COMBINATORIAL RIGIDITY AND FLEXIBILITY OF SIMPLICIAL 2-COMPLEXES WITH FEW VERTICES

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ABSTRACT. We study the problem of reconstruction of a simplicial 2-complex from its 1-skeleton together with the prescribed quantities of 2-simplices at each 1-simplex, under the restriction that these quantities are bounded above by 2. It is a known fact that a 2-complex is uniquely reconstructible, or "combinatorially rigid", if it has 5 or fewer vertices. In this paper "combinatorially flexible" 2-complexes (that is, non-uniquely reconstructible from their 1-skeletons) with 6 vertices are characterized in terms of necessary 2-subcomplexes.

Keywords: simplicial 2-complex, graph, triangulation, 2-manifold, combinatorial flexibility, combinatorial rigidity.

1. INTRODUCTION

A simplicial 2-complex L can be viewed as a 1-complex ("graph") fitted with a collection of 2-simplices ("triangles"). If there is a different collection of 2-simplices in which the 1-simplices ("edges") occur with the same multiplicities, L is said to be (combinatorially) "flexible", otherwise "rigid". In this paper we study the reconstruction of L from the edge-triangle incidence numbers. We restrict our attention to the case where each edge is incident with at most 2 triangles, and our motivation comes from studying triangulations of 2-manifolds.

Formally, the question is whether a given simplicial 2-complex is uniquely reconstructible from its 1-skeleton together with the quantities of 2-simplices at each

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1-simplex. Originally this question emerged from the Newtonian potential inverse problem for polyhedra; the uniqueness of reconstruction of a polyhedron from its graph guarantees the uniqueness of the solution to this problem. The answer is generally negative and, as usual in the theory of graphs and hypergraphs, requires some characterization of the nonuniqueness (resp. uniqueness) in terms of necessary (resp. forbidden) 2-subcomplexes. In this paper we are largely concerned with deriving a characterization, in terms of necessary 2-subcomplexes, of *(combinatorially) flexible* 2-complexes, that is, 2-complexes which are reconstructed non-uniquely. Then we reinterpret this result for characterizing the graphs which are uniquely embeddable in a given closed 2-manifold.

Our graph-theoretic terminology and notation are fairly standard. The term "graph" always means a "simple graph," that is, a graph without loops or multiple edges. The term "cycle" always means a "simple cycle", that is, a closed walk without repeated vertices. An *n-cycle* is a cycle of length n. A *triangulation* of a 2-manifold Σ with a graph G is an embedding $T : G \to \Sigma$ whose each face is bounded by a 3-cycle of G. Thus a triangulation T naturally gives rise to an abstract simplicial 2-complex L whose 1-skeleton is isomorphic to G and whose underlying space is homeomorphic to Σ. As matter of language, the 1-skeleton of 2-complex L is called the *graph* of L and denoted by $G = G(L)$. The set of vertices (0-simplices), edges (l-simplices), and triangles (2-simplices) of L are denoted by $V(L)$, $E(L)$, and $F(L)$, respectively. Combinatorially, $F(L)$ corresponds to the set of 3-cycles of G bounding triangles of L. The *star* of v in T, denoted by $\mathrm{st}(v, T)$, is defined to be the family of all simplices in T of which v is a vertex. For an edge uv of G, the degree of uv in L, denoted by $\deg(uv, L)$, is defined to be the number of triangles of L of which uv is an edge. Note that in any triangulation of a closed 2-manifold the degree of each edge equals 2.

Let L_1 and L_2 be 2-complexes, both with the same graph G. We say that L_1 and L_2 are *(combinatorially) equivalent* if $F(L_1) = F(L_2)$, that is, any 3-cycle of G bounds a triangle either in both L_1 and L_2 or in neither. Otherwise we say that L_1 and L_2 are (combinatorially) *distinct*. Therefore, a 2-complex L is completely determined by its graph $G(L)$ together with the collection of triangles $F(L)$. Furthermore, L_1 and L_2 are said to be *isomorphic* if there exists a graph-isomorphism $\pi : G(L_1) \to G(L_2)$ such that a 3-cycle (u, v, w, u) bounds a triangle of L_1 if and only if the 3-cycle $(\pi(u), \pi(v), \pi(w), \pi(u))$ bounds a triangle of L_2. Note that distinct 2-complexes are distinguishable at the vertex-labeled level and that they may be isomorphic, that is, identical once the labels of the vertices are neglected.

