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## ON WEAKLY tcc-SUBGROUPS OF FINITE GROUPS

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**ABSTRACT.** The subgroups  $A$  and  $B$  are said to be *cc-permutable*, if  $A$  is permutable with  $B^x$  for some  $x \in \langle A, B \rangle$ . A subgroup  $A$  of a finite group  $G$  is called *weakly tcc-subgroup* (*wtcc-subgroup*, for brevity) in  $G$ , if there exists a subgroup  $Y$  of  $G$  such that  $G = AY$  and  $A$  has a chief series  $1 = A_0 \leq A_1 \leq \dots \leq A_{s-1} \leq A_s = A$  such that every  $A_i$  is cc-permutable with all subgroups of  $Y$  for all  $i = 1, \dots, s$ . In this paper, we studied the influence of given systems of wtcc-subgroups on the structure of a group  $G$ .

**Keywords:** Finite group, cc-permutable subgroups, Sylow subgroups, maximal subgroups, supersoluble group.

### 1. INTRODUCTION

Throughout this paper, all groups are finite and  $G$  always denotes a finite group. We use the standard notations and terminology of [1, 2]. The notation  $H \leq G$  means that  $H$  is a subgroup of a group  $G$ . If  $H \leq G$  and  $H \neq G$ , we write  $H < G$ . The notation  $H \trianglelefteq G$  means that  $H$  is a normal subgroup of a group  $G$ .

We say that the subgroups  $A$  and  $B$  of a group  $G$  are *permutable* if  $AB = BA$ . Note that the equality  $AB = BA$  is equivalent to  $AB \leq G$ . In accordance with [2], subgroups  $A$  and  $B$  of a group  $G$  are called *totally permutable* if every subgroup of  $A$  is permutable with every subgroup of  $B$ .

In [3] the following concept was introduced: the subgroups  $A$  and  $B$  are said to be *cc-permutable*, if  $A$  is permutable with  $B^x$  for some  $x \in \langle A, B \rangle$ . It is clear that if  $A$  and  $B$  are permutable, then  $A$  and  $B$  are cc-permutable. Naturally total permutability was generalized to tcc-permutability, i.e. the subgroups  $A$  and  $B$  of  $G$  are said to be *tcc-permutable* [4] if every subgroup of  $A$  is cc-permutable with every subgroup of  $B$ . Results related to the study of groups with given systems

of cc-permutable or tcc-permutable subgroups were provided in [5], see also the literature in [4].

We follow [6, 7] and collect the following concepts:

(1) A subgroup  $A$  of a group  $G$  is called *tcc-subgroup* in  $G$ , if there exists a subgroup  $T$  of  $G$  such that:

- (1.1)  $G = AT$ ;
- (1.2)  $A$  and  $T$  are tcc-permutable.

(2) A subgroup  $A$  of a group  $G$  is said to be *NS-supplemented* in  $G$ , if there exists a subgroup  $B$  of  $G$  such that:

- (2.1)  $G = AB$ ;
- (2.2) whenever  $X$  is a normal subgroup of  $A$  and  $p \in \pi(B)$ , there exists a Sylow  $p$ -subgroup  $B_p$  of  $B$  such that  $XB_p = B_pX$ .

By [6, Lemma 3.1 (3)], every tcc-subgroup of  $G$  is NS-supplemented in  $G$ , but the converse does not hold. For example, the alternating group  $G = A_4$  has NS-supplemented Sylow 3-subgroup  $P$  and  $P$  is not tcc-subgroup in  $G$ . In [6, 7] proved the supersolubility of a group  $G$  under the condition that factors, Sylow subgroups, maximal subgroups, minimal subgroups, maximal subgroups of every Sylow subgroups of  $G$  are NS-supplemented or tcc-subgroups in  $G$ .

In the present paper, we study another generalization of tcc-subgroup in a new way.

