SEQUENTIAL LABYRINTH FRACTALS

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Abstract

This paper introduces the concept of sequential labyrinth fractal constructed on a unit square using two sequences. It also explains how the obtained fractal differs from the classical labyrinth fractal, mixed labyrinth fractal and super-mixed labyrinth fractals. The Hausdorff and the box-counting dimension of sequential labyrinth fractals, which are constructed using convergent sequences, are also examined. Besides that, it gives the dimension of fractals on the unit square, which are generated from converging sequences, but without having the labyrinth conditions.

Keywords: Fractals; Labyrinth fractals; Hausdorff dimension; Box-counting dimension; Sequences.

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1 INTRODUCTION

There are various methods for constructing a fractal in literature. Some such techniques include fractals generated by iterated function systems, fractals generated by finite subdivisions, and so on. This paper deals with fractals constructed on a unit

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square by the finite subdivision method, where the division is characterised by a sequence. Besides that, we are imposing extra conditions on this construction to produce a special type of fractal called sequential labyrinth fractal.

A fractal using finite subdivision on the unit square was introduced and studied by Bedford[1] and McMullen[2] simultaneously in 1984, and such fractals are known as Bedford-McMullen carpet. The ratio of division is preserved at each stage of construction. It can be seen in literature that there exist a collection of fractals using different kinds of subdivisions in the unit square such as Lalley-Gatzouras carpets[3], Baranski carpets[4] and so on. These can be considered as the generalization of the Bedford-McMullen carpet.

Cristea and Stiensky introduced the concept of labyrinth fractal on the unit square by the finite subdivision of unit square into m^2 squares and by choosing a set of squares with tree property, corner property, and exit property[5],[6]. This process is continued on each chosen square to obtain the classical labyrinth fractal. This kind of fractal has further extensions, such as mixed[7] and supermixed[8] labyrinths. In mixed labyrinths, different division ratios and different patterns are used at different stages of iteration and in supermixed case, different patterns are used within a particular stage. But in each of these cases, the ratio of division remains the same at a particular stage or a specific pattern. In [9], Cristea and Leobacher studied some properties of mixed labyrinth fractal. Labyrinth fractals are further constructed in triangles[10] and quadrilaterals[11]. In this paper, a labyrinth fractal is constructed on the unit square using two sequences of tuples.

Labyrinth fractals on a square have broad applications in Physics. They are used in the study of planar nanostructures[12], the fractal reconstruction of complicated images, signals and radar backgrounds [13], and the construction of prototypes of ultra-wide band radar antennas[14] . Besides, labyrinth fractals have applications in telecommunication [15].

The organization of the paper is as follows: Section 2 gives the construction of sequential labyrinth fractal using two sequences of k-tuple and l-tuple. The construction precisely uses the finite subdivision on the unit square, which results in a set of rectangles. From the obtained smaller rectangles of each stage, a set of rectangles are chosen such that they satisfy tree property, corner property, and exit property. This collection is said to be a sequential labyrinth set of respective order, and its limit is defined as the sequential labyrinth fractal. Sections 3 and 4 mainly deals with the converging sequence. In section 3, we begin with two sequences of k-tuple, which converge to the k-tuple of $1/k$, and examine whether the corresponding fractal coincides with the classical $k \times k$ labyrinth fractal given in [5] and [6]. Section 4 is devoted to finding both the box-counting dimension and the Hausdorff dimension of the sequential labyrinth fractal, which are generated from converging sequences. The lower bound for the Hausdorff dimension is found by the mass distribution principle [16] and the upper bound for the box-counting dimension is found by considering the number of boxes required to cover the fractal.

2 Labyrinth fractal on a unit square generated by two sequence

This section gives the construction of sequential labyrinth fractals on a unit square. It can be seen that the classical labyrinth fractal is a particular case of the sequential labyrinth fractal.

Let I denotes the unit square $[0,1] \times [0,1]$ throughout the paper.

Definition 2.1. Let $\{\lambda_n\}$ be a sequence of k-tuples ($k \geq 3$), say, $\lambda_n = (\lambda_n^1, \lambda_n^2, ..., \lambda_n^k)$ with $0 < \lambda_n^i < 1$ for $i = 1, 2, ..., k$ and $\{\gamma_n\}$ be a sequence of l-tuples ($l \geq 3$), say, $\gamma_n = (\gamma_n^1, \gamma_n^2, ..., \gamma_n^l)$ with $0 < \gamma_n^j < 1$ with $j = 1, 2, , ..., l$ such that $\Sigma_{i=1}^k \lambda_n^i = 1$ and $\Sigma_{j=1}^l \gamma_n^j = 1$ for all n.

Definition 2.2. Let $\{\lambda_n\}$ and $\{\gamma_n\}$ be the sequences of k-tuple and l-tuple as given in Definition 2.1. For $1 \leq i \leq k$,$1 \leq j \leq l$, and $n \geq 1$ define the function $P_n^{i,j} : I \to I$ as,

$P_n^{i,j}(x, y) = (\lambda_n^i x, \gamma_n^j y) + (\lambda_n^1 + \lambda_n^2 + ... + \lambda_n^{i-1}, \gamma_n^1 + \gamma_n^2 + ... + \gamma_n^{j-1})$.

Theorem 2.1. Each $P_n^{i,j}$ is a contraction.

