FRACTIONAL ANALYSIS OF AN SIS EPIDEMIC MODEL WITH AGE STRUCTURE

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Abstract

The age is one of the most important characteristic in modeling population and infectious disease. In this paper, we investigate a fractional order epidemic model spreading in an age structured population where the transmission coefficient depends on age. The model is derived for an SIS disease, that is a susceptible individual who contract the disease will become infective and assumes that recovered individuals become susceptible again immediately without immunity. We formulate the basic model as an abstract fractional Cauchy problem on a Banach space to prove the existence, uniqueness of local mild solution and ensure global existence of solution. Moreover, we examine existence for the steady state and under certain conditions, uniqueness is also shown.

Key Words:Age structure, Fractional epidemic model, Cauchy problem, SIS epidemic model.

1 Introduction

Many infectious diseases such as malaria, influenza, gonorrhea, and childhood disease can be analyzed using *SIS* epidemiological models. In these models, *S* denotes susceptible individuals, while *I* represents those who are infected.

In recognizing the impact of individuals demographic behaviors, scholars have come to acknowledge that age-structured epidemic models offer a more realistic representation. This is primarily because any disease prevention policy is contingent on the age distribution of the host population, and both instant death and infection rates are intricately tied to age. Since the groundbreaking work of McKendrick [20], many researchers have delved into various age-structured epidemic models [3, 5, 28–30]. Additionally, individuals of different ages may also have different behaviours, and behavioural changes are crucial in control and prevention of many infectious diseases. Young individuals tend to be more active in interactions with or between populations, and in disease transmissions. Therefore, various age-structured epidemic models have been investigated by many authors and a number of papers have been published of finding the threshold conditions for the disease to become endemic, describing the stability of steady state solutions and analyzing the global behavior of these age-structured epidemic models [1, 3, 4, 8, 9, 19, 23–25].

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Fractional-order derivatives play a crucial role in epidemic modelling due to their capacity to incorporate memory and hereditary characteristics. By utilizing these non-integer derivatives, epidemic models become more realistic and reflective of real-world dynamics. The memory effect allows the models to retain and integrate past information, enabling them to make more accurate predictions and interpretations of how epidemics unfold over time. As a result, the use of fractional-order derivatives provides a valuable tool for understanding and effectively responding to infectious disease outbreaks. As a result, many authors [31–35] have begun to study epidemic models using fractional differential equations.

In this paper, we generalized the most basic classical epidemic model *SIS* with age structured population with age-dependent transmission coefficient [3, 36] into fractional derivative in Caputo sense, that is the population is subdivided into two subpopulations: infected individuals and susceptible individuals and assume that the recovered individuals become susceptible again immediately without immunity, and it is relatively easy to show that the long time behavior of its solution is completely determined by a threshold value number $\mathcal{R}_0$: If $\mathcal{R}_0 \leq 1$ there is no endemic steady state(the disease free equilibrium), while if $\mathcal{R}_0 > 1$ the endemic steady state exist and under the proportionate mixing assumption (that is, the transmission kernel is given by the type of separation of variable) as a series of paper [4, 37, 38], there exist a unique of endemic steady state. The threshold value $\mathcal{R}_0$ is called the basic reproduction number and it plays an important role as an indicator of the intensity of diseases since it implies the expected number of secondary cases produced by a typical infected individual during its entire period of infectiousness in a fully susceptible population. The reader may refer to [5, 12, 15] for the original implications of $\mathcal{R}_0$. See also [2] for a practical approach to the computation of $\mathcal{R}_0$.

The organization of the remainder of this article is as follows: In section 2, we give some known preliminary results to be used later. In section 3, we consider the fractional order $(0 < \alpha \leq 1)$ *SIS* epidemic model with age structure with Caputo derivative. In section 4, we formulate the fractional *SIS* epidemic model with age structure as an abstract fractional Cauchy problem on a Banach space and show the existence and uniqueness of its local mild solution and we will ensure global existence of the solution. In section 5, we shall prove the existence of endemic states and under certain conditions, uniqueness is also shown. In section 6, we ends the paper by giving some numerical examples.

2 Preliminaries

In this section we introduce notations, definitions and preliminary facts which are used throughout this paper. We denoted by $E = L^1(0,\omega)$ the Banach space with the norm $|\cdot|$. Let $C([0,T],E)$ be the Banach space of continuous function from $[0,T]$ into E with the norm $\|u\| = \sup_{t\in[0,T]} |u|$ where $u \in C([0,T],E)$. We need some basic definitions and properties of the fractional calculus theory. For more details, see [16, 26, 27].

Definition 2.1. ([16]). The fractional integral of the function $h \in L^1([a,b])$ of order $\alpha \in \mathbb{R}^+$is defined by

$$I_a^\alpha h(t) = \frac{1}{\Gamma(\alpha)} \int_a^t (t-s)^{\alpha-1} h(s)ds,$$

where Γ is the gamma function.

Definition 2.2. ([16]). For a function h given on the interval $[a,b]$, the Caputo fractional order derivative of h, is given by

$$^cD^\alpha h(t) = \frac{1}{\Gamma(n-\alpha)} \int_a^t (t-s)^{n-\alpha-1} h^{(n)}(s)ds,$$

where $n = [\alpha]+1$ and $[\alpha]$ denote the integer part of α.

Let $(T(t))_{t\geq 0}$ a C_0 semigroup and $M=\sup_{t\geq 0}\|T(t)\|$ and define

$$\mathcal{S}_\alpha(t)=\int_0^\infty h_\alpha(\theta)T\left(t^\alpha\theta\right)d\theta,\quad \mathcal{P}_\alpha(t)=\alpha\int_0^\infty \theta h_\alpha(\theta)T\left(t^\alpha\theta\right)d\theta,\quad t\geq 0$$

$$h_\alpha(\theta)=\frac{1}{\alpha}\theta^{-1-1/\alpha}\psi_\alpha\left(\theta^{-1/\alpha}\right)\geq 0$$

$$\psi_\alpha(\theta)=\frac{1}{\pi}\sum_{n=1}^{\infty}(-1)^{n-1}\theta^{-n\alpha-1}\frac{\Gamma(n\alpha+1)}{n!}\sin(n\pi\alpha),\quad \theta\in(0,\infty)$$

where h_α is a probability density function defined on $(0,\infty)$, that is

$$h_\alpha(\theta)\geq 0,\quad \theta\in(0,\infty),\quad \int_0^\infty h_\alpha(\theta)d\theta=1.$$

For $\gamma\in[0,1]$,

$$\int_0^\infty \theta^\gamma h_\alpha(\theta)\mathrm{d}\theta=\frac{\Gamma(1+\gamma)}{\Gamma(1+\alpha\gamma)}.$$

Lemma 2.3. *([27]).*

(i) *For any fixed $t\geq 0$ and any $x\in E$,*

$$\|\mathcal{S}_\alpha(t)x\|\leq M\|x\| \text{ and } \|\mathcal{P}_\alpha(t)x\|\leq M\|x\|/\Gamma(\alpha).$$

(ii) *$\{\mathcal{S}_\alpha(t):t\geq 0\}$ and $\{\mathcal{P}_\alpha(t):t\geq 0\}$ are strongly continuous.*

(iii) *For each $t>0, \mathcal{S}_\alpha(t)$ and $\mathcal{P}_\alpha(t)$ are compact operators if $T(t)$ is compact.*

We recall a generalization of Gronwall's lemma that we will use in the sequel.