We begin with a general discussion of the problem of reconstruction of a 2-complex L from its graph. It is a previously known result [23] that the reconstruction problem has a unique solution when $|V(L)| \leqslant 5$. The central result of this paper, Theorem 1 in Section 4, is a characterization of combinatorially flexible 2-complexes on 6 vertices. Finally, we apply these results to characterizing the combinatorial uniqueness of the triangulability of a fixed closed 2-manifold by a given graph.

2. The reconstruction problem

For interpretation of 3-dimentional gravity (or magnetic) anomalies, geophysicists often use polytopal approximations of the sources causing the anomalies and face the inverse Newtonian potential problem for polyhedra. The volume occupied

by mass with a constant density in 3-space $\mathbb{R}^3$ is approximated by a bounded homogeneous 3-dimensional polytope D in $\mathbb{R}^3$. Then, it is a result of M. A. Brodsky [1, 2] that the solution to the inverse potential problem, for the exterior gravitational potential created by D, is unique (that is, is D itself) if and only if the boundary 2-complex of D is uniquely reconstructed from its 1-skeleton (in geometric sense). In connection with this result, it is worthwhile to mention a theorem of M. A. Brodsky and V. N. Strakhov [3] which states that any geometric 2-dimensional polyhedron is uniquely reconstructable from its 1-skeleton whenever that polyhedron bounds a topological 3-ball embedded in $\mathbb{R}^3$.

The reconstruction problem of 2-dimensional topology that arose from this interpretation is posed as follows:

> **Given:** A nontrivial edge-weighted graph G with weights in $\{0, 1, 2\}$, that is, a pair $\{G, \omega(\,\cdot\,, G)\}$, where $\omega(\,\cdot\,, G) : E(G) \to \{0, 1, 2\}$ is a given function;
> **Find:** All 2-complexes $L = L(G, \omega(\,\cdot\,, G))$, up to equivalence, satisfying the following two conditions:
>
> - *edge condition:* $G(L) = G$,
> - *edge-degree condition:* $\deg(uv, L) = \omega(uv, G)$ for all $uv \in E(G)$.

The limitation, $\omega(\forall, G) \leqslant 2$, is suggested by the study of 2-manifolds; boundary edges have weight 1, interior edges have weight 2. By writing "$\omega(\forall, G)\ldots$" we mean "$\omega(uv, G)\ldots$ for all $uv \in E(G)$".

We are primarily concerned with 2-complexes in which each edge has degree $\leqslant 2$, and L always denotes such an object. L is (combinatorially) *rigid* if it is uniquely reconstructed from its graph $G(L)$; more precisely, the reconstruction problem for the edge-weighted graph $\{G(L), \deg(\,\cdot\,, L)\}$ has a unique solution, that is, L itself. Otherwise L is (combinatorially) *flexible*.

The sum modulo 2 of two 2-complexes L_1 and L_2, both with the same graph G, is defined to be a 2-complex, denoted by $L_1 \oplus L_2$, which also has G as its graph and has a triangle uvw whenever uvw is a triangle of L_1 but not of L_2 or vice versa. Let $KK(G)$ denote the *complete* 2-*complex* on G, that is, the 2-complex with graph G in which every 3-cycle of G bounds a triangle. It is easy to see that the operation $\oplus$ makes the family of the 2-complexes having G as graph into an abelian group. This group is isomorphic to the group of 2-chains $C_2(KK(G))$ over the ring of integers modulo 2. Note that $L_1 \oplus L_2$ degenerates into a 1-complex if and only if $L_1 \oplus L_2 = 0$, or if and only if L_1 and L_2 are equivalent, written $L_1 = L_2$.

Lemma 1. *Assume that L_1 and L_2 are two solutions to the reconstruction problem. Then the simplicial 2-complex $L_1 \oplus L_2$ has the following three properties:*

(i) The degree of each edge of $L_1 \oplus L_2$ is even.

(ii) If each edge of $L_1 \oplus L_2$ has degree less than four, then $L_1 \oplus L_2$ is triangle-2-colorable: one color for the triangles in L_1 and the other for the triangles in L_2.

(iii) Under the hypothesis of part (ii), $L_1 \oplus L_2$ is Grünbaum colorable, that is, the edges of $L_1 \oplus L_2$ can be assigned three fixed colors so that the edges bounding any triangle of $L_1 \oplus L_2$ are assigned three different colors.

Proof. (i) is obvious.