Proof.

$$\begin{aligned} d(P_n^{i,j}(x,y), P_n^{i,j}(x',y')) &= d((\lambda_n^i x, \gamma_n^j y), (\lambda_n^i x', \gamma_n^j y')) \\ &= ||(\lambda_n^i(x-x'), \gamma_n^j(y-y'))|| \\ &\leq max\{\lambda_n^i, \gamma_n^j\}||(x-x', y-y')|| \\ &= max\{\lambda_n^i, \gamma_n^j\}d((x,y),(x',y')) \end{aligned}$$

Since $0 < \lambda_n^i, \gamma_n^j < 1$, $0 < max\{\lambda_n^i, \gamma_n^j\} < 1$ and hence $P_n^{i,j}$ is a contraction for all i, j, and n. □

Construct two sequences $\{\mu_n\}$ and $\{\delta_n\}$ from the sequences $\{\lambda_n\}$ and $\{\gamma_n\}$ respectively as follows; $\{\mu_n\}$ (respectively $\{\delta_n\}$) is a sequence of sequences and for each μ_n (respectively δ_n), after the $(k^n)th$ (respectively $(l^n)th$) element, all entries will be zero. So $\{\mu_n\}$ and $\{\delta_n\}$ are sequences of elements from c_{00}, where,
$c_{00} = \{x \in \ell^p$: all but finitely many $x(j)'$s are equal to zero$\}$. The sequences $\{\mu_n\}$ and $\{\delta_n\}$ are explicitly defined as following; The i^{th} term of μ_n is,

$$\mu_n^i = \begin{cases} \lambda_1^{i_1}\lambda_2^{i_2}...\lambda_{n-1}^{i_{n-1}}\lambda_n^{i-sk}, & \text{if } 1 \leq i \leq k^n \\ 0, & \text{otherwise} \end{cases} \tag{1}$$

where, $i_t = Rem(\frac{[\frac{i-1}{k^{n-t}}]}{k}) + 1$ for $t = 1, 2, ..., n-1$ and $s = [\frac{i-1}{k}]$.
The j^{th} term of δ_n is,

$$\delta_n^j = \begin{cases} \gamma_1^{j_1}\gamma_2^{j_2}...\gamma_{n-1}^{j_{n-1}}\gamma_n^{j-vk}, & \text{if } 1 \leq j \leq l^n \\ 0, & \text{otherwise} \end{cases} \tag{2}$$

where, $j_t = Rem(\frac{[\frac{j-1}{l^{n-t}}]}{l}) + 1$ for $t = 1, 2, ..., n-1$ and $v = [\frac{j-1}{l}]$. Here, Rem denotes the remainder function.

Proposition 2.1. For each $n \geq 1$, $\Sigma_{i=1}^{\infty}\mu_n^i = 1$ and $\Sigma_{j=1}^{\infty}\delta_n^j = 1$.

The proof of Proposition 2.1 easily follows and it is given in Appendix 5.

Definition 2.3. Let $\{\mu_n\}$ and $\{\delta_n\}$ be the sequences as given in Equations 1 and 2, which are defined using $\{\lambda_n\}$ and $\{\gamma_n\}$. For $1 \leq i \leq k^n, 1 \leq j \leq l^n$, and $n \geq 1$, define the function $Q_n^{i,j} : I \to I$ as,

$Q_n^{i,j}(x,y) = (\mu_n^i x, \delta_n^j y) + (\mu_n^1 + \mu_n^2 + ... + \mu_n^{i-1}, \delta_n^1 + \delta_n^2 + ... + \delta_n^{j-1})$.

Theorem 2.2. Each $Q_n^{i,j}$ is a contraction.

Proof.

$$\begin{aligned} d(Q_n^{i,j}(x,y), Q_n^{i,j}(x',y')) &= d((\mu_n^i x, \delta_n^j y), (\mu_n^i x', \delta_n^j y')) \\ &= ||(\mu_n^i(x-x'), \delta_n^j(y-y'))|| \\ &\leq max\{\mu_n^i, \delta_n^j\}||(x-x', y-y')|| \\ &= max\{\mu_n^i, \delta_n^j\} d((x,y),(x',y')) \end{aligned}$$

Since each μ_n^i and δ_n^j can be seen as $\mu_n^i = \lambda_1^{i_1}\lambda_2^{i_2}...\lambda_n^{i_n}$ and $\delta_n^j = \gamma_1^{j_1}\gamma_2^{j_2}...\gamma_n^{j_n}$, where $i_1, i_2, ..., i_n \in \{1,2,...,k\}$ and $j_1, j_2, ..., j_n \in \{1,2,...,l\}$, it is obvious that $0 < \mu_n^i, \delta_n^j < 1$ for all i, j, and n. Hence $0 < max\{\mu_n^i, \delta_n^j\} < 1$. Thus $Q_n^{i,j}$ is a contraction for all i, j, and n. □

Remark 2.1. Denote the set $\{Q_m^{i,j}(I) : 1 \leq i \leq k^m, 1 \leq j \leq l^m\}$ by $\mathcal{S}_m$ for any $m \geq 1$. It is easy to see that

$\mathcal{S}_1 = \{Q_1^{i,j}(I) : 1 \leq i \leq k, 1 \leq j \leq l\} = \{P_1^{i,j}(I) : 1 \leq i \leq k, 1 \leq j \leq l\}$

Definition 2.4. For an $m \geq 1$, let $\mathcal{W} \subseteq \mathcal{S}_m$. The graph corresponding to $\mathcal{W}$ is denoted by $\mathcal{G}(\mathcal{W})$ and is defined as follows: The vertices of $\mathcal{G}(\mathcal{W})$ are elements of $\mathcal{W}$ and there exists an edge between two vertices of $\mathcal{G}(\mathcal{W})$ if and only if the corresponding elements in $\mathcal{W}$ have a common side.

Definition 2.5. Let $A_m = \{(i,j) : 1 \leq i \leq k^m, 1 \leq j \leq l^m\}$. Then, for a fixed $\mathcal{W} \subseteq \mathcal{S}_m$, let, $B_m = \{(i,j) \in A_m : Q_m^{i,j}(I) \in \mathcal{W}\}$.