Lemma 2.4. *(Generalized Gronwall inequality [15]). Let $v:[0,b]\longrightarrow[0,+\infty)$ be a real function and $\omega(\cdot)$ be a nonnegative, locally integrable function on $[0,b]$. Suppose that there exist $a>0$ and $0<\alpha<1$ such that*

$$v(t)\leq\omega(t)+a\int_0^t(t-s)^{-\alpha}v(s)ds.$$

Then there exists a constant $m=m(\alpha)$ such that

$$v(t)\leq\omega(t)+ma\int_0^t(t-s)^{-\alpha}\omega(s)ds,\quad \text{for}\quad t\in[0,b].$$

Here we summarize basic definitions from positive operator theory [15]. Let E be a real or complex Banach space and let E^* be its dual space. Then, E^* is a space of all linear functionals on E. In the following, we write the value of $f\in E^*$ at $\psi\in E$ as $\langle f,\psi\rangle$. A closed subset $C\subset E$ is called the cone (or positive cone) if the following conditions hold:

1. $C+C\subset C$,
2. $\lambda\geq 0\Rightarrow\lambda C\subset C$,
3. $C\cap(-C)=\{0\}$,
4. $C\neq\{0\}$.

With respect to the cone C, we write $x \leq y$ if $y-x \in C$ and $x < y$ if $y-x \in C \backslash \{0\}$. If the set $\{\psi - \phi \mid \psi, \phi \in C\}$ is dense in E, the cone C is said to be total. If $E = C - C$, C is called a reproducing cone. Let $B(E)$ be a set of bounded linear operators from E into itself. Let $r(T)$ be the spectral radius of $T \in B(E)$ and let $P_\sigma(T)$ be the point spectrum of T. The dual cone C^* is a subset of E^* composed of all positive linear functionals. $f \in C^*$ is called a positive linear functional if $\langle f, \psi \rangle \geq 0$ for all $\psi \in C$. $\psi \in C$ is called a quasi-interior point or nonsupporting point if $\langle f, \psi \rangle > 0$ for all $f \in C^* \backslash \{0\}$. A positive linear functional $f \in C^*$ is called strictly positive if $\langle f, \psi \rangle > 0$ for all $\psi \in C^+$. A nonzero operator $T \in B(E)$ is called positive if $T(C) \subset C$. If $(T-S)(C) \subset C$ for $T, S \in B(E)$, we write $S \leq T$. A positive operator $T \in B(E)$ is called semi-nonsupporting if, for any $\psi \in C^+$and $f \in C^* \backslash \{0\}$, there exists an integer $p = p(\psi, f)$ such that $\langle f, T^p \psi \rangle > 0$. A positive operator $T \in B(E)$ is called nonsupporting if, for any $\psi \in C^+$and $f \in C^* \backslash \{0\}$, there exists an integer $p = p(\psi, f)$ such that $\langle f, T^n \psi \rangle > 0$ for all $n \geq p$. A positive operator $T \in B(E)$ is called strictly nonsupporting if, for any $\psi \in C^+$, there exists a positive integer $p = p(\psi)$ such that $T^n \psi$ is a quasi-interior point of C for all $n \geq p$.

3 The basic model

First as we consider a host population, we consider a closed one sex age structured population under the demographic stable growth. Let $P(t,a)$ be the age density at time t of the host population, $\mu(a)$ the age specific natural death rate and $f(a)$ the age specific fertility rate. Then we assume that the host population dynamics is described by the McKendrick equation as follows:

$$\left(\frac{\partial}{\partial t} + \frac{\partial}{\partial a}\right) P(t,a) = -\mu(a) P(t,a), \tag{3.1}$$

$$P(t,0) = \int_0^\omega f(a) P(t,a) da, \tag{3.2}$$

$$P(0,a) = P_0(a), \tag{3.3}$$

where $P_0(a)$ is given initial data and $\omega < \infty$ is the upper bound of age. The system (3.1)-(3.3) is well known as the stable population model in demography.
It follows from the stable population theory [11, 14] that the system (3.1) and (3.2) has a unique persistent age profile as

$$\psi(a) := \frac{e^{-r_0 a} \ell(a)}{\int_0^\omega e^{-r_0 a} \ell(a) da},$$

where $\ell(a)$ is the survival rate defined by

$$\ell(a) := \exp\left(-\int_0^a \mu(\sigma) d\sigma\right),$$

and r_0 called the intrinsic rate of natural increase, is given by the dominant real root of the Euler-Lotka characteristic equation:

$$\int_0^\omega e^{-ra} f(a) \ell(a) da = 1. \tag{3.4}$$

Since ω is the maximum attainable age, that is, $\ell(\omega) = 0$, we assume that $\mu \in L^1_{+,loc}(0,\omega)$ and $\int_0^\omega \mu(\sigma) d\sigma = \infty$.
Subsequently let us divide the host population into two subpopulations; the susceptible class and the infective class, the age-density functions of each class are denoted by $S(t,a)$ and $I(t,a)$. Let $\beta(a,\sigma)$ be the transmission rate between the susceptible individuals aged a and the infective individuals

aged σ, $\gamma(a)$ the cure rate at age a. Moreover, we can assume that the host population in steady state is a demographic stationary population given by

$$P(t,a) = P_0(a) := B\ell(a) \quad \forall t > 0,$$

where B is the birth rate (number of new borns per unit time). Hence, $\psi(a) = b_0\ell(a)$, where $b_0 = \frac{1}{\int_0^\omega \ell(a)da}$ denotes the crude birth rate in the stationary population. Then we obtain the following system of equations that describe the dynamics of the model:

$$\begin{aligned}
\left(\frac{\partial^\alpha}{\partial t^\alpha} + \frac{\partial}{\partial a}\right) S(t,a) &= -\lambda(t,a)S(t,a) + \gamma(a)I(t,a) - \mu(a)S(t,a), \\
\left(\frac{\partial^\alpha}{\partial t^\alpha} + \frac{\partial}{\partial a}\right) I(t,a) &= \lambda(t,a)S(t,a) - \gamma(a)I(t,a) - \mu(a)I(t,a), \\
S(t,0) &= \int_0^\omega f(a)S(t,a)da, \\
I(t,0) &= 0, \\
S(0,a) &= S_0(a), \\
I(0,a) &= I_0(a),
\end{aligned} \tag{3.5}$$

where $\frac{\partial^\alpha}{\partial t^\alpha}$ is the Caputo fractional derivative of order $0 < \alpha \leq 1$ and the force of infection $\lambda(t,a)$ is given by

$$\lambda(t,a) = \frac{1}{P(t)} \int_0^\omega \beta(a,\sigma)I(t,\sigma)d\sigma,$$

where $P(t) := \int_0^\omega P(t,a)da$ is the total size of the population and

$$P(t,a) = S(t,a) + I(t,a).$$

We consider the normalization of the solution of the system (3.5):

$$s(t,a) := \frac{S(t,a)}{P(t,a)}, \quad i(t,a) := \frac{I(t,a)}{P(t,a)}.$$

Then the new system is given by:

$$\begin{cases}
\left(\frac{\partial^\alpha}{\partial t^\alpha} + \frac{\partial}{\partial a}\right) s(t,a) = -\lambda(t,a)s(t,a) + \gamma(a)i(t,a), \\
\left(\frac{\partial^\alpha}{\partial t^\alpha} + \frac{\partial}{\partial a}\right) i(t,a) = \lambda(t,a)s(t,a) - \gamma(a)i(t,a), \\
s(t,0) = 1, \\
i(t,0) = 0, \\
s(0,a) = s_0(a), \\
i(0,a) = i_0(a).
\end{cases} \tag{3.6}$$