(ii) Observe that, by (i), the degree of each edge of $L_1 \oplus L_2$ is either 0 or 2. In the latter case, by the edge-degree condition, one of the two triangles of $L_1 \oplus L_2$ at that edge is in L_1 while the other in L_2.

(iii) In fact, the existence of Grünbaum coloring follows from the triangle-2-colorability stated in (ii); see [13, 15]. □

A *pseudosurface* is a topological space Π which fails to be a 2-manifold at a finite number of points, called *singular points*; more precisely, a small neighborhood of a singular point is homeomorphic to the union of two or more cones having only their vertex in common. (In this context, a 2-manifold may be thought of as a pseudosurface with 0 singular points.) A triangulation of a pseudosurface Π is a 2-complex whose underlying space is homeomorphic to Π. Clearly, in any triangulation T of Π, every singular point of Π is occupied by a vertex of T; furthermore, if Π has empty boundary, every edge of T has degree 2.

Lemma 2. [23] *Any simplicial 2-complex L having at most five vertices and with each edge of degree at most 2 is combinatorially rigid.*

Although the content of this lemma is a known result, here we include a proof using Lemma 1, which is more elegant than the original proof [23].

Proof. Suppose for a contradiction that L_1 and L_2 are two distinct solutions to the reconstruction problem for $\{G(L), \deg(\,\cdot\,, L)\}$. By Lemma 1(i) and the hypothesis, $|V(L)| \leqslant 5$, we have $\deg(uv, L_1 \oplus L_2) \in \{0, 2\}$ for all $uv \in E(G)$. Let M be the 2-subcomplex obtained from $L_1 \oplus L_2$ by deleting the edges of degree 0. Certainly, then, the underlying space of M is a pseudosurface without boundary. Furthermore, it is actually a closed 2-manifold because the property of a vertex to be singular obviously requires $\geqslant 6$ incident edges. Therefore, M is actually a triangulation of some closed 2-manifold and, since $|V(M)| \leqslant 5$, that 2-manifold may only be the 2-sphere. On the other hand, it is easy to see that there are, in all, two triangulations of the 2-sphere with $\leqslant 5$ vertices, both with a vertex of degree 3, which is impossible by Lemma 1(ii). □

3. Examples

There are results of S. Negami [16, 18] and the first author [10, 8, 11, 9] that the complete graphs K_6 and K_7 triangulate respectively the projective plane $\tilde{\Sigma}_1$ (that is, the nonorientable 2-manifold of genus 1) and the torus Σ_1 (that is, the orientable 2-manifold of genus 1) uniquely up to isomorphism. Figs. 1a (or 1b) and 1c present triangulations $K_6 \to \tilde{\Sigma}_1$ and $K_7 \to \Sigma_1$, respectively. The 2-manifolds $\tilde{\Sigma}_1$ and Σ_1 are presented as quotient spaces of the respective polygons. The triangulations in each pair—(a,a'), (b,b'), and (c,c')—are distinct; they have only the unshaded triangles in common.

Two weight functions for the graph K_6 with vertices labeled by 0 to 5 are defined by:

$$\alpha(uv) = \begin{cases} 0 & uv \in \{0\,1,\ 2\,3,\ 4\,5\} \\ 1 & \text{otherwise} \end{cases}$$