Definition 2.6. A subset $\mathcal{W} \subseteq \mathcal{S}_m$ is said to be a sequential labyrinth set of order m corresponding to the sequences $\{\mu_n\}$ and $\{\delta_n\}$ if it satisfies the following three properties.

1. Tree property : $\mathcal{G}(\mathcal{W})$ is a tree.

2. Exit property : There exist exactly one i such that $Q_m^{i,1}(I), Q_m^{i,l^m}(I) \in \mathcal{W}$ and exactly one j such that $Q_m^{1,j}(I), Q_m^{k^m,j}(I) \in \mathcal{W}$. $Q_m^{i,1}(I)$ is called the bottom exit, $Q_m^{i,l^m}(I)$ is called the top exit, $Q_m^{1,j}(I)$ is called the left exit and $Q_m^{k^m,j}(I)$ is called the right exit.

3. Corner property: There exist atmost one element in $\mathcal{W}_m$ from each of the sets $\{Q_m^{1,1}(I), Q_m^{k^m,l^m}(I)\}$ and $\{Q_m^{k^m,1}(I), Q_m^{1,l^m}(I)\}$. $Q_m^{1,1}(I)$ is called the bottom-left corner, $Q_m^{k^m,l^m}(I)$ is called the top-right corner, $Q_m^{k^m,1}(I)$ is called the bottom-right corner and $Q_m^{1,l^m}(I)$ is called the top-left corner.

In this case, an element $W \in \mathcal{W}_m \subseteq \mathcal{S}_m$ is said to be a white rectangle of order m.

Definition 2.7. Let $\mathcal{W}_1 \subseteq \mathcal{S}_1$ and let B_1 is taken as in Definition 2.5. Define
$\mathcal{W}_m = \{P_1^{i_1,j_1} \circ P_2^{i_2,j_2} \circ ..... \circ P_m^{i_m,j_m}(I) : (i_1, j_1), (i_2, j_2), ..., (i_m, j_m) \in B_1\}$.

Theorem 2.3. If $\mathcal{W}_1$ is a sequential labyrinth set of order 1 corresponding to the sequence $\{\mu_n\}$ and $\{\delta_n\}$, then $\mathcal{W}_m$ will be a sequential labyrinth set of order m corresponding to $\{\mu_n\}$ and $\{\delta_n\}$.

Proof. Here we need to prove the below 4 conditions.

1. $\mathcal{W}_m \subseteq \mathcal{S}_m$

   To prove this, choose an element, $P_1^{k_1,l_1} \circ P_2^{k_2,l_2} \circ ..... \circ P_m^{k_m,l_m}(I) \in \mathcal{W}_m$. We should be able to write this as a $Q_m^{i,j}(I)$ for some $1 \leq i \leq k^m, 1 \leq j \leq l^m$.

For $(x,y) \in I$,

$$
\begin{aligned}
&P_1^{k_1,l_1} \circ P_2^{k_2,l_2} \circ ..... \circ P_m^{k_m,l_m}(x,y) \\
&\quad = P_1^{k_1,l_1} \circ P_2^{k_2,l_2} \circ ..... \circ P_{m-1}^{k_{m-1},l_{m-1}}((\lambda_m^{k_m}x, \gamma_m^{l_m}y)+ \\
&\qquad\qquad (\lambda_m^1 + ... + \lambda_m^{k_m-1}, \gamma_m^1 + ... + \gamma_m^{l_m-1})) \\
&\quad = P_1^{k_1,l_1} \circ P_2^{k_2,l_2} \circ ..... \circ P_{m-2}^{k_{m-2},l_{m-2}}((\lambda_{m-1}^{k_{m-1}}\lambda_m^{k_m}x, \gamma_{m-1}^{l_{m-1}}\gamma_m^{l_m}y) \\
&\qquad + (\lambda_{m-1}^{k_{m-1}}(\lambda_m^1 + ... + \lambda_m^{k_m-1}), \gamma_{m-1}^{k_{m-1}}(\gamma_m^1 + ... + \gamma_m^{l_m-1})) \\
&\qquad + (\lambda_{m-1}^1 + ... + \lambda_{m-1}^{k_{m-1}-1}, \gamma_{m-1}^1 + ... + \gamma_{m-1}^{l_{m-1}-1})) \\
&\quad .... \\
&\quad = (\lambda_1^{k_1}\lambda_2^{k_2}...\lambda_m^{k_m}x, \gamma_1^{l_1}\gamma_2^{l_2}...\gamma_m^{l_m}y)+ \\
&\quad (\lambda_1^{k_1}\lambda_2^{k_2}...\lambda_{m-1}^{k_{m-1}}(\lambda_m^1 + ... + \lambda_m^{k_m-1}), \gamma_1^{l_1}\gamma_2^{l_2}...\gamma_{m-1}^{l_{m-1}}(\gamma_m^1 + ... + \gamma_m^{l_m-1}))+ \\
&\quad (\lambda_1^{k_1}\lambda_2^{k_2}...\lambda_{m-2}^{k_{m-2}}(\lambda_{m-1}^1 + ... + \lambda_{m-1}^{k_{m-1}-1}), \gamma_1^{l_1}\gamma_2^{l_2}...\gamma_{m-2}^{l_{m-2}}(\gamma_{m-1}^1 + ... + \gamma_{m-1}^{l_{m-1}-1})) \\
&\quad + ... + (\lambda_1^{k_1}\lambda_2^{k_2}(\lambda_3^1 + ... + \lambda_3^{k_3-1}), \gamma_1^{l_1}\gamma_2^{l_2}(\gamma_3^1 + ... + \gamma_3^{l_3-1}))+ \\
&\quad + (\lambda_1^{k_1}(\lambda_2^1 + ... + \lambda_2^{k_2-1}), \gamma_1^{l_1}(\gamma_2^1 + ... + \gamma_2^{l_2-1})) \\
&\quad + (\lambda_1^1 + ... + \lambda_1^{k_1-1}, \gamma_1^1 + ... + \gamma_1^{l_1-1}) \\
&\quad = (\mu_m^p x, \delta_m^q y) + (\mu_m^1 + \mu_m^2 + ... + \mu_m^{p-1}, \delta_m^1 + \delta_m^2 + ... + \delta_m^{q-1}) \\
&\quad = (\mu_m^p x, \delta_m^q y) + (\mu_m^1 + \mu_m^2 + ... + \mu_m^{p-1}, \delta_m^1 + \delta_m^2 + ... + \delta_m^{q-1}) \\
&\quad = Q_m^{p,q}(x,y)
\end{aligned}
$$