Moreover of course, it follows from the definition that

$$s(t,a) + i(t,a) = 1. \tag{3.7}$$

In the following, we mainly consider the normalized system (3.6) under the condition (3.7) and the following technical assumption:

Assumption 3.1. $\beta \in L^\infty_+((0,\omega)\times(0,\omega))$ and $\gamma, f \in L^\infty_+(0,\omega)$.

From the normalized condition (3.7), the system can be reduced to a single equation for $i(t,a)$:

$$\begin{cases} \left(\frac{\partial^\alpha}{\partial t^\alpha} + \frac{\partial}{\partial a}\right) i(t,a) = \lambda[a|i](1-i(t,a)) - \gamma(a)i(t,a), \\ i(t,0) = 0, \\ i(0,a) = i_0(a), \end{cases} \tag{3.8}$$

where $\lambda[a|i]$ is given by the integral operator defined by

$$\lambda[a|\phi] := \int_0^\omega \beta(a,\sigma)\psi(\sigma)\phi(\sigma)d\sigma, \qquad \phi \in L^1(0,\omega).$$

The state space of the system (3.8) is

$$\Omega := \{i \in E_+, 0 \le i \le 1\},$$

where E_+ is the positive cone of $E = L^1(0,\omega)$.
Let us define operators A and F on E as follows:

$$(A\phi)(a) := -\phi'(a), \qquad D(A) = \{\phi \in E, \phi \in AC(0,\omega), \phi(0) = 0\},$$

where $AC(0,\omega)$ denotes the set of absolutely continuous functions on the interval $(0,\omega)$.

$$F(\phi)(a) = \lambda[a \mid \phi](1-\phi(a)) - \gamma(a)\phi(a).$$

Let us define $u(t) := i(t,.)$. Thus (3.8) can be expressed as the following Cauchy problem in E:

$$\frac{d^\alpha u(t)}{dt^\alpha} = Au(t) + F(u(t)), u(0) = u_0. \tag{3.9}$$

The operator A generates a C_0 semigroup $(T(t))_{t\ge0}$ such that $T(t) = e^{tA}$ and $M = \sup_{t\ge0}\|T(t)\|$, and $F : \Omega \longrightarrow E$ is lipschitz continuous;

$$\|F(u) - F(v)\| \le L\|u - v\|,$$

and

$$N = \sup_{u\in E}\|F(u)\|.$$

4 Main results

4.1 Existence and uniqueness of the local solution

It is suitable to rewrite the Cauchy problem (3.9) in the equivalent integral equation.

$$u(t) = u_0 + \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}Au(s)ds + \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}F(u(s))ds, \tag{4.1}$$

for $t \in [0,T]$.
Note that the Laplace transform of an abstract function $f \in L^1(\mathbb{R}^+, X)$ is defined by $\widehat{f}(\lambda) := \int_0^\infty e^{-\lambda t} f(t)dt \quad (\lambda > 0)$. Applying the Laplace transform to (4.1) we get

$$\widehat{u}(\lambda) = \frac{u_0}{\lambda} - \frac{1}{\lambda^\alpha}A\widehat{u}(\lambda) + \frac{\widehat{F}(u(\lambda))}{\lambda^\alpha}$$

that is,

$$\widehat{u}(\lambda)=\lambda^{\alpha-1}\left(\lambda^{\alpha}+A\right)^{-1}u_0+\left(\lambda^{\alpha}+A\right)^{-1}\widehat{F}(u(\lambda)).$$

$$\widehat{u}(\lambda)=\lambda^{\alpha-1}\int_0^{\infty}e^{-\lambda^{\alpha}}T(t)u_0dt+\int_0^{\infty}e^{-\lambda^{\alpha}t}T(t)\widehat{F}(\lambda)dt. \tag{4.2}$$

Consider the one-sides stable probability density

$$\psi_{\alpha}(\theta)=\frac{1}{\pi}\sum_{n=1}^{\infty}(-1)^{n-1}\theta^{-n\alpha-1}\frac{\Gamma(n\alpha+1)}{n!}\sin(n\pi\alpha).$$

Whose Laplace transform is given by

$$\int_0^{\infty}e^{-\lambda\theta}\psi_{\alpha}(\theta)d\theta=e^{-\lambda^{\alpha}} \qquad \alpha\in(0,1). \tag{4.3}$$

Using (4.2) and (4.3), we get

$$\begin{aligned}\widehat{u}(\lambda)=&\int_0^{\infty}e^{-\lambda t}\Big[\int_0^{\infty}\psi_{\alpha}(\theta)T\left(\frac{t^{\alpha}}{\theta^{\alpha}}\right)d\theta\\&+\alpha\int_0^{t}\int_0^{\infty}\psi_{\alpha}(\theta)T\left(\frac{(t-s)^{\alpha}}{\theta^{\alpha}}\right)(t-s)^{\alpha-1}\frac{(t-s)^{\alpha-1}}{\theta^{\alpha}}F(u(s))d\theta ds\Big]dt.\end{aligned}$$

Now we can invert the last Laplace transform to get

$$u(t)=\mathcal{S}_{\alpha}(t)u_0+\int_0^{t}(t-s)^{\alpha-1}\mathcal{P}_{\alpha}(t-s)F(u(s))ds, \quad t\in[0,T].$$

We give the following definition of the mild solution of (3.9).

Definition 4.1. By a mild solution of problem (3.9), we mean a function $u\in C([0,T];E)$ satisfying

$$u(t)=\mathcal{S}_{\alpha}(t)u_0+\int_0^{t}(t-s)^{\alpha-1}\mathcal{P}_{\alpha}(t-s)F(u(s))ds, \quad t\in[0,T]. \tag{4.4}$$

Theorem 4.2. *Let $u_0\in\Omega$ and assume that hypothesis $\frac{MLT^{\alpha}}{\Gamma(\alpha+1)}<1$ and $M\|u_0\|+\frac{MNT^{\alpha}}{\Gamma(\alpha+1)}<1$ hold, then the fractional problem* (3.9) *has a unique mild solution defined on* $[0,T]$.