$$\beta(uv) = \begin{cases} 2 & uv \in \{0\,1,\ 2\,3,\ 4\,5\} \\ 1 & \text{otherwise} \end{cases}$$
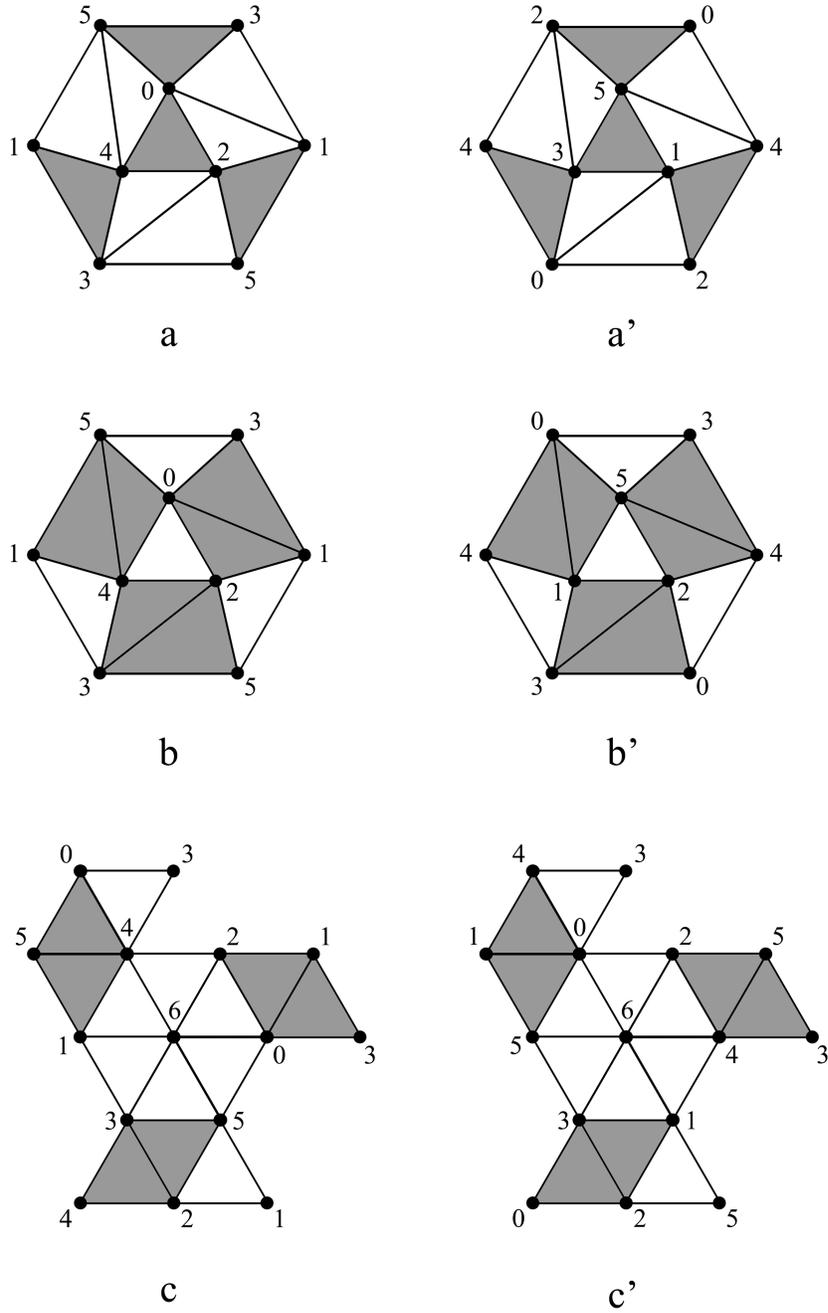
Figure 1.

It can be easily seen that the corresponding reconstruction problems have exactly two solutions each, isomorphic in pairs.

One solution to the 1st problem, that is, with weight function α, denote it by BT_1, is the 2-complex determined by the triangles shaded in Fig. 1a; the other

solution, BT_2, in shaded in Fig 1a'. Similarly, one solution to the 2nd problem, that is, with weight function β, denote it by BS_1, is the 2-complex shaded in Fig. 1b (and also in Fig. 1c); the other solution, BS_2, is shaded in Fig. 1b' (and also in Fig. 1c').

The *bunch of triangles*, denoted by BT, is defined to be the abstract 2-complex isomorphic to the two solutions to the 1st problem. The *bunch of squares*, denoted by BS, is the 2-complex isomorphic to the solutions to the 2nd problem.

Note that the 2-complexes shaded in Figs. 1c and 1c' are also bunches of squares, the former is equivalent to the 2-complex shaded in Fig. 1b, and the latter to the 2-complex shaded in Fig. 1b'.

Remark 1. The bunch of triangles BT may also be thought of as either of the two color classes of a triangle-2-coloring of an octahedron. This observation gives a way of constructing many flexible complexes with all edge-weights 1 as follows: Take any triangle-2-colorable triangulation of a closed 2-manifold, and take the two color classes.

Thanks to the identity $\alpha(uv) + \beta(uv) = 2$ (which holds for all edges uv of K_6), the 2-complexes