where,

$p = (k_1 - 1)k^{m-1} + (k_2 - 1)k^{m-2} + ... + (k_{m-1} - 1)k + k_m$

$q = (l_1 - 1)l^{m-1} + (l_2 - 1)l^{m-2} + ... + (l_{m-1} - 1)l + l_m$

2. Exit property : If $P_1^{i,1}(I)$ and $P_1^{i,l}(1) \in \mathcal{W}_1$ forms unique bottom and top exits respectively in the first stage, $P_1^{i,1} \circ P_2^{i,1} \circ ... \circ P_m^{i,1}(I)$ and $P_1^{i,l} \circ P_2^{i,l} \circ ... \circ P_m^{i,l}(I)$ forms the bottom and top exits respectively in the m^{th} stage.

3. $\mathcal{G}(\mathcal{W}_m)$ is a tree: The connectedness is ensured by exit property. Suppose there exist a cycle in the m^{th} stage. Let $C = \{a_0, a_1, ...a_s\}$ be a cycle in $\mathcal{G}(\mathcal{W}_m)$. For each $a \in \mathcal{W}_m$ let $t(a)$ be an element in $\mathcal{W}_{m-1}$, which contains a. Let

$j_0 = 0, b_0 = t(a_0)$ and $j_k = \min\{i : t(a_i) \neq b_{k-1}, j_{k-1} < i \leq s\}$, $b_k = t(a_{j_k})$ for $k \geq 1$. Choose r minimal such that the set $\{i : t(a_i) \neq b_r, j_r < i \leq s\}$ is empty. For $i = 1, 2, ..., r$ the vertex b_{i-1} is a neighbour of b_i in $\mathcal{G}(\mathcal{W}_{m-1})$. The set $\{b_0, b_1, ..., b_r\}$ can not contain a cycle by the induction hypothesis. Also, if $r = 0$, then all a_i's are contained in b_0 and it will contradicts the fact that $\mathcal{G}(\mathcal{W}_1)$ is a tree. Hence $r \geq 1$. Thus the graph induced by $\mathcal{G}(\mathcal{W}_{m-1})$ on the set $\{b_0, b_1 ..., b_r\}$ is a tree with more than one vertex. This implies that the cycle C returns to a_0 through the same side where it leaves a_0. But the exit property gives that $a_i = a_j$ for some $j \neq i$; $a_i, a_j \in \{a_0, a_1, ..., a_s\}$, which is a contradiction. Hence $\mathcal{G}(\mathcal{W}_m)$ is a tree.

4. Corner property : Since, there exist atmost one element from opposite corners, there will exist atmost one element from opposite corner in $\mathcal{W}_m$ also. i.e., if $P_1^{1,1}(I) \in \mathcal{W}_1$ from $\{P_1^{1,1}(I), P_1^{k,l}(I)\}$, then $P_1^{1,1} \circ P_2^{1,1} \circ ..... \circ P_m^{1,1}(I)$ will be the corner in $\mathcal{W}_m$.

□

Theorem 2.4. If $\mathcal{W}_1$ is a sequential labyrinth set of order 1 corresponding to the sequences $\{\mu_n\}$ and $\{\delta_n\}$, then $\mathcal{W}_m$ is a sequential labyrinth set of order 1 corresponding to the subsequence $\{\mu_{nm}\}$ and $\{\delta_{nm}\}$.

Proof. It is clear that $\mathcal{W}_m$ is a sequential labyrinth set from the proof of theorem 2.3. Also $\mathcal{W}_m$ is corresponds to the term μ_m and δ_m and they are the first term of the subsequence $\{\mu_{nm}\}$ and $\{\delta_{nm}\}$ respectively. So the Theorem follows trivially. □

Definition 2.8. If $\mathcal{W}_1$ is a sequential labyrinth set of order 1, the m^{th} stage pre-fractal associated with $\mathcal{W}_1$ is denoted by $\mathcal{L}_m$ and is defined as $L_m = \bigcup_{W_m \in \mathcal{W}_m} W_m$ for all $m \geq 1$ and the sequential labyrinth fractal is denoted by L_∞ and is defined as $L_\infty = \bigcap_{m=1}^{\infty} L_m$.

Example 2.1. Consider the sequences

$\{\lambda_n\} = \{(\frac{1}{4}, \frac{1}{8}, \frac{1}{4}, \frac{3}{8}), (\frac{1}{6}, \frac{1}{6}, \frac{1}{3}, \frac{1}{3}), (\frac{1}{4}, \frac{1}{8}, \frac{1}{4}, \frac{3}{8}), (\frac{1}{6}, \frac{1}{6}, \frac{1}{3}, \frac{1}{3}), ...\}$ and

$\{\gamma_n\} = \{(\frac{1}{4}, \frac{1}{4}, \frac{1}{2}), (\frac{1}{3}, \frac{1}{3}, \frac{1}{3}), (\frac{3}{5}, \frac{1}{5}, \frac{1}{5}), (\frac{1}{4}, \frac{1}{4}, \frac{1}{2}), ...\}$.