Proof.
Consider the mapping H given by

$$(Hu)(t)=\mathcal{S}_{\alpha}(t)u_0+\int_0^{t}(t-s)^{\alpha-1}\mathcal{P}_{\alpha}(t-s)F(u(s))ds, \quad t\in[0,T].$$

We see that $(Hu)(t)\in C([0,T];E)$.
First, we show that $H(\Omega)\subset\Omega$. Let $t\in[0,T]$ and $u\in\Omega$

$$\begin{aligned}|(Hu)(t)|&\leq|\mathcal{S}_{\alpha}(t)u_0|+\int_0^{t}(t-s)^{\alpha-1}|\mathcal{P}_{\alpha}(t-s)F(u(s))|ds\\&\leq M\mid u_0\mid+\frac{M}{\Gamma(\alpha)}\int_0^{t}(t-s)^{\alpha-1}\mid F(u(s))\mid ds.\end{aligned}$$

$$\|H(u)\|\leq M\|u_0\|+\frac{MNT^{\alpha}}{\Gamma(\alpha+1)}<1.$$

Then H maps Ω into itself.
Next, we shall show that H is a strict contraction on Ω which will ensure the existence of a unique

mild solution.
Let u and v two elements in Ω, by the assumption on F and $\frac{MLT^{\alpha}}{\Gamma(\alpha+1)} < 1$, we have:

$$\begin{aligned}
|(Hu)(t)-(Hv)(t)| &\leq \int_0^t (t-s)^{\alpha-1}|\mathcal{P}_\alpha(t-s)(F(u(s))-F(v(s)))|ds \\
&\leq \frac{ML}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}|u(s)-v(s)|ds \\
\|(Hu)-(Hv)\| &\leq \frac{MLT^{\alpha}}{\Gamma(\alpha+1)}\|u-v\|.
\end{aligned}$$

This yields that H is a contraction on Ω. So H has a unique fixed point $u \in \Omega$ by the Banach Fixed point Theorem, which is a mild solution to problem (3.9) on $[0,T]$.

Remark 4.3. If $u_0 \in D(A)$, the mild solution becomes a classical solution.

4.2 Global existence of the solution

For concrete application, the global existence of the solution of the fractional differential equation always becomes a main concern, which thereby is discussed in this section.

Theorem 4.4. *Let $u_0 \in \Omega$, the mapping $F : \Omega \longrightarrow E$ is Lipschitz continuous and there exist a positive constants c_1 et c_2 such that*

$$\|F(u)\| \leq c_1 + c_2\|u\|. \tag{4.5}$$

Then the problem (3.9) *has a global mild solution.*

Proof.
We have

$$F(u)(a) = \lambda\,[a|u]\,(1-u(a)) - \gamma(a)u(a).$$

If we define $\lambda^+ = \sup\lambda$ and $\gamma^+ = \sup\gamma$, we obtain

$$\begin{aligned}
|F(u)(a)| &\leq \lambda^+(1+|u(a)|) + \gamma^+|u(a)| \\
&\leq \lambda^+ + (\lambda^+ + \gamma^+)|u(a)|
\end{aligned}$$

we choose $c_1 = \lambda^+$, $c_2 = \lambda^+ + \gamma^+$ and we go to the sup, we know that (4.5) holds.
Next, we assume that the mild solution u admits a maximal existence interval $(0, T_{max})$ (T_{max} is the maximum time of existence). Suppose there is a sequence $t_n \longrightarrow T_{max}$ such that $|u(t_n| \longrightarrow \infty$. Then for $0 < t < T_{max}$, substitution of (4.5) in the equation (4.4) leads to the following estimation.

$$|u(t)| \leq |\mathcal{S}_\alpha(t)u_0| + \int_0^t (t-s)^{\alpha-1}|\mathcal{P}_\alpha(t-s)|(c_1 + c_2|u(s)|)ds.$$

Therefore,

$$|u(t)| \leq M|u_0| + \frac{Mc_1t^{\alpha}}{\Gamma(\alpha+1)} + \frac{Mc_2}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}|u(s)|ds.$$

If we take

$$\omega(t) = M|u_0| + \frac{Mc_1t^{\alpha}}{\Gamma(\alpha+1)},$$

which is bounded, and

$$a = \frac{Mc_2}{\Gamma(\alpha)},$$

it follows, in accordance with Lemma 2.4, that $v(t) = |u(t)|$ is bounded. Which contradicts the fact that $\lim_{t\to T_{max}}|u(t)| = \infty$. So $T_{max} = \infty$.

5 Existence of steady states

We now consider the existence of endemic steady states. First note that the endemic steady state $(s^*(a), i^*(a))^{\mathrm{T}}$ satisfies the following ODE system:

$$
\begin{aligned}
&\frac{d}{da}s^*(a) = -\lambda^*(a)s^*(a) + \gamma(a)i^*(a),\\
&\frac{d}{da}i^*(a) = \lambda^*(a)s^*(a) - \gamma(a)i^*(a),\\
&s^*(0) = 1, \quad i^*(0) = 0,
\end{aligned}
\tag{5.1}
$$

where

$$
\lambda^*(a) := \int_0^\omega \beta(a,\sigma)\psi(\sigma)i^*(\sigma)d\sigma, \tag{5.2}
$$

$$
\delta(a) := \exp\left(-\int_0^a \gamma(\sigma)d\sigma\right).
$$

Formally solving the above ODEs, we have the following expressions:

$$
s^*(a) = e^{-\int_0^a \lambda^*(\sigma)d\sigma} + \int_0^a e^{-\int_\sigma^a \lambda^*(\sigma)d\sigma}\gamma(\sigma)i^*(\sigma)d\sigma, \tag{5.3}
$$

$$
i^*(a) = \int_0^a \frac{\delta(a)}{\delta(\sigma)}\lambda^*(\sigma)s^*(\sigma)d\sigma. \tag{5.4}
$$

Inserting the above expression into (5.3), we obtain

$$
s^*(a) = e^{-\int_0^a \lambda^*(s)ds} + \int_0^a e^{-\int_\sigma^a \lambda^*(s)ds}\gamma(\sigma)\int_0^\sigma \frac{\delta(\sigma)}{\delta(\eta)}\lambda^*(\eta)s^*(\eta)d\eta d\sigma. \tag{5.5}
$$

Let $b^*(a) := \lambda^*(a)s^*(a)$ be the density of newly infecteds at steady state and define a nonlinear operator Δ given by

$$
\Delta[\phi](a,\sigma) := \phi(a)e^{-\int_\sigma^a \phi(s)ds}, \quad \phi \in E. \tag{5.6}
$$

Then,

$$
b^*(a) = \Delta[\lambda^*](a,0) + \int_0^a \Delta[\lambda^*](a,\sigma)\int_0^\sigma \Delta[\gamma](\sigma,0)\frac{b^*(\eta)}{\delta(\eta)}d\eta d\sigma.
$$

We define a linear operator T in E as

$$
(T[\lambda^*]\phi)(a) := \int_0^a \Delta[\lambda^*](a,\sigma)\int_0^\sigma \Delta[\gamma](\sigma,0)\phi(\eta)d\eta d\sigma. \tag{5.7}
$$

Then we have

$$
b^*(a) = \Delta[\lambda^*](a,0) + (T[\lambda^*]b^*)(a). \tag{5.8}
$$

Lemma 5.1.
There exists a number $k \in (0,1)$ such that $|T[\lambda^*]| \le k$ uniformly for $\lambda^* \in E_+$.