**Definition.**

A subgroup  $A$  of a group  $G$  is called *weakly tcc-subgroup* (*wtcc-subgroup*, for brevity) in  $G$ , if there exists a subgroup  $Y$  of  $G$  such that:

- (1)  $G = AY$ ;
- (2)  $A$  has a chief series

$$1 = A_0 \leq A_1 \leq \dots \leq A_{s-1} \leq A_s = A \quad (1)$$

such that every  $A_i$  is cc-permutable with all subgroups of  $Y$  for all  $i = 1, \dots, s$ .

In this definition, we say that  $Y$  is a *wtcc-supplement* to  $A$  in  $G$  and a chief series (1) is a *wtcc-series* of  $A$ .

It is clear that every tcc-subgroup of  $G$  is wtcc-subgroup in  $G$ , but the converse does not hold. For example, the group  $G = Z_3 \times S_3$  (IdGroup=[18,3], see [8]) has Sylow 3-subgroup  $P$  such that  $P$  is a wtcc-subgroup in  $G$ , but  $P$  is not tcc-subgroup in  $G$ . Besides, the groups  $A_4$  and  $Z_3 \times S_3$  show that there is no inclusion-relationship between the concepts to be wtcc-subgroup and to be NS-supplemented.

In Section 3, we give some properties of wtcc-subgroups. In Section 4, we study the structure of a group  $G$  in which the maximal subgroups, Sylow subgroups, minimal subgroups, 2-maximal subgroups, maximal subgroups of every Sylow subgroup, factors are wtcc-subgroups. Besides, we obtain the generalization of some result of [6] without using the properties of tcc-permutability.

**Theorem.** *Let  $G$  be a group.*

1. Then  $G$  is supersoluble if one of the following statements holds:
  - (1.1) every maximal subgroup of  $G$  is a wtcc-subgroup in  $G$ .
  - (1.2) every Sylow subgroup of  $G$  is a wtcc-subgroup in  $G$ .
  - (1.3) let  $H$  be a proper normal subgroup of  $G$ . Suppose that  $G/H$  is supersoluble and every cyclic subgroup of prime order or order 4 of  $H$  is a wtcc-subgroup in  $G$ .
  - (1.4) every 2-maximal subgroup of  $G$  is a wtcc-subgroup in  $G$ .
  - (1.5) all maximal subgroups of every non-cyclic Sylow subgroup of  $G$  are wtcc-subgroups in  $G$ .

2. Let  $A$  and  $B$  be wtcc-subgroups in a soluble group  $G$  and  $G = AB$ . If  $A$  and  $B$  are supersoluble, then  $G$  is supersoluble.

## 2. PRELIMINARIES

In this section, we give some definitions and basic results which are essential in the sequel.

Denote by  $Z(G)$ ,  $F(G)$  and  $\Phi(G)$  the centre, Fitting and Frattini subgroups of  $G$  respectively, and by  $O_p(G)$  and  $O_{p'}(G)$  the greatest normal  $p$ - and  $p'$ -subgroups of  $G$  respectively. Denote by  $\pi(G)$  the set of all prime divisors of order of  $G$ . We use  $E_{p^t}$  to denote an elementary abelian group of order  $p^t$  and  $Z_m$  to denote a cyclic group of order  $m$ . The semidirect product of a normal subgroup  $A$  and a subgroup  $B$  is written as follows:  $A \rtimes B$ . If  $H$  is a subgroup of  $G$ , then  $H_G = \bigcap_{x \in G} H^x$  is called the core of  $H$  in  $G$ . Recall that  $H^G = \langle H^g \mid g \in G \rangle$  is the smallest normal subgroup of  $G$  containing  $H$ .

A group whose chief factors have prime orders is called *supersoluble*. Recall that a  $p$ -closed group is a group with a normal Sylow  $p$ -subgroup.

**Lemma 1.** ([1, VI.9])

- (1) Every minimal normal subgroup of a supersoluble group has prime order.
- (2) Let  $N$  be a normal subgroup of  $G$  and assume that  $G/N$  is supersoluble. If  $N$  is either cyclic, or  $N \leq Z(G)$ , or  $N \leq \Phi(G)$ , then  $G$  is supersoluble.
- (3) Each supersoluble group has an ordered Sylow tower of supersoluble type.
- (4) The derived subgroup of a supersoluble group is nilpotent.
- (5) A group  $G$  is supersoluble if and only if every maximal subgroup of  $G$  has prime index.
- (6) The class  $\mathfrak{U}$  is a hereditary saturated formation.