Then the corresponding $\{\mu_n\}$ and $\{\delta_n\}$ are,

$\{\mu_n\} = \{(\frac{1}{4}, \frac{1}{8}, \frac{1}{4}, \frac{3}{8}, 0, 0, ...), (\frac{1}{24}, \frac{1}{24}, \frac{1}{12}, \frac{1}{12}, \frac{1}{48}, \frac{1}{48}, \frac{1}{24}, \frac{1}{24}, \frac{1}{24}, \frac{1}{24}, \frac{1}{12}, \frac{1}{12}, \frac{3}{48}, \frac{3}{48}, \frac{3}{24}, \frac{3}{24}, 0, 0, ...), ...\}$

$\{\delta_n\} = \{(\frac{1}{4}, \frac{1}{4}, \frac{1}{2}, 0, 0, ...), (\frac{1}{12}, \frac{1}{12}, \frac{1}{12}, \frac{1}{12}, \frac{1}{12}, \frac{1}{12}, \frac{1}{6}, \frac{1}{6}, \frac{1}{6}, 0, 0, ...), ...\}$

Choose $\mathcal{W}_1 = \{P_1^{1,1}(I), P_1^{1,2}(I), P_1^{2,2}(I), P_1^{3,1}(I), P_1^{3,2}(I), P_1^{3,3}(I), P_1^{4,2}(I)\}$, a sequential labyrinth set of order 1 and take $\mathcal{W}_m$ as in Definition 2.7. Then $L_\infty = \bigcap_{m=1}^{\infty} \bigcup_{W_m \in \mathcal{W}_m} W_m$ is a sequential labyrinth set. The first 3 stages of this sequential labyrinth fractal is given in Figure 1.

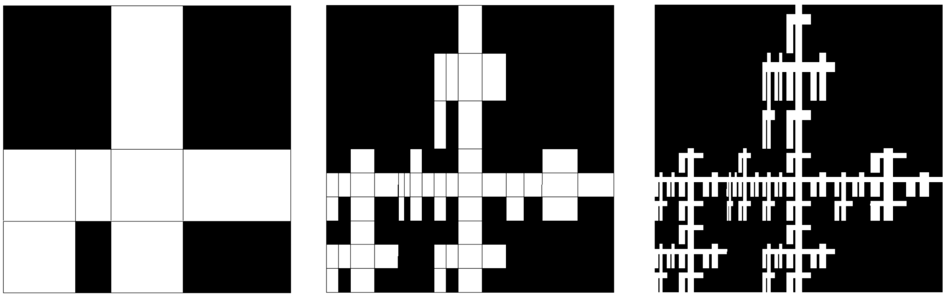

Figure 1: An example for the first three stages of a sequential labyrinth fractal

Remark 2.2. If $\{\lambda_n\}$ and $\{\gamma_n\}$ are constant sequences of k-tuples with each coordinate $\frac{1}{k}$ and let $\{\mu_n\}$ and $\{\delta_n\}$ be sequences constructed from $\{\lambda_n\}$ and $\{\gamma_n\}$ as in 1 and 2. Then the corresponding sequential labyrinth fractal is same as the classical $k \times k$-labyrinth fractal on a unit square.

Remark 2.3. We claim that the sequential labyrinth fractal defined above does not fall in any of the cases of sequential labyrinth fractal on the unit square in literature. The classical labyrinth fractal given in [5] and [6] uses a same division factor, say $\frac{1}{m}$, $m \geq 4$. $m \in \mathbb{N}$ at each stage and the same pattern is used throughout the construction. The mixed labyrinth fractal defined in [7] uses a division factor $\frac{1}{m_k}$ for the k^{th} stage, where $m_k \in \mathbb{N}$, $m_k \geq 4$ and different patterns at different stage. The supermixed labyrinth fractal[8] uses different patterns within a particular stage, but again the division factor is of the form $\frac{1}{m}$, $m \in \mathbb{N}$, $m \geq 4$ at every stage. Note that the sequential labyrinth fractal does not assume any such restriction. The division depends on sequence of tuples, where the entries in tuple can be any number between

0 and 1 such that sum of entries of each tuple should be 1. So this is a more general class of labyrinth fractal where the division along the horizontal and the vertical axis are completely different and at each stage the division factor, that depends on the sequence, keeps on changing. But the pattern remains same throughout. So unlike the earlier cases, we obtained a set of different-sized rectangles at every stage instead of squares.

3 Sequential labyrinth fractals generated from converging sequences

Let $\{\lambda_n\}$ and $\{\gamma_n\}$ are two sequences of k-tuples such that both $\{\lambda_n\}$ and $\{\gamma_n\}$ converges to $(\frac{1}{k}, \frac{1}{k}, ..., \frac{1}{k})$. In this section, we shall prove by an example that the sequential labyrinth fractal generated from this sequences and the classical $k \times k$-labyrinth fractal (i.e., the fractal generated from constant sequences) need not be same.