Proof.
For given $\lambda^* \in E_+$,

$$\begin{aligned}\int_0^\omega \left(T\left[\lambda^*\right]\phi\right)(a)da &= \int_0^\omega da \int_0^a \Delta[\lambda^*](a,\sigma)\int_0^\sigma \Delta[\gamma](\sigma,0)\phi(\eta)d\eta d\sigma \\ &= \int_0^\omega da \int_0^a \lambda^*(a)e^{-\int_\sigma^a \lambda^*(s)ds}\int_0^\sigma \gamma(\sigma)e^{-\int_0^\sigma \gamma(s)ds}\phi(\eta)d\eta d\sigma \\ &= \int_0^\omega \gamma(\sigma)e^{-\int_0^\sigma \gamma(s)ds}d\sigma \int_\sigma^\omega \lambda^*(a)e^{-\int_\sigma^a \lambda^*(s)ds}da \int_0^\sigma \phi(\eta)d\eta \\ &\le \left(1-e^{-|\gamma|}\right)\left(1-e^{-|\lambda^*|}\right)|\phi|.\end{aligned}$$

which shows that $|T\left[\lambda^*\right]| \le k$ with $k := 1-e^{-|\gamma|}$.
If $\lambda^* \in E_+ = L^1_+(0,\omega)$ is given, (5.8) is a Volterra integral equation with respect to b^*. As the Volterra operator has the spectral radius zero, (5.8) is solved as following:

$$b^* = \left(I - T\left[\lambda^*\right]\right)^{-1}\Delta\left[\lambda^*\right](\cdot,0). \tag{5.9}$$

Therefore, we obtain a fixed point equation for the force of infection λ^* :

$$\begin{aligned}\lambda^*(a) = \left(\Psi\lambda^*\right)(a) &:= \int_0^\omega \beta(a,\sigma)\psi(\sigma)i^*(\sigma)d\sigma \\ &= \int_0^\omega \beta(a,\sigma)\psi(\sigma)\int_0^\sigma \frac{\delta(\sigma)}{\delta(\eta)}b^*(\eta)d\eta d\sigma \\ &= \int_0^\omega \int_\eta^\omega \beta(a,\sigma)\psi(\sigma)\frac{\delta(\sigma)}{\delta(\eta)}d\sigma\left(\left(I-T\left[\lambda^*\right]\right)^{-1}\Delta\left[\lambda^*\right](\cdot,0)\right)(\eta)d\eta,\end{aligned} \tag{5.10}$$

where Ψ is a nonlinear operator from E_+ into itself defined by the right hand side of (5.10). Then the Fréchet derivative of Ψ at zero is given by

$$\left(\Psi'[0]\phi\right)(a) = \int_0^\omega \int_\eta^\omega \beta(a,\sigma)\psi(\sigma)\frac{\delta(\sigma)}{\delta(\eta)}d\sigma\phi(\eta)d\eta. \tag{5.11}$$

Now we can define $K := \Psi'[0]$ as the next generation operator (NGO) and its spectral radius $\mathcal{R}_0 := r(K)$ the basic reproduction number for the normalized system, although K is a similar operator of the next generation operator for the original system (3.5). If K and $\mathcal{R}_0$ are defined for the original system (3.5). In order to solve the fixed point problem $\lambda^* = \Psi\lambda^*$ in $E := L^1(0,\omega)$, we use a corollary of the well-known theorem by Krasnoselskii ([17]):

Theorem 5.2. *Suppose that E is a real Banach space and E_+ is a positive cone of E. Let Ψ is a positive operator on E_+ which has a strong Fréchet derivative at the origin $K = \Psi'[0]$, satisfies $\Psi(0) = 0$ and $\Psi(E_+)$ is bounded. Moreover, K has a positive eigenvector $v_0 \in E_+$ associated with eigenvalue $\lambda_0 > 1$, but has no eigenvector in E_+ with unity. Then, Ψ has at least one nonzero fixed point in $\Psi(E_+)$.*

According to [13], we adopt the following technical assumptions for the transmission coefficient $\beta(a,\sigma)$, which is a natural assumption to make the next generation operator becomes nonsupporting and compact.

Assumption 5.3.

1. There exist numbers $\delta_0 \in (0,\omega)$ and $\underline{\beta} > 0$ such that

$$\beta(a,\eta) \ge \underline{\beta} \quad \text{for almost all } (a,\eta) \in (0,\omega)\times(\omega-\delta_0,\omega). \tag{5.12}$$

2. $\beta \in L^\infty_+((0,\omega)\times(0,\omega))$ is extended into $L^\infty_+\left(\mathbb{R}^2\right)$ by $\beta(a,\sigma)=0$ for $(a,\sigma)\notin(0,\omega)\times(0,\omega)$ and satisfies

$$\lim_{h\to 0}\int_0^\omega |\beta(a+h,\eta)-\beta(a,\eta)|da = 0 \quad \text{uniformly for } \eta\in\mathbb{R}. \tag{5.13}$$

Lemma 5.4. *The next generation operator K is nonsupporting and compact.*

Proof.
Define the positive linear functional $f_0 \in E^*_+$ by

$$\langle f_0,\phi\rangle := \int_0^\omega \int_\eta^\omega \beta_0(\sigma)\psi(\sigma)\frac{\delta(\sigma)}{\delta(\eta)}d\sigma\phi(\eta)d\eta,$$

where

$$\beta_0(\sigma)=\begin{cases}\beta & \text{for } \sigma\in(\omega-\delta_0,\omega),\\ 0 & \text{otherwise}.\end{cases} \tag{5.14}$$

Then $K\phi \geq \langle f_0,\phi\rangle e$, for all $\phi\in E_+$, where $e=1\in E_+$, which implies

$$K^{n+1}\phi \geq \langle f_0,\phi\rangle \langle f_0,e\rangle^n e, \quad \forall n\in\mathbb{N}.$$

Thus for arbitrary $F\in E^*_+\backslash\{0\}, \phi\in E_+\backslash\{0\}$ and $n\geq 1$,

$$\langle F,K^n\phi\rangle \geq \langle f_0,\phi\rangle \langle f_0,e\rangle^{n-1}\langle F,e\rangle > 0.$$

This shows K is nonsupporting. Next we show the compactness of K. Let C be an arbitrary bounded subset in $L^1_+(0,\omega)$, and take $M>0$ such that $\sup_{\phi\in C}|\phi|\leq M$. For all $\phi\in C$, using Assumption (5.3),

$$\begin{aligned}
&\lim_{h\to 0}\int_{\mathbb{R}}|K\phi(a+h)-K\phi(a)|da\\
&\quad\leq \lim_{h\to 0}\int_{\mathbb{R}}\int_0^\omega\int_\eta^\omega |\beta(a+h,\sigma)-\beta(a,\sigma)|\psi(\sigma)\frac{\delta(\sigma)}{\delta(\eta)}d\sigma\phi(\eta)d\eta da\\
&\quad\leq \lim_{h\to 0}\int_{\mathbb{R}}\int_0^\omega \phi(\eta)d\eta\int_0^\omega |\beta(a+h,\sigma)-\beta(a,\sigma)|b_0 d\sigma da\\
&\quad\leq b_0 M\lim_{h\to 0}\int_{\mathbb{R}}\int_0^\omega |\beta(a+h,\sigma)-\beta(a,\sigma)|d\sigma da = 0.
\end{aligned}$$

By the Fréchet-Kolmogorov criterion for the compactness of sets in $L^p(\mathbb{R})$ [7,22], $\Psi(C)$ is relatively compact. This shows that K is compact.