**Lemma 2.** ([9, Theorem 1]) Let  $G$  be a group and  $A, B, S \leq G$  such that  $S \leq A \cap B$  and  $AB = BA$ . If  $S$  is subnormal in  $A$  and in  $B$ , then  $S$  is subnormal in  $AB$ .

The class of all supersoluble groups is denoted by  $\mathfrak{U}$ .

A simple check proves the following lemma.

**Lemma 3.** Let  $G$  be a soluble group. Assume that  $G \notin \mathfrak{U}$ , but  $G/K \in \mathfrak{U}$  for every non-trivial normal subgroup  $K$  of  $G$ . Then:

- (1)  $G$  contains a unique minimal normal subgroup  $N$ ,  $N = F(G) = O_p(G) = C_G(N)$  for some  $p \in \pi(G)$ ;
- (2)  $Z(G) = O_{p'}(G) = \Phi(G) = 1$ ;
- (3)  $G$  is primitive;  $G = N \rtimes M$ , where  $M$  is maximal in  $G$  with trivial core;
- (4)  $N$  is an elementary abelian subgroup of order  $p^n$ ,  $n > 1$ .

**Lemma 4.** ([10], [1, p. 721]) Let  $G$  be a minimal non-supersoluble group. Then the following statements hold:

- (1)  $G$  is soluble and  $|\pi(G)| \leq 3$ ;
- (2)  $G$  has a unique normal Sylow subgroup  $P$  and  $P = G^{\mathfrak{U}}$ . Besides,  $P$  has exponent  $p$  if  $p \neq 2$  and exponent at most 4 if  $p = 2$ ;
- (3)  $P/\Phi(P)$  is a minimal normal subgroup of  $G/\Phi(P)$  and  $|P/\Phi(P)| > p$ ;
- (4) if  $Q$  is a complement to  $P$  in  $G$ , then  $Q/Q \cap \Phi(G)$  is a minimal non-abelian group or a cyclic group of prime power order;
- (5) if  $|\pi(Q)| = 2$ , then  $Q$  is a non-cyclic group with cyclic Sylow subgroups;

(6) if  $G$  is not a Schmidt group, then  $G$  possesses an ordered Sylow tower of supersoluble type.

**Lemma 5.** ([11, Theorem 2]) *Let  $G$  be a group with  $p \in \pi(G)$  and  $p \neq 3$ . If  $G$  has a Hall  $\{p, r\}$ -subgroup for every  $r \in \pi(G)$ , then  $G$  is  $p$ -soluble.*

### 3. PROPERTIES OF wtcc-SUBGROUPS

**Lemma 6.** *Let  $A$  be a wtcc-subgroup in  $G$  and  $Y$  be a wtcc-supplement to  $A$  in  $G$  and  $1 = A_0 \leq A_1 \leq \dots \leq A_{s-1} \leq A_s = A$  be a wtcc-series of  $A$ . Then the following statements hold:*

- (1)  $A$  is a wtcc-subgroup in  $H$  for any subgroup  $H$  of  $G$  such that  $A \leq H$ ;
- (2) if  $A$  is supersoluble, then  $AN/N$  is a wtcc-subgroup in  $G/N$  for any  $N \trianglelefteq G$ ;
- (3) for every  $i = 1, \dots, s$  and arbitrary  $X \leq Y$  there exists an element  $y \in Y$  such that  $A_i X^y \leq G$ . In particular,  $A_i M \leq G$  for some maximal subgroup  $M$  of  $Y$  and  $A_i H \leq G$  for some Hall  $\pi$ -subgroup  $H$  of soluble  $Y$  and any  $\pi \subseteq \pi(G)$ ;
- (4)  $A_i K \leq G$  for every  $i = 1, \dots, s$  and every subnormal subgroup  $K$  of  $Y$ ;
- (5) for every  $i = 1, \dots, s$  and every subnormal subgroup  $K$  of  $Y$  we have  $A_i K^g \leq G$  for any  $g \in G$ ;
- (6) if  $Y$  is soluble and  $A$  is  $r$ -closed, then Sylow  $r$ -subgroup  $A_r$  of  $A$  is subnormal in  $G$ , where  $r$  is the greatest prime in  $\pi(G)$ .