Example 3.1. Let $\{\lambda_n\} = \{(\frac{1}{4} - \frac{1}{4(n+1)}, \frac{1}{4} + \frac{1}{4(n+1)}, \frac{1}{4}, \frac{1}{4})\}$ and $\{\gamma_n\} = \{(\frac{1}{4} - \frac{1}{2^{n+1}}, \frac{1}{4}, \frac{1}{4} - \frac{1}{2^{n+1}}, \frac{1}{4})\}$. Then both the sequences $\{\lambda_n\}$ and $\{\gamma_n\}$ converges to $(\frac{1}{4}, \frac{1}{4}, \frac{1}{4}, \frac{1}{4})$. Let $\{\mu_n\}$ and $\{\delta_n\}$ be the sequences constructed as in 1 and 2 . We will consider the first stage of a sequential labyrinth fractal generated from the sequences $\{\mu_n\}$ and $\{\delta_n\}$. Note that $\mu_1 = (\frac{1}{8}, \frac{3}{8}, \frac{1}{4}, \frac{1}{4}, 0, 0, ...)$ and $\delta_1 = (\frac{1}{8}, \frac{1}{4}, \frac{3}{8}, \frac{1}{4}, 0, 0, ...)$. Choose $\mathcal{W}_1 = \{P_1^{1,1}(I), P_1^{1,2}(I), P_1^{2,2}(I), P_1^{3,2}(I), P_1^{4,2}(I), P_1^{2,3}(I), P_1^{2,4}(I), P_1^{1,4}(I)\}$ as the first stage of sequential labyrinth set and let L_∞ be the labyrinth fractal corresponding to $\{\mu_n\}$ and $\{\delta_n\}$. Let L^*_∞ be the sequential labyrinth fractal generated from constant sequence (i.e., the classical labyrinth fractal) with position of whites in the first stage is same as that of position of whites in the first stage of sequential labyrinth set. It can be explicitely given as follows: Take $\{\lambda^*_n\} = \{\gamma^*_n\} = \{(\frac{1}{4}, \frac{1}{4}, \frac{1}{4}, \frac{1}{4})\}$ and $\{\mu^*_n\}$ and $\{\delta^*_n\}$ are defined as in 1 and 2 . Take the first stage as
$\mathcal{W}^*_1 = \{P*_1^{1,1}(I), P*_1^{1,2}(I), P*_1^{2,2}(I), P*_1^{3,2}(I), P*_1^{4,2}(I), P*_1^{2,3}(I), P*_1^{2,4}(I), P*_1^{1,4}(I)\}$
and all these functions are defined as in Definition 2.2 and corresponding to $\{\lambda^*_n\}$ and $\{\gamma^*_n\}$. Even though the sequence $\{\lambda^*_n\}$ and $\{\gamma^*_n\}$ converges to $\{(\frac{1}{4}, \frac{1}{4}, \frac{1}{4}, \frac{1}{4})\}$, the frac-

tals L_∞ and L^*_∞ are different. For this, consider the first stage of both L_∞ and L^*_∞, which are given in the Figure 2.

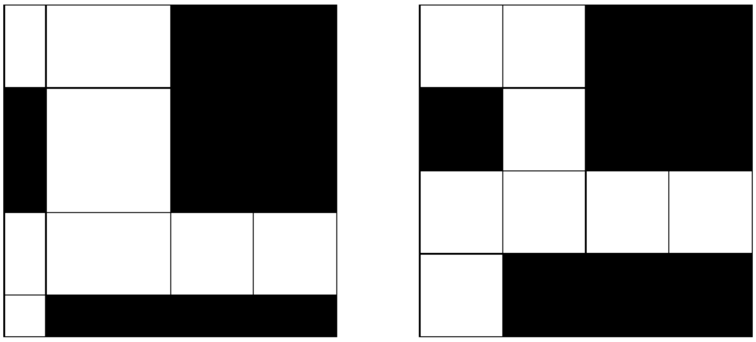

Figure 2: The first stage of L_∞ and L^*_∞ respectively

Since $(\frac{1}{2}, \frac{1}{4})$ is the bottom-left corner of $P_1^{3,2}(I)$, and bottom left corner of first stage is a white rectangle, $(\frac{1}{2}, \frac{1}{4}) \in L_m$ for each m and hence $(\frac{1}{2}, \frac{1}{4}) \in L_\infty$. But $(\frac{1}{2}, \frac{1}{4})$ does not belongs to first stage of L^*_∞ and hence cannot belong to L^*_∞. Hence, L_∞ and L^*_∞ are different.

Even though these fractals are different, in the next section it can be seen that the dimension of both the fractals coincides.

4 Fractal dimension of sequential labyrinth fractal generated from converging sequences

In this section, the dimension of sequential labyrinth fractal corresponding to the converging sequences are examined.

Theorem 4.1. Let L_∞ be a sequential labyrinth fractal generated from the non-constant sequences $\{\lambda_n\} = \{(\lambda_n^1, \lambda_n^2, ..., \lambda_n^k)\}$ and $\{\gamma_n\} = \{(\gamma_n^1, \gamma_n^2, ..., \gamma_n^k)\}$, both converges to the k-touple $(\frac{1}{k}, \frac{1}{k}, ..., \frac{1}{k})$, $k \geq 3$. Let $\{\mu_n\}$ and $\{\gamma_n\}$ be defined as in Equations 1 and 2. Let T be the number of white rectangles in the first stage of L_∞.

Then, the box-counting dimension of L_∞ is bounded above by $\frac{\log T}{\log k}$.

Proof. Define,

$$\alpha_n = \max\left\{\sup_{1\le i\le k^n} \mu_n^i, \sup_{1\le i\le k^n} \delta_n^i\right\}$$