Lemma 5.5. *Let $E_+ = L^1_+(0,\omega)$ and $\Omega_M := \{\phi\in E_+ : |\phi|\leq M\}$. There exists a number $M>0$ such that $\Psi(E_+)\subset\Omega_M$.*

Proof.
Define the nonlinear operator $\Lambda:\lambda^*\to b^*$ in E_+ by $\Lambda\phi = (I-T[\phi])^{-1}\Delta[\phi](\cdot,0)$. From Lemma 5.1, it follows that

$$|\phi|\leq |(I-T[\phi])^{-1}| \quad |\Delta[\phi](\cdot,0)|\leq \frac{1}{1-k}.$$

Then we have $\Psi(E_+)\subset\Omega_M$ with $M:=\frac{1}{1-k}\int_0^\omega da\int_0^\omega \beta(a,\sigma)\psi(\sigma)d\sigma$

From the well-known Krein-Rutman's Theorem, we know that $r(K)$ is a positive eigenvalue if $r(K)>0$, and it is a pole of the resolvent because K is compact. Then we can apply Sawashima's results for nonsupporting operator to obtain the following properties [18,21]:

Proposition 5.6. *Suppose that the cone E_+ is total, K is compact, nonsupporting with respect to E_+ and $r(K) > 0$. Then the following holds:*

1. *$r(K) \in P_\sigma(K)\backslash\{0\}$ and $r(K)$ is a simple pole of the resolvent $(\lambda I - K)^{-1}$.*
2. *The eigenspace corresponding to $r(K)$ is one-dimensional and its eigenvector $v_0 \in E_+$ is a quasi-interior point. Any eigenvector in $E_+$is proportional to v_0.*
3. *The adjoint eigenspace corresponding to $r(K)$ is one-dimensional and its eigenfunctional $f \in E^*\backslash\{0\}$ is strictly positive.*

As is seen above, the idea of being nonsupporting for positive operator is an infinite-dimensional extension of the primitivity of nonnegative matrices in the finite-dimensional case. Using the above facts, we can show the main theorem in this section:

Theorem 5.7. *If $\mathcal{R}_0 > 1$, there exists at least one endemic steady state, while there is no endemic steady state if $\mathcal{R}_0 \leq 1$.*

Proof.
Supposing that $\mathcal{R}_0 = r(K) \leq 1$, it is easily checked that $K\phi - \Psi(\phi) \in E_+\backslash\{0\}$ for $\phi \in E_+\backslash\{0\}$. If there exists an $\phi_0 \in E_+\backslash\{0\}$ being a solution of $\phi = \Psi(\phi)$, then $\phi_0 = \Psi(\phi_0) \leq K(\phi_0)$. Let $F_0^* \in E_+^*\backslash\{0\}$ be the adjoint eigenvector of K corresponding to $r(K)$. Taking duality pairing, we find

$$\langle F_0^*, K(\phi_0) - \phi_0\rangle = \langle (K^* - I^*) F_0^*, \phi_0\rangle = (r(K) - 1)\langle F_0^*, \phi_0\rangle > 0,$$

because $K(\phi_0) - \phi_0 \in E_+\backslash\{0\}$ and F_0^* is strictly positive. Then we have $r(K) > 1$, which is a contradiction. Next we assume that $\mathcal{R}_0 = r(K) > 1$. We define an operator Ψ_r by:

$$\Psi_r(x) = \begin{cases} \Psi(x), & \text{if } |x| \geq r, x \in E_+, \\ \Psi(x) + (r - |x|)x_0, & \text{if } |x| \leq r, x \in E_+, \end{cases}$$

where x_0 is the positive eigenvector of K corresponding to $r(K) > 1$.
Let

$$\Omega_r = \{x \in E_+, |x| \leq M + r|x_0|\}.$$

Then Ψ_r is completely continuous and transforms the set Ω_r into itself. Since Ω_r is bounded, convex and closed in E, Ψ_r has a fixed point $x_r \in \Omega_r$ (Schauder's principle). Observe that the Frechet derivative of $\Psi(x)$ at $x = 0$ is the operator K and K does not have in E_+ eigenvectors corresponding to the eigenvalue one. Then we apply the method of [17, Theorem 4.11], and it can be shown that the norms of these fixed points are greater than r if r is sufficiently small. That is, Ψ has a positive fixed point. This completes the proof.

We adopt the following assumption called separable mixing assumption which means that there is no correlation between the age of the infected individuals and that of the susceptible individuals.

Assumption 5.8. There exist $\beta_1, \beta_2 \in L_+^\infty(0, \omega)$ such that

$$\beta(a, \sigma) = \beta_1(a)\beta_2(\sigma).$$

with $\beta_1(a)$ reflecting the susceptibility and $\beta_2(\sigma)$ indicating the infectivity. In particular, if $\beta_1 = \beta_2$, we have the proportionate mixing assumption. (Traditionally, these assumptions are often called the proportionate mixing assumption without distinguishing between separable and proportional [6, 10]). Under the separable mixing assumption, (5.10) can be written as

$$\lambda^*(a) = \beta_1(a)\int_0^\omega \int_\eta^\omega \beta_2(\sigma)\psi(\sigma)\frac{\delta(\sigma)}{\delta(\eta)}d\sigma\left(\left(I - T[\lambda^*]\right)^{-1}\Delta[\lambda^*](\cdot, 0)\right)(\eta)d\eta. \tag{5.15}$$

Then (5.15) becomes a one-dimensional equation and its solution must be written as $\lambda^*(a) = c\beta_1(a)$, where c is a constant. Inserting this expression into (5.15), we arrive at an equation for the unknown number c.

$$1 = \int_0^\omega \int_\eta^\omega \beta_2(\sigma)\psi(\sigma)\frac{\delta(\sigma)}{\delta(\eta)}d\sigma \left(I - T\left[c\beta_1\right]\right)^{-1} \beta_1(\eta)e^{-c\int_0^\eta \beta_1(s)ds}d\eta. \tag{5.16}$$

The right-hand side of (5.16) is strictly decreasing function of c that goes to zero as $c \longrightarrow \infty$. Therefore, if the condition

$$\int_0^\omega \beta_1(\eta)\int_\eta^\omega \beta_2(\sigma)\psi(\sigma)\frac{\delta(\sigma)}{\delta(\eta)}d\sigma d\eta > 1, \tag{5.17}$$

holds, the characteristic equation (5.16) has a unique positive solution.
The left-hand side of the threshold condition (5.17) gives the basic reproduction number $\mathcal{R}_0$ of the system (3.6). Hence, we can summarize the above argument as follows:

Proposition 5.9. *For the normalized epidemic system* (3.6) *with the separable mixing assumption, there exists a unique endemic steady state if and only if the basic reproduction number is greater than unity.*