*Proof 1.* Since  $Y$  is a wtcc-supplement to  $A$  in  $G$ , it follows that  $G = AY$ . By Dedekind's identity,  $H = H \cap AY = A(H \cap Y)$ . Since  $H \cap Y \leq Y$ , then for any  $i = 1, \dots, s$  and any  $Z \leq H \cap Y$  there exists an element  $u \in \langle A_i, Z \rangle$  such that  $A_i Z^u \leq G$ . Hence  $A$  is a wtcc-subgroup in  $H$ .

2. Since  $G = AY$ , it follows that  $G/N = (AN/N)(YN/N)$ . Let  $X/N$  be an arbitrary subgroup of  $YN/N$ . Since  $N \leq X \leq YN$ , we have by Dedekind's identity,  $X = X \cap YN = (X \cap Y)N$ .

Consider the series

$$1 = A_0 N/N \leq A_1 N/N \leq \dots \leq A_{s-1} N/N \leq A_s N/N = AN/N \quad (2)$$

of  $AN/N$ . Since  $A_i$  is normal in  $A$ , it follows that  $A \leq N_G(A_i N)$  and  $A_i N/N$  is normal in  $AN/N$ . Obviously,

$$(A_i N/N)/(A_{i-1} N/N) \simeq A_i N/A_{i-1} N \simeq A_i/A_i \cap A_{i-1} N \simeq A_i/A_{i-1}(A_i \cap N).$$

Since  $A$  is supersoluble,  $|A_i/A_{i-1}|$  is a prime and therefore  $|(A_i N/N)/(A_{i-1} N/N)|$  is a prime. Hence the series (2) is a chief series of  $AN/N$ .

Because  $X \cap Y$  is a group of  $Y$ , we have for any  $i = 1, \dots, s$  there exists an element  $u \in \langle A_i, X \cap Y \rangle$  such that  $A_i (X \cap Y)^u \leq G$ . Hence

$$(A_i N/N)(X/N)^{uN} = A_i (X \cap Y)^u N/N \leq G/N$$

for  $uN \in \langle A_i, X \cap Y \rangle N/N \subseteq \langle A_i, X \rangle N/N = \langle A_i N/N, X/N \rangle$ . Consequently,  $AN/N$  is a wtcc-subgroup in  $G/N$ .

3. Since  $A$  is a wtcc-subgroup in  $G$ , for every  $i = 1, \dots, s$  and  $X \leq Y$  there exists an element  $u \in \langle A_i, X \rangle$  such that  $A_i X^u \leq G$ . Because  $u \in G = AY = YA$ , it follows that  $u = ya$  for some  $y \in Y$  and  $a \in A$ . Then

$$A_i X^u = A_i X^{ya} = A_i (X^y)^a = A_i^a (X^y)^a = (A_i X^y)^a \leq G.$$

Hence there is a subgroup  $A_i X^y$  in  $G$  for some  $y \in Y$ . Clearly, that if  $X$  is a Hall  $\pi$ -subgroup of  $Y$ , then  $H = X^y$  is a Hall  $\pi$ -subgroup of  $Y$ . Thus  $A_i H \leq G$ . Similarly,

for maximal subgroup  $X$  of  $Y$ . Then  $M = X^y$  is a maximal subgroup of  $Y$  and  $A_i M \leq G$ .

4. Since  $K$  is subnormal in  $Y$ , there is a chain of subgroups

$$Y = K_0 \geq K_1 \geq \dots \geq K_{n-1} \geq K_n = K$$

such that  $K_{i+1}$  is normal in  $K_i$  for all  $i$ . We use induction by  $n$ . By (3), there exists an element  $y \in Y$  such that  $A_j K_1^y = A_j K_1 \leq G$  for every  $j = 1, \dots, s$ . Hence the statement holds for  $n = 0$  and  $n = 1$ . Therefore  $n \geq 2$ . By (1),  $A$  is a wtcc-subgroup in  $AK_1$  and  $K_1$  is a wtcc-supplement to  $A$  in  $AK_1$ . Since the length of subnormal chain between  $K$  and  $K_1$  less than  $n$ , it follows that by induction, there is a subgroup  $A_i K$  of  $AK_1$ . Consequently  $A_i K \leq G$ .