Consider a sequence $\{\lambda_1^{i_1}, \lambda_2^{i_2}, \lambda_3^{i_3}, ...\}$, where $i_1, i_2, i_3, ... \in \{1, 2, 3, ..., k\}$. It is clear that this sequence converges to $\frac{1}{k}$. Choose $\epsilon = \frac{1}{k}$. Since, $\{\lambda_1^{i_1}, \lambda_2^{i_2}, \lambda_3^{i_3}, ...\}$ converges to $\frac{1}{k}$, there exist an N such that for all $n \ge N$, $|\lambda_n^{i_n} - \frac{1}{k}| < \frac{1}{k}$. Thus, for all $n \ge N$, we have $\lambda_n^{i_n} \le |\lambda_n^{i_n} - \frac{1}{k}| + \frac{1}{k} \le \frac{2}{k}$. Hence $\prod\limits_{n=N}^{\infty} \lambda_n^{i_n} \le \prod\limits_{n=N}^{\infty} \frac{2}{k} = 0$, since $k > 2$. Now consider, $\prod\limits_{n=1}^{\infty} \lambda_n^{i_n} = \lambda_1^{i_1}\lambda_2^{i_2}....\lambda_{N-1}^{i_{N-1}} \prod\limits_{n=N}^{\infty} \lambda_n^{i_n} \le \Lambda^{N-1} \prod\limits_{n=N}^{\infty} \lambda_n^{i_n} = 0$, where $\Lambda = \max\{\lambda_1^{i_1}, \lambda_2^{i_2}, ..., \lambda_{N-1}^{i_{N-1}}\}$. Note that each μ_n^i can be written as $\mu_n^i = \lambda_1^{i_1}\lambda_2^{i_2}...\lambda_n^{i_n}$ for some $i_1, i_2, ..., i_n \in \{1, 2, ..., k\}$ Thus, $\lim\limits_{n\to\infty} \mu_n^i = 0$ for any $1 \le i \le k^n$. Similarly, $\lim\limits_{n\to\infty} \delta_n^i = 0$ for any $1 \le i \le k^n$. Hence $\lim\limits_{n\to\infty} \alpha_n = 0$.
Now, $d(\alpha_n, \frac{1}{k^n}) \le d(\alpha_n, 0) + d(0, \frac{1}{k^n}) \to 0$ as $n \to \infty$.

The minimum number of squares of side length α_n which covers L_∞ cannot be exceeded T^n, where T is the number of white rectangles in first stage. Hence box counting dimension is bounded above by $\lim\limits_{n\to\infty} \frac{\log T^n}{-\log \alpha_n}$. Since $\{d(\alpha_n, \frac{1}{k^n})\}$ is a positive sequence which converges to 0, choose a decreasing subsequence, say, $\{d(\alpha_{n_t}, \frac{1}{k^{n_t}})\}$, such that, $d(\alpha_{n_t}, \frac{1}{k^{n_t}}) \le \frac{1}{k^{n_t}}$. So the upper bound for the box-counting dimension of L_∞ is, $\dim_B(L_\infty) \le \lim\limits_{t\to\infty} \frac{\log T^{n_t}}{-\log \alpha_{n_t}}$.
Now, $d(\alpha_{n_t}, \frac{1}{k^{n_t}}) = |\alpha_{n_t} - \frac{1}{k^{n_t}}| \le \frac{1}{k^{n_t}} \implies \alpha_{n_t} = |\alpha_{n_t}| \le |\alpha_{n_t} - \frac{1}{k^{n_t}}| + |\frac{1}{k^{n_t}}| \le \frac{2}{k^{n_t}}$
Hence, $\log(\alpha_{n_t}) \le \log \frac{2}{k^{n_t}} \implies \frac{1}{-\log(\alpha_{n_t})} \le \frac{1}{-\log \frac{2}{k^{n_t}}}$
$\implies \frac{\log T^{n_t}}{-log(\alpha_{n_t})} \le \frac{\log T^{n_t}}{-\log \frac{2}{k^{n_t}}}$
$\implies \lim\limits_{t\to\infty} \frac{\log T^{n_t}}{-\log(\alpha_{n_t})} \le \lim\limits_{t\to\infty} \frac{\log T^{n_t}}{-\log \frac{2}{k^{n_t}}} = \lim\limits_{t\to\infty} \frac{n_t \log T}{n_t \log k - \log 2} = \frac{\log T}{\log k}$
Thus, the box counting dimension of L_∞, $\dim_B(L_\infty) \le \frac{\log T}{\log k}$, where T is the number of white rectangles of first stage. □

Theorem 4.2. Let L_∞ be a sequential labyrinth fractal generated from the non-constant sequences $\{\lambda_n\} = \{(\lambda_n^1, \lambda_n^2, ..., \lambda_n^k)\}$ and $\{\gamma_n\} = \{(\gamma_n^1, \gamma_n^2, ..., \gamma_n^k)\}$, both converges to the k-touple $(\frac{1}{k}, \frac{1}{k}, ..., \frac{1}{k})$, $k \ge 3$. Let $\{\mu_n\}$ and $\{\gamma_n\}$ be defined as in Equations 1 and 2. Let T be the number of white rectangles in the first stage of L_∞. Then, $\dim_H(L_\infty) \ge \frac{\log T}{\log k}$, where $\dim_H(L_\infty)$ denotes the Hausdorff dimension of L_∞.

Proof. The proof is using mass distribution principle [16].

Let,

$$\alpha_n = \max\left\{\sup_{1\le i\le k^n} \mu_n^i, \sup_{1\le i\le k^n} \delta_n^i\right\}$$

and

$$\beta_n = \min\left\{\inf_{1\le i\le k^n} \mu_n^i, \inf_{1\le i\le k^n} \delta_n^i\right\}$$

Since T is the number of white rectangles in the first stage, T^n will be the number of white rectangles in the n^{th} stage. The zeroth stage is the unit square. Suppose the unit square has unit mass. The unit mass is distributed at each iteration such that every white rectangle in $\mathcal{W}_m$ has a mass of $\frac{1}{T^m}$. Define a measure m on L_∞ such that each n^{th} stage white rectangle carries a mass of $\frac{1}{T^n}$. Now let U be a set with $|U| < 1$ and n be an integer such that $\frac{1}{k^{n+1}} \le |U| < \frac{1}{k^n}$. Note that $\beta_n \le \frac{1}{k^n} \le \alpha_n$ for all n. Then, $\beta_{n+1} \le \frac{1}{k^{n+1}} \le |U| < \frac{1}{k^n} \le \alpha_n$ i.e., $\beta_{n+1} \le |U| < \alpha_n$.