$\mathcal{R}_0$ for the separable mixing case is given by

$$\mathcal{R}_0 = \int_0^\omega \int_\eta^\omega \beta_2(\sigma)\psi(\sigma)\frac{\delta(\sigma)}{\delta(\eta)}d\sigma\beta_1(\eta)d\eta. \tag{5.18}$$

6 Numerical simulation

In this section, we provide numerical examples that support and complete our theoretical results. The system (3.5) is numerically integrated by using the forward finite difference approximations to discretize the time-fractional derivative and age derivative [39]. Let us fix the following demographic parameters:

$$\mu(a) = 0.0005, \quad \gamma(a) = 0.1, \quad \beta(a,\sigma) = \beta(a) = e^{-a}.$$

In what follows, we fix the following initial data

$$I(0,a) = (\sin(0.055a))^2 + 0.001e^{-0.5a}, \qquad S(0,a) = 0.01e^{-0.09a} + 0.7(\sin(0.05a))^2.$$

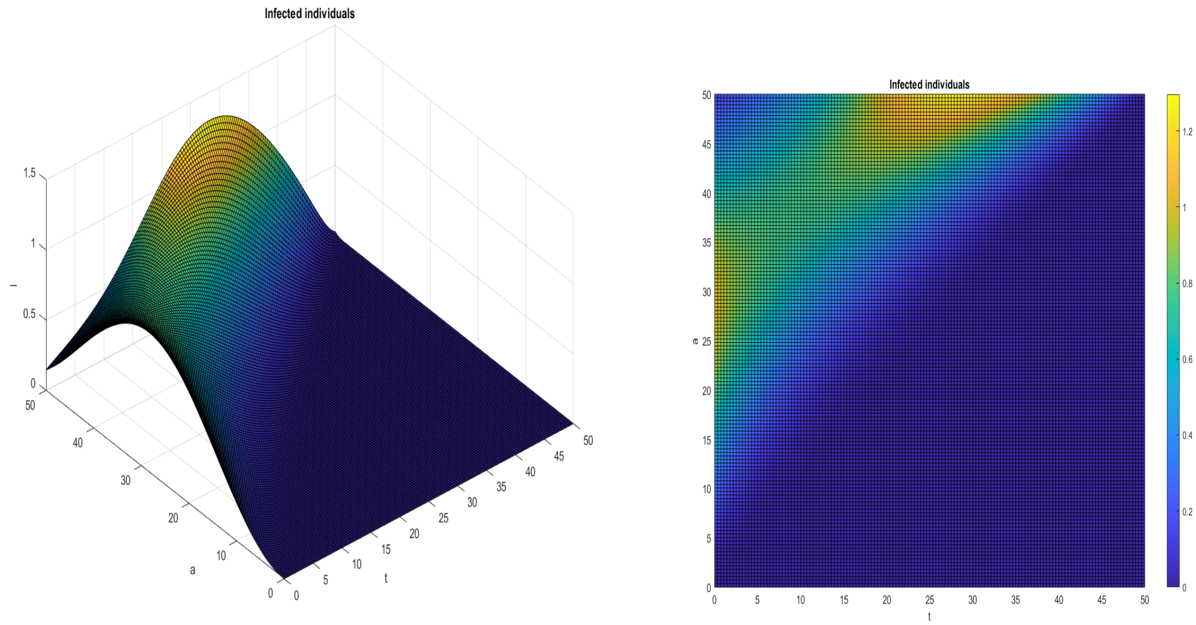

Figure 1. The dynamics of infected population for $\alpha = 1$ with $\mathcal{R}_0 \leq 1$.

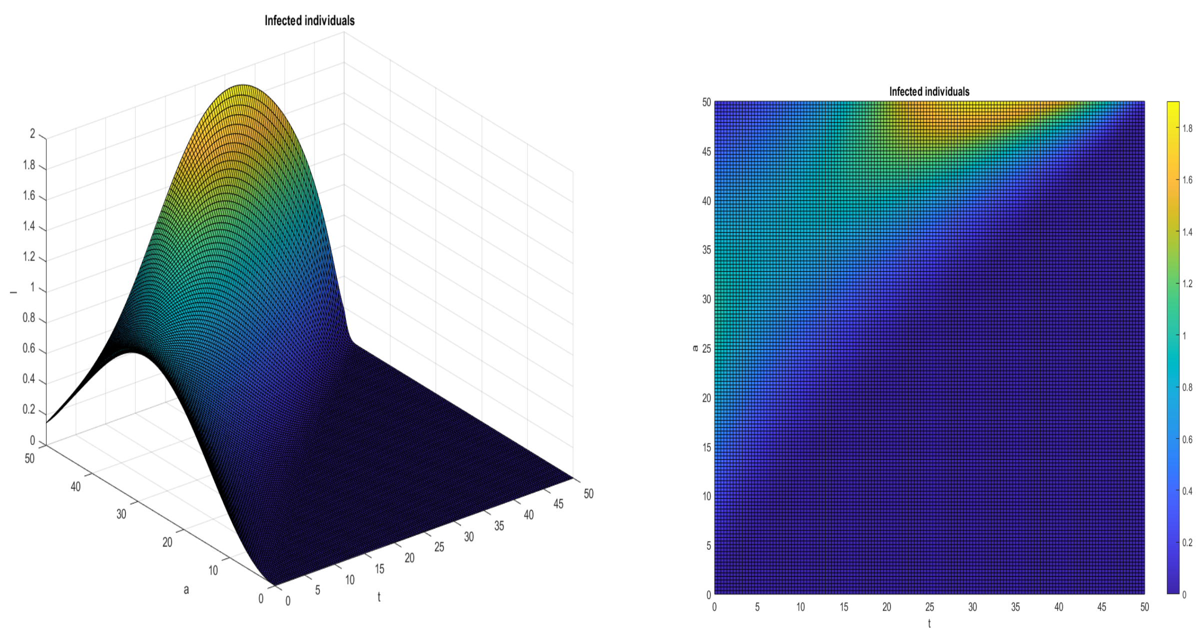

Figure 2. The dynamics of infected population for $\alpha = 0.8$ with $\mathcal{R}_0 \leq 1$.

7 Conclusion

In this paper, our primary focus revolved around examining the qualitative dynamics exhibited by solutions within an *SIS* epidemic model that incorporates age structure and is subject to the influence of the fractional derivative α. Our investigation began by delving into the global behavior of an expanded set of scenarios, extending beyond the fundamental *SIS* model. These extensions introduced memory effects characterized by the Caputo fractional derivative, with the parameter α constrained to the range $0 < \alpha \leq 1$. Notably, as α converges to 1, our model aligns with the classical variant devoid of memory. Secondly, we have shown that the basic reproduction number $\mathcal{R}_0$ plays the role of a threshold value for the existence and uniqueness and of each steady state such that the disease will be naturally eradicated if $\mathcal{R}_0 \leq 1$, while it is strongly persistent and endemic steady states exists if $\mathcal{R}_0 > 1$. The paper concludes with a series of numerical simulations, which serve to further enhance our comprehension of both our proposed model and the fractional derivative's influence. Through these simulations, we have successfully determined the stability of the disease-free equilibrium, not only for the traditional integer derivative case ($\alpha = 1$) but also across the entire range of fractional derivatives where $0 < \alpha \leq 1$. This comprehensive analysis firmly establishes the broad applicability and generality of our system. Furthermore, the incorporation of fractional derivatives has yielded additional insights into predicting the disease's progression. In certain scenarios, it has even demonstrated the capacity to influence the time required for the system to attain stable states.