$$(1) \qquad\qquad PP_{ij} = BT_i \oplus BS_j \qquad\qquad (i, j \in \{1, 2\})$$

have underlying spaces homeomorphic to the projective plane $\tilde{\Sigma}_1$ and correspond to distinct triangulations $K_6 \to \tilde{\Sigma}_1$ (as defined in the Introduction). In fact, Figs. 1a (or 1b), 1a', and 1b' present triangulations PP_{11}, PP_{21}, and PP_{12}, respectively; PP_{22} is obtained from Fig. 1a' by "flipping" the diagonals in the three unshaded "squares." There are 3 choices of the edges of K_6 as diagonals of the three squares of BS: A first diagonal can be picked arbitrarily, say, $0\,1$; a second and a third diagonals can be any pair of independent edges in the graph $K_4 = K_6 - \{0, 1\}$ which has three 1-factors. For each of the 3 choices, there are four triangulations, $K_6 \to \tilde{\Sigma}_1$, of the form of Eq. (1). Therefore, the reconstruction problem for $G = K_6$ with $\omega(\forall, G) = 2$ has exactly 12 distinct solutions, all isomorphic. (It is a folk result that K_6 admits twelve distinct triangulations of the projective plane; see [18, 11, 9, 20, 12, 22].)

To conclude this section, we briefly address *self-complementary* 2-complexes, that is, 2-complexes isomorphic to their complements. The *complement*, $\overline{L}$, of a 2-complex L is the 2-complex with the same graph $G(L)$ in which any 3-cycle bounds a triangle whenever it does not bound a triangle in L, or in other words, $L \oplus \overline{L} = KK(G)$. Algebraically,

$$PP_{11} \oplus PP_{22} = (BT_1 \oplus BS_1) \oplus (BT_2 \oplus BS_2)$$
$$= (BT_1 \oplus BT_2) \oplus (BS_1 \oplus BS_2) = OH \oplus \overline{OH} = KK(K_6),$$

where OH denotes the usual octahedron with diagonals (all with degee 0). Hence, PP_{11} and PP_{22} (and, similarly, PP_{12} and PP_{21}) are complements of each other (in pairs). Therefore, $K_6 \to \tilde{\Sigma}_1$ is a self-complementary simplicial 2-complex; moreover, this 2-complex is the only such 2-complex (up to isomorphism) whose underlying space is homeomorphic to the projective plane [7].

4. On the nonuniqueness of reconstruction

The reader may note that in the examples given, the flexibility is ensured by the existence of a 2-subcomplex isomorphic to BT or BS. We now show that it is a general rule.

Theorem 1. *Any combinatorially flexible simplicial* 2-*complex L with six vertices and with all edges of degree at most* 2 *contains either the bunch of triangles BT or the bunch of squares BS as a* 2-*subcomplex.*

Proof. Let L_2 be another solution (if any) to the reconstruction problem for the edge-weighted graph $\{G(L), \deg(\,\cdot\,, L)\}$, distinct from L. We make the following two assumptions:

Assumption 1. $F(L) \cap F(L_2) = \emptyset$.

Assumption 2. $\deg(\forall, L) \geqslant 1$.

Assumptions 1 and 2 limit the number of triangles from above and below, respectively. These simplifying assumptions are justified since we can delete common triangles and null-weighted edges in both L and L_2, which would lead to two still distinct solutions to a "smaller" reconstruction problem, but still with 6 vertices, by Lemma 2; then we could repeat the argument for those smaller solutions. We have two cases to consider:

Case 1. $G(L) \neq K_6$.

Case 2. $G(L) = K_6$.

Consider Case 1. By Assumption 1, $\deg(v, G(L))) \geqslant 4$ for all $v \in V(G(L))$, since the triangles at a vertex of degree 3 are obviously reconstructed uniquely, whence each vertex of $G(L)$ has degree at least 4. Therefore the degree of each vertex of the complement $\overline{G}(L)$ is at most 1, whence $\overline{G}(L)$ is a collection of independent edges and isolated vertices. We suppose without loss of generality that $G(L)$ is either $K_6 - \{0\,1\}$ or $K_6 - \{0\,1, 2\,3\}$ or $K_6 - \{0\,1, 2\,3, 4\,5\}$. Hence, each edge at vertex 0 occurs in at most three 3-cycles of $G(L)$ and, by Lemma 1(i), along with Assumptions 1 and 2, each edge at 0 has degree 1 (in L). This is only possible when vertex 0 is in precisely two triangles of L (and similarly in two triangles of L_2) which are edge-disjoint. Those two triangles determine a pair of independent edges in the subgraph $\langle 2, 3, 4, 5 \rangle$ of $G(L)$ induced by these four vertices. This induced subgraph is in fact a subgraph of $K_4 = K_6 - \{0, 1\}$, not necessarily proper but certainly with $\geqslant 4$ edges; when it has exactly 4 edges, it is necessarily a 4-cycle. The same argument applies to vertex 1. Therefore, we have two pairs of independent edges in $\langle 2, 3, 4, 5 \rangle$, one corresponding to vertex 0, and the other to vertex 1. These two pairs either form a 4-cycle or coincide. In the former event we already have a desired 2-complex BT in L. The latter is in tact vacuous because fitting the induced subgraph $\langle 2, 3, 4, 5 \rangle$ even with a single triangle (if posible) would produce eldges of degree $\geqslant 3$.