Since, $|U| < \alpha_n$, U can intersect with atmost three white rectangles of n^{th} stage. i.e.,

$$m(U) \le \frac{3}{T^n} = 3T^{-n} = 3(k^{\frac{\log T}{\log k}})^{-n} = 3(k^{-n})^{\frac{\log T}{\log k}} \le 3.k^{\frac{\log T}{\log k}}|U|^{\frac{\log T}{\log k}} \tag{3}$$

Hence for $s = \frac{\log T}{\log k}$, there exist $c = 3.k^{\log T/\log k} > 0$ and $\epsilon = 1 > 0$, such that $m(U) \le c|U|^s$. By mass distribution principle[16], $\dim_H(L_\infty) \ge \frac{\log T}{\log k}$.

□

Theorem 4.3. The Hausdorff dimension and the box-counting dimension of sequential labyrinth fractal, L_∞, is $\frac{\log T}{\log k}$. i.e, $\dim_H(L_\infty) = \dim_B(L_\infty) = \frac{\log T}{\log k}$, where T is the number of white rectangles of first stage.

Proof. From Theorem 4.1 and Theorem 4.2, we have $\frac{\log T}{\log k} \le \dim_H(L_\infty)$ and $\dim_B(L_\infty) \le \frac{\log T}{\log k}$. Also, $\dim_H(L_\infty) \le \dim_B(L_\infty)$.

Thus $\frac{\log T}{\log k} \le \dim_H(L_\infty) \le \dim_B(L_\infty) \le \frac{\log T}{\log k}$

$\implies \dim_H(L_\infty) = \dim_B(L_\infty) = \frac{\log T}{\log k}$, where T is the number of whites in the first stage. □

Remark 4.1. The selection of white rectangles do not play a role in finding the dimension. Even if we are choosing the white rectangles which do not satisfy the tree property, the corner property, and the exit property, the Hausdorff dimension and the box-counting dimension of the limiting set will be $\frac{\log T}{\log k}$, where T is the number of whites in the first stage. The proof for this is the same as that in the labyrinth case above, except that $M(U) \leq \frac{3}{T^n}$ will be replaced by $M(U) \leq \frac{4}{T^n}$. But in any case, the convergence of sequences is used to find the dimension.

5 Conclusion

This paper mainly deals with constructing the sequential labyrinth fractal and finding the dimension of the sequential labyrinth fractal generated from converging sequences. Besides that, we can construct fractals on the unit square from sequences without imposing the labyrinth condition. The dimension of the sequential labyrinth fractal generated from non-converging sequences is yet to be explored.

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APPENDIX : Proof of Proposition 2.1

Consider the case $n = 1$.

Clearly, $\Sigma_{i=1}^{\infty}\mu_1^i = \Sigma_{i=1}^{k}\mu_1^i = \Sigma_{i=1}^{k}\lambda_1^i = 1$.

For any n, consider,

$$
\begin{aligned}
\sum_{i=1}^{\infty}\mu_n^i &= \sum_{i=1}^{k^n}\mu_n^i \\
&= \sum_{i=1}^{k}\mu_n^i + \sum_{i=k+1}^{2k}\mu_n^i + ... + \sum_{i=(k-1)k+1}^{k^2}\mu_n^i + \sum_{i=k^2+1}^{k^2+k}\mu_n^i + ... \\
&\quad + \sum_{i=2k^2-k+1}^{2k^2}\mu_n^i + ... + \sum_{i=k^n-k^{n-1}+1}^{k^n-k^{n-1}+k}\mu_n^i + ... + \sum_{i=k^n-k+1}^{k^n}\mu_n^i \\
&= \sum_{i=1}^{k}\lambda_1^1\lambda_2^1...\lambda_{n-1}^1\lambda_n^i + \sum_{i=k+1}^{2k}\lambda_1^1\lambda_2^1...\lambda_{n-2}^1\lambda_{n-1}^2\lambda_n^{i-k} + ... \\
&\quad + \sum_{i=(k-1)k+1}^{k^2}\lambda_1^1\lambda_2^1...\lambda_{n-2}^1\lambda_{n-1}^k\lambda_n^{i-(k-1)k} + \sum_{i=k^2+1}^{k^2+k}\lambda_1^1\lambda_2^1...\lambda_{n-2}^2\lambda_{n-1}^1\lambda_n^{i-k^2} \\
&\quad + ... + \sum_{i=2k^2-k+1}^{2k^2}\lambda_1^1\lambda_2^1...\lambda_{n-2}^2\lambda_{n-1}^k\lambda_n^{i-(2k^2-k)} + ... \\
&\quad + \sum_{i=k^n-k^2+1}^{k^n-k^2+k}\lambda_1^k\lambda_2^k...\lambda_{n-1}^1\lambda_n^{i-(k^n-k^2)} + ... + \sum_{i=k^n-k+1}^{k^n}\lambda_1^k...\lambda_{n-1}^k\lambda_n^{i-(k^n-k)} \\
&= \sum_{i=1}^{k}\lambda_1^1\lambda_2^1...\lambda_{n-2}^1\lambda_{n-1}^i + \sum_{i=1}^{k}\lambda_1^1\lambda_2^1...\lambda_{n-3}^1\lambda_{n-2}^2\lambda_{n-1}^i + ... \\
&\quad + \sum_{i=1}^{k}\lambda_1^k\lambda_2^k....\lambda_{n-2}^k\lambda_{n-1}^i \\
&\qquad ..... \\
&= \lambda_1^1 + \lambda_1^2 + ...\lambda_1^k \\
&= 1
\end{aligned}
$$

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