References

[1] Allen, L.J.S. Discrete and continuous models of populations and epidemics. Journal of Mathematical Systems, Estimation, and Control, 1(3): 335-369 (1991).

[2] Barril, C., Calsina,À. and Rippol, J. A practical approach to $\mathcal{R}_0$ in continuous time ecological models, Math. Meth. Appl. Sci., 41(2017)8432-8445.

[3] Busenberg, S., Iannelli, M., Thieme, H. Global behavior of an age-structured epidemic model. SIAM J. Math. Anal., 22(4): 1065-1080 (1991).

[4] Cha, Y., Iannelli, M., Milner, E. Existence and uniqueness of endemic states for the age-structured SIR epidemic model. Maththematical Biosciences, 150: 177-190 (1998).

[5] Diekmann, O., Heesterbeek, J. A. P., Metz, J. A. J., On the deflnition and the computation of the basic reproduction ratio R in models for infectious diseases in heterogeneous populations, J. Math. Biol., 28 (1990), 365-382.

[6] Dietz, K., Schenzle, D.: Proportionate mixing models for age-dependent infection transmission. J. Math. Biol. 22, 117-120 (1985).

[7] Dunford, N. and Schwartz, J. T. Linear Operators Part I: General Theory, New York: Interscience publishers, 1958.

[8] El-Doma, M. Analysis of an age-dependent SIS epidemic model with vertical transmission and proportionate mixing assumption. Math. Comput. Model., 29: 31-43 (1999).

[9] Greenhalgh, D. Analytical threshold and stability results on age-structured epidemic models with vaccination. Theoretical Population Biology, 33: 266-290 (1988).

[10] Greenhalgh, D., Dietz, K.: Some bounds on estimates for reproductive ratios derived from the age-specific force of infection. Math. Biosci. 124, 9-57 (1994).

[11] Iannelli, M., Mathematical Theory of Age-Structured Population Dynamics, Giardini Editori e Stampatori in Pisa, (1995).

[12] Inaba, H. on a new perspective of the basic reproduction number in heterogeneous environments. J.Math.Biol., 65(2012) 309-348.

[13] Inaba, H. Threshold and stability results for an age-structured epidemic model. J. Math. Biol., 28: 149-175(1990).

[14] Inaba, H. A semigroup approach to the strong ergodic theorem of the multistate stable population process, Math. Popul. Studies, 1 (1988), 49-77.

[15] Inaba, H. Age-Structured Population Dynamics in Demography and Epidemiology, Springer, Singapore, (2017).

[16] Kilbas, A. A., Srivastava, H. H., Trujillo, J. J. Theory and Applications of Fractional Differential Equations, (2006).

[17] Krasnoselskii, M.A. Positive Solutions of Operator Equations. Groningen, Noordhoff, (1964).

[18] Marek, I. Frobenius theory of positive operators: comparison theorems and applications. SIAM J. Appl. Math., 19(3): 607-628 (1970).

[19] May, R.M., Anderson, R.M. Endemic infections in growing populations. Math. Biosci., 77: 141-156 (1985).

[20] Mckendrick, A. Applications of mathematics to medical problems. Proc. Edinburgh Math. Soc., 44: 98-130 (1926).

[21] Sawashima, I. On spectral properties of some positive operators. Nat. Sci. Dep. Ochanomizu Univ., 15 : 53-64 (1964). Sin. 15, 45-53 (1999).

[22] Smith, H. L. and Thieme, H. R. Dynamical Systems and Population Persistence, Graduate Studies in Mathematics 118, Amer. Math. Soc. Providence, Rhode Island, (2011).

[23] Tudor, D.W. An age-dependent epidemic model with applications to measles. Math. Biosci., 73: 131-147 (1985).

[24] Webb, G.B. Theory of Nonlinear Age-dependent Population Dynamics. New York and Basel: Marcel Dekker, (1985).

[25] Yang, H.M. Directly transmitted infections modeling considering an age-structured contact rate. Math. Comput. Model., 29: 39-48 (1999).

[26] Ye, H, Gao, J, Ding, Y: A generalized Gronwall inequality and its application to a fractional differential equation. J. Math. Anal. Appl. 328(2), 1075-1081 (2007).

[27] Zhou Y., Jiao F., Existence of mild solutions for fractional neutral evolution equations, Comput. Math. Appl., (2010), 59(3), 1063-1077.

[28] Bacaer N. and Guernaoui S., The epidemic threshold of vector-borne diseases with seasonality, Journal of Mathematical Biology, vol. 53, no. 3, pp. 421-436, 2006.

[29] Bacaer, N., and Ouifki, R. (2007). Growth rate and basic reproduction number for population models with a simple periodic factor. Mathematical Biosciences, 210(2), 647-658.

[30] Feng, Z., Huang, W., and Castillo-Chavez, C. (2005). Global behavior of a multi-group SIS epidemic model with age structure. Journal of Differential Equations, 218(2), 292-324

[31] Delavari, H., Baleanu, D., and Sadati, J. Stability analysis of Caputo fractional-order nonlinear systems revisited. Nonlinear Dynamics 64, 4 (2012), 2433-2439

[32] Kumar, A. Stability of a fractional-order epidemic model with nonlinear incidences and treatment rates. Iranian Journal of Science and Technology, Transactions A: Science volume 44, 5 (2020), 1505-1517.

[33] Li, H. L., Zhang, L., Hu, C., Jiang, Y. L., and Teng, Z. Dynamical analysis of a fractional-order predator-prey model incorporating a prey refuge. Journal of Applied Mathematics and Computing 54, 1 (2017), 435-449

[34] Zbair, M., Qaffou, A., Cherkaoui, F. and Hilal, K. (2021, June). Bayesian inference of a discrete fractional seird model. In International Conference on Partial Differential Equations and Applications, Modeling and Simulation (pp. 138-146). Cham: Springer International Publishing.

[35] Cherkaoui, F., Hilal, K. and Qaffou, A. (2023). Analysis of an age structured SIR epidemic model with fractional Caputo derivative. International Journal of Nonlinear Analysis and Applications.

[36] Iannelli, M., Milner, F. A., and Pugliese, A. (1992). Analytical and numerical results for the age-structured SIS epidemic model with mixed inter-intracohort transmission. SIAM journal on mathematical analysis, 23(3), 662-688.

[37] Cha, Y., Iannelli, M. and F. Milner, A., emphAre multiple endemic equilibria possible?, Advances in Mathematical Population Dynamics -Molecules, Cells and Man, O. Arino, D. Axelrod and M. Kimmel (eds.), World Scientiflc, Singapore, (1997), 779-788.

[38] Cha, Y., Iannelli, M. and F. Milner, A., Stability change of an epidemic model, Dynamic Systems and Applications, 9 (2000), 361-376.

[39] Lin, Y. and Xu, C. Finite difference/spectral approximations for the time-fractional diffusion equation. J. Comput. Phys. 225(2), 1533-1552.