5. Since  $g \in G = AY = YA$ , it follows that  $g = ya$  for some  $y \in Y$  and  $a \in A$ . Then

$$A_i K^g = A_i K^{ya} = A_i (K^y)^a = (A_i K^y)^a.$$

Since  $K$  is subnormal in  $Y$ , we have  $K^y$  is subnormal in  $Y$ . By (4),  $A_i K^y \leq G$ . Therefore,  $AK^g \leq G$ .

6. We proceed by induction on  $|G|$ . By (3),  $AY_1 \leq G$  for some Hall  $r'$ -subgroup  $Y_1$  of  $Y$ . If  $AY_1 < G$ , then by (1),  $A$  is a wtcc-subgroup in  $AY_1$  and by induction,  $A_r$  is subnormal in  $AY_1$ . Besides,  $A_r$  is subnormal in some Sylow  $r$ -subgroup  $G_r$  of  $G$ . Let  $Y_r \leq R$ , where  $R$  is a Sylow  $r$ -subgroup of  $G$  and  $R^g = G_r$  for some  $g \in G$ . By Lemma 2,  $A_r$  is subnormal in  $G = AY = AY_1 Y_r = (AY_1) Y_r^g = (AY_1) G_r$ .

Hence we consider that  $G = AY_1$ . By (3),  $AQ \leq G$  for some Sylow  $q$ -subgroup  $Q$  of  $Y_1$ . If  $AQ < G$ , then  $A$  is a wtcc-subgroup in  $AQ$  and by induction,  $A_r$  is subnormal in  $AQ$ . Therefore  $A_r$  is normal in  $AQ$  and  $Q \leq N_G(A_r)$ . Since it is true for any  $q \in \pi(Y_1)$ , it follows that  $A_r$  is normal in  $G = AY_1$ .

Hence  $G = AQ$ . By (4),  $Q$  is a minimal wtcc-supplement to  $A$  in  $G$ . By (3),  $AM < G$  for some maximal subgroup  $M$  of  $Q$ . Because  $A$  is a wtcc-subgroup in  $AM$ , we have by induction,  $A_r$  is subnormal in  $AM$  and hence  $A_r$  is normal in  $AM$ . Since  $|G : AM| = q$ , it follows that  $G/(AM)_G$  is isomorphic to a subgroup of symmetric group  $S_q$ . Hence  $G_r \leq (AM)_G \leq AM$  and  $A_r = G_r$  is subnormal in  $G$ . Lemma is proved.

#### 4. MAIN RESULTS

(1.1) Let  $M$  be an arbitrary maximal subgroup of  $G$ . By Lemma 6 (3),  $MY_p \leq G$  for some Sylow  $p$ -subgroup  $Y_p$  of  $Y$ . Since  $M$  is maximal in  $G$ , it follows that either  $MY_p = M$ , or  $MY_p = G$ . If  $MY_p = M$  for all  $p \in \pi(Y)$ , then  $Y \leq M$  and  $G = MY = M$ , a contradiction. Therefore there exists  $q \in \pi(Y)$  such that  $MY_q = G$  and  $Y_q$  is a wtcc-supplement to  $M$  in  $G$ . By Lemma 6 (4), we can consider that  $Y_q$  is a minimal wtcc-supplement to  $M$  in  $G$ . By Lemma 6 (3),  $MS < G$  and  $|G : MS| = q$  for some maximal subgroup  $S$  of  $Y_q$ . Since  $M$  is a maximal subgroup of  $G$ , we have  $|G : M| = |G : MS| = q$ . By Lemma 1 (6),  $G$  is supersoluble.