Consider Case 2. We process the 2-complex L as follows. Choose an edge of degree 1 (if any). By Assumption 2, every edge of $G(L) = K_6$ is assigned weight 1 or 2, and hence there is another edge of degree 1 adjacent to the chosen one. Proceed along the edges of degree 1, starting from the chosen edge, until some vertex is reached for a second time, which creates a cycle. Paste a 2-disk along that

cycle, which raises, from 1 to 2, the degrees of the edges traversed. Repeat this process until each edge has degree equal to 2. Therefore the underlying space of L is homeomorphic to a bordered 2-manifold with a number of holes corresponding to the 2-disks pasted. By construction, those holes are bounded by edge-disjoint cycles. Let L' denote the 2-complex obtained from L by closing the triangular holes (if any) by new triangles; consequently, there are no triangular holes in L'.

If the underlying space of L' is homeomorphic to a closed 2-manifold, L' is necessarily isomorphic to the unique (up to isomorphism) triangulation $K_6 \to \tilde{\Sigma}_1$ (Fig. 1a,b) and L is its 2-subcomplex obtained by deleting a collection of edge-disjoint triangles. Then the reader can easily verify that L always contains BT or BS.

If the underlying space of L' is not a closed 2-manifold, there are some polygonal holes in L', bounded by edge-disjoint cycles of length $\geqslant 4$. Let r, h, b, and χ denote respectively the number of triangles, holes, boundary edges in L', and the Euler characteristic of the 2-manifold resulting from L' by closing all holes by 2-disks (it is indeed a 2-manifold because a singular vertex would require $\geqslant 6$ incident edges). By a standard counting argument involving Euler's polyhedral formula,

$$(2) \qquad r + h = \chi - |V(K_6)| + |E(K_6)| = \chi + 9,$$

$$(3) \qquad 3r + b = 2|E(K_6)| = 30,$$

$$(4) \qquad 4h \leqslant b \leqslant 12,$$

so that $h \leqslant 3$. The upper bound on b follows because the boundary edges form an eulerian subgraph of K_6 in which every vertex has even degree, and hence degree at most 4. We consider the possible values of h (that is, 1, 2, and 3) in the sequel.

Suppose first that $h = 1$. By Eq. (3), b must be a multiple of 3. Furthermore, by Inequality (4), $b \geqslant 4$ and, since the hole is bounded by a cycle of length $\leqslant |V(K_6)| = 6$, it follows that $b = 6$. Whence, by Eqs. (2) and (3), $\chi = 0$. Thus, adding a vertex in the 2-disk closing the hole, and connecting that vertex with edges to the boundary vertices result in the unique (up to isomorphism) triangulation $K_7 \to \Sigma_1$ (Fig. 1c), since a theorem of Philip Franklin [6] states that K_7 is not embeddable on the Klein bottle $\tilde{\Sigma}_2$ (that is, the nonorientable 2-manifold of genus 2). Whence L' is isomorphic to the 2-complex obtained from Fig. 1c by deleting vertex 6 together with the incident triangles. Observe that each shaded triangle in Fig.1c has some boundary edge of L' and, by Assumption 2, must be also present in L. Therefore, since the shaded 2-complex is BS, we are done in this case.

Suppose next that $h = 2$. By Inequality (4), $8 \leqslant b \leqslant 12$. Since b is a multiple of 3, it follows that b is either 9 or 12. The case $b = 12$, and hence $\chi = -1$, is also impossible, for otherwise we could convent L' into a triangulation of $\tilde{\Sigma}_3$ (that is, the nonorientable 2-manifold of genus 3) with 8 vertices by replacing the two holes by the stars of two new vertices as in the preceding paragraph; however, the minimum number of vertices in a triangulation of $\tilde{\Sigma}_3$ is 9, see G. Ringel [19]. The case $b = 9$, and hence $\chi = 0$, is vacuous too; here is the proof, by contradiction: The two holes must be bounded by a 4-cycle and a 5-cycle, respectively. Put a vertex in each of 2-disks closing the holes and connect it with edges to the boundary vertices. This converts L' into a triangulation of the torus Σ_1 or the Klein bottle $\tilde{\Sigma}_2$ with 8 vertices having a pair of nonadjacent vertices with degrees 4 and 5. We have checked

all the triangulations of Σ_1 and $\tilde{\Sigma}_2$ with 8 vertices, listed in [10, 8] and in [14, 21], respectively, and identified only one of Σ_1 and only one of $\tilde{\Sigma}_2$ with this property. However, in both of them the stars of any pair of nonadjacent vertices with degrees 4 and 5 are not edge-disjoint, a contradiction.

