{-1} \dots x_n^{-1} \end{array} \right]$$

The first homology group is torsion free, we get Alexander-Fox matrix, sending generators x, y, z, x_2, \dots, x_n to t , and x_1 to 1.

$$A_{U_1} = \begin{bmatrix} 0 & 0 & 0 & 1-t & 0 & \dots & 0 \\ t-1 & 0 & 0 & 0 & 1-t & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ t-1 & 0 & 0 & 0 & 0 & \dots & 1-t \\ t-1 & 0 & 1-t & 0 & 0 & \dots & 0 \\ t-1 & 1-t & 0 & 0 & 0 & \dots & 0 \\ 0 & 1-t & t-1 & -t^{n-1} & -t^{n-2} & \dots & -1 \end{bmatrix},$$

With a simple computation via induction by the number of component, we have the Alexander polynomial for the link L_n is $\Delta_{L_n} = (t-1)^{n+1}$

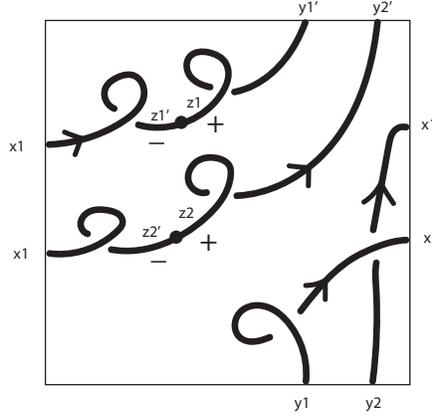


FIG. 15. Diagram of K

Example 4. Let K is the knot in T^3 with diagram shown in the Fig. 15. It has the knot group $\pi_1(T^3 \setminus K)$ in T^3

$$\begin{aligned} \langle x, y, z, x_1, x'_1, x_2, x'_2, y_1, y'_1, y_2, y'_2, z_1, z'_1, z_2, z'_2 | x_1 = z'_1, x_2 = z'_2, z_1 = y'_1, z_2 = y'_2, \\ y_1 = x'_2, (x'_2)y_2(x'_2)^{-1}(x'_1)^{-1} = 1, xx_1x^{-1}(x'_1)^{-1} = 1, xx_2x^{-1}(x'_2)^{-1} = 1, \\ yy_1y^{-1}(y'_1)^{-1} = 1, yy_2y^{-1}(y'_2)^{-1} = 1, zz_1z^{-1}(z'_1)^{-1} = 1, zz_2z^{-1}(z'_2)^{-1} = 1, \\ zxz^{-1}x^{-1}(y_1)^{-1}(y_2)^{-1} = 1, yzy^{-1}z^{-1}(x_1)^{-1}(x_2)^{-1} = 1, \\ yxy^{-1}x^{-1}(z_1)^{-1}(z_2)^{-1} = 1, \rangle \end{aligned}$$

The first homology group is commutative additive group $H = \langle x, y, z, g | 2g = 0 \rangle \cong \mathbb{Z}^3 \oplus \mathbb{Z}_2$.

The twisted homomorphism $\sigma_1 : \mathbb{Z}[H] \rightarrow \mathbb{Z}[t, t^{-1}]$ is determined by $\sigma_1(x) = \sigma_1(y) = \sigma_1(z) = t$ and $\sigma_1(g) = 1$. We get the twisted Alexander polynomial $\Delta_K^{\sigma_1} = (t+1)(t-1)^2(t^2-t+1)$. The twisted homomorphism $\sigma_2 : \mathbb{Z}[H] \rightarrow \mathbb{Z}[t, t^{-1}]$ is defined as $\sigma_2(x) = \sigma_2(y) = \sigma_2(z) = t$ and $\sigma_2(g) = -1$. Then twisted Alexander polynomial is $\Delta_K^{\sigma_2} = (t-1)^2(t^2-t+1)$.

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