Suppose finally that $h = 3$. As above, we derive that $b = 12$. Hense, L' is one of the solutions to the reconstruction problem with 12 edges of K_6 assigned weight 1 and $|E(K_6)| - 12 = 3$ edges assigned weight 2. Since there must be an even number of edges of weight 1 incident with each vertex, the three edges of weight 2 must be independent. As known from Section 3, L' is then isomorphic to BS; furthermore, each triangle of L' has some edge of weight 1 and, by Assumption 2, must be present also in L, which finishes this case. (A more subtle analysis shows that this case is vacuous, though.) The theorem has been proved. $\square$

Generally, it would be worthwhile to characterize flexible 2-complexes in terms of necessary 2-subcomplexes, for a given number of vertices p. Theorem 1 settles the case of $p = 6$.

5. On the uniqueness of graph embedding

Let $T : G \to \Sigma$ be a triangulation of a closed 2-manifold Σ with a graph $G = G(T)$. Consider the reconstruction problem for G with $\omega(\forall, G) = 2$. The "rigidity / flexibility of T in Σ" is defined word for word in the same way as the usual (combinatorial) rigidity / flexibility of the 2-complex T under restricting to the solutions whose underlying spaces are homeomorphic to Σ. In the graph-theoretic language, the rigidity of T in Σ corresponds to the uniqueness of embedding $G \to \Sigma$; throughout this section, the term "uniqueness" means the uniqueness in the vertex-labeled sense. Whitney's Theorem [24] implies that any triangulation of the 2-sphere is rigid in the 2-sphere, but fails in 2-manifolds of higher genera in which flexibility occurs. In [5, 4] a global approach is elaborated to characterizing flexible triangulations (with arbitrarily many vertices). In this section we develop a local approach to characterizing rigid triangulations.

Let v be a vertex of T. If each triangle in the star $\mathrm{st}(v, T)$ is encountered in any triangulation $G \to \Sigma$, the vertex v is called *stable*, and *unstable* otherwise. For instance, a vertex of degree 3 is always stable. Let $S(G) = \{s_1, \ldots, s_m\}$ be a set of vertices which are known to be stable; especially, we may set

$$S(G) = \{v \in V(G) \mid \langle N(v, G) \rangle \text{ has a unique hamilton cycle}\},$$

where $N(v, G)$ denotes the set of neighbors of v in G, and, as usual, $\langle X \rangle$ denotes the subgraph of G induced by a given subset $X \subseteq V(G)$. Denote the set $V(G) \setminus S(G)$ by $U^0(G) = \{u_1, \ldots, u_n\}$, and by $U^1(T)$ and $U^2(T)$ the 1-subcomplex and the 2-subcomplex of T (respectively) induced by the set $U^0(G)$. Note that $U^1(T)$ is in fact a subgraph of G.

By $\mathrm{st}(S(G))$ we denote the 2-complex determined by the triangles of $\mathrm{st}(s_1), \ldots, \mathrm{st}(s_m)$ together with all 0- and 1-simplices incident with those triangles. Note that $\mathrm{st}(S(G))$ may also contain some unstable vertices and note that it is possible to have $uv \in E(U^1(T))$ and yet $\deg(uv, \mathrm{st}(S(G))) = 2$. Clearly, $\deg(uv, \mathrm{st}(S(G))) = 2$ whenever $u \in S(G)$ or $v \in S(G)$. For $uv \in E(U^1(T))$, the *slack* of uv in $U^1(T)$ is defined by:

$$\text{slack}(uv, U^1(T)) = \left\{ \begin{array}{ll} 2 & \text{if} \quad uv \notin \text{st}(S(G)) \\ 2 - \deg(uv, \text{st}(S(G))) & \text{if} \quad uv \in \text{st}(S(G)) \end{array} \right.$$

Note that $\text{slack}(uv, U^1(T)) \in \{0, 1, 2\}$. This allows a fruitful re-interpretation of the type of results on the uniqueness of the solution to the reconstruction problem, which leads to a number of results, old and new, on the uniqueness of embedding, addressed below.

Applying Lemma 2 to the edge-weighted graph $\{U^1(T), \text{slack}(\,\cdot\,, U^1(T))\}$leads to the following result:

Theorem 2. [23] *Let $T : G \to \Sigma$ be a triangulation of a closed 2-manifold Σ with a graph G. If the graph $U^1(T)$ has at most five vertices in each component, then G triangulates Σ uniquely in the vertex-labeled sense.*

As we know from Section 3, K_6 admits distinct triangulations of the projective plane, so Theorem 2 is best possible in terms of the number of vertices. Theorem 2 improves an earlier result of S. Negami [17] stating in fact that if $U^1(T)$ has at most four vertices, then G triangulates Σ uniquely.

Corollary 1. *Any combinatorially flexible triangulation of a closed 2-manifold has at least six unstable vertices.*

The next corollary follows from Theorems 1 and 2.

Corollary 2. *Any combinatorially flexible triangulation of the projective plane which contains neither the bunch of triangles nor the bunch of squares has more than six flexible vertices.*

It is a result of [4] that "more than six" in Corollary 2 can be replaced by "exactly seven." The next theorem also follows from Theorems 1 and 2.

Theorem 3. *Let $T : G \to \Sigma$ be a triangulation of a closed 2-manifold Σ with a graph G. If the simplicial 2-complex $U^2(T)$ has at most six vertices in each component and contains neither the bunch of triangles nor the bunch of squares, then G triangulates Σ uniquely in the vertex-labeled sense.*

Finally, we mention a known local condition for the uniqueness of embedding $G \to \Sigma$. A *pyramidal* vertex of G is one of degree $|V(G)| - 1$.

Theorem 4. [23] *Let G be a connected graph which is not a complete graph. Assume that G triangulates the projective plane, torus or Klein bottle and also assume that $v \in S(G)$ for each non-pyramidal vertex v; in particular, this is true if $\langle N(v, G) \rangle$ has a unique hamilton cycle for each nonpyramidal vertex v. Then G triangulates the corresponding named 2-manifold uniquely in the vertex-labeled sense.*
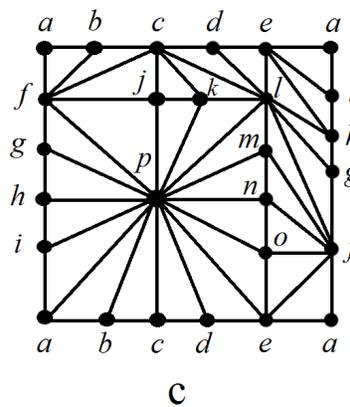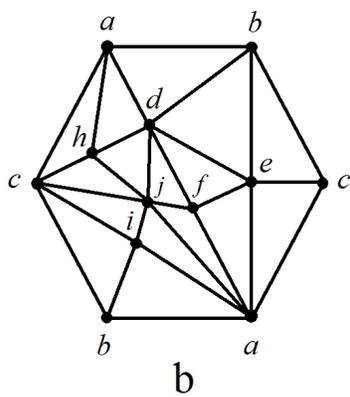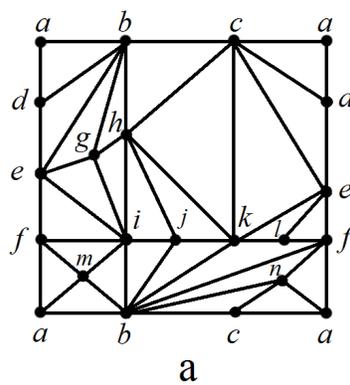
Figure 2.

Thanks to this theorem, to check whether a given graph G triangulating the torus (or the projective plane, or the Klein bottle) is uniquely embeddable in the torus, we do not need to check the condition $v \in S(G)$ for the vertices of degree $|V(G)| - 1$ as well as the vertices of degree 3.

Finally, to understand better the worth of Theorem 4, we construct graphs to which this theorem is applicable but in which the subgraph induced by the neighbors of some pyramidal vertex has more than one hamilton cycle. Such examples are presented in Figs. 2a for the torus, 2b for the projective plane, and 2c for the Klein bottle (identify the boundary edges of each fundamental poilygon in pairs as the boundary labels prescribe). These three graphs have exactly one pyramidal vertex each: vertex b, vertex a, and vertex p, respectively. (The example of Fig. 2a is due to [23] which paper also contains a similar example for the double torus Σ_2.)

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