(1.2) We show that  $G$  is soluble. Let  $R$  be a Sylow  $r$ -subgroup of  $G$ . Then  $R$  is a wtcc-subgroup in  $G$ . Let  $T$  be a wtcc-supplement to  $R$  in  $G$ . By Lemma 6 (3),  $RQ \leq G$  for some Sylow  $q$ -subgroup  $Q$  of  $T$  and for any  $q \in \pi(T) \setminus \{p\}$ . The subgroup  $RQ$  is a Hall  $\{r, q\}$ -subgroup of  $G$ . By Lemma 5,  $G$  is  $r$ -soluble for  $r \neq 3$ . Let  $t$  be the smallest prime in  $\pi(G)$ . If  $t > 2$ , then  $G$  is soluble. If  $t = 2$ , then by the above,  $G$  is  $t$ -soluble and consequently,  $G$  is soluble.

Next we show that  $G$  is supersoluble. Assume that the claim is false and let  $G$  be a minimal counterexample. Let  $N$  be a non-trivial normal subgroup of  $G$  and  $RN/N$  be a Sylow  $r$ -subgroup of  $G/N$ . By Lemma 6 (2),  $RN/N$  is a wtcc-subgroup in  $G/N$ . Then  $G/N$  is supersoluble by the choice of  $G$ .

Let  $P$  be a Sylow  $p$ -subgroup of  $G$ , where  $p$  is the greatest prime in  $\pi(G)$ . By Lemma 6 (6),  $P$  is subnormal in  $G$  and consequently,  $P$  is normal in  $G$ . By Lemma 3,  $G$  has a unique minimal normal subgroup  $N$  such that  $N = C_G(N) = O_p(G) = F(G) = P$  and  $N$  is an elementary abelian subgroup of order  $p^n$ ,  $n > 1$ .

Let  $T$  be a wtcc-supplement to  $P$  in  $G$  and

$$1 = P_0 \leq P_1 \leq \dots \leq P_{s-1} \leq P_s = P-$$

be a wtcc-series of  $P$ . It is clear  $|P_1| = p$ . By Lemma 6 (3), for every  $r \in \pi(T)$  there exists a Sylow  $r$ -subgroup  $R$  of  $T$  such that  $P_1R \leq G$ . If  $p \neq r$ , then

$$P \cap P_1R = P_1(P \cap R) = P_1$$

is normal in  $P_1R$  and  $R \leq N_G(P_1)$ . Since this inclusion holds for any  $r \in \pi(T) \setminus \{p\}$ , we have  $T_1 \leq N_G(P_1)$  for some Hall  $p'$ -subgroup  $T_1$  of  $T$ . Hence  $P_1$  is normal in  $G = PT = PT_pT_1 = PT_1$ , a contradiction.

(1.3) Assume that the theorem is false and let  $G$  be a minimal counterexample. Let  $K$  be a proper subgroup of  $G$ . It clear that  $K \cap H \trianglelefteq K$  and  $K/K \cap H \simeq KH/H$  is supersoluble. By Lemma 6 (1), every cyclic subgroup of prime order or order 4 of  $K \cap H$  is a wtcc-subgroup in  $K$ . Then by induction,  $K$  is supersoluble and hence  $G$  is a minimal non-supersoluble group. Besides,  $H$  is also supersoluble. Let  $q$  be the greatest prime in  $\pi(H)$ . Then by Lemma 1, a Sylow  $q$ -subgroup  $Q$  of  $H$  is normal in  $H$  and consequently,  $Q$  is normal in  $G$ . Let  $\overline{Q_1} = Q_1/Q$  be a cyclic subgroup of prime order or order 4 of  $H/Q$ . Then

$$\overline{Q_1} = \langle xQ \rangle = \langle x \rangle Q/Q \simeq \langle x \rangle / \langle x \rangle \cap Q \simeq \langle x \rangle,$$

because  $x \notin Q$ . Since  $x \in H$ , by Lemma 6 (2),  $\overline{Q_1}$  is a wtcc-subgroup in  $G/Q$ . Since  $(G/Q)/(H/Q)$  is supersoluble, by induction,  $G/Q$  is supersoluble.

By Lemma 4,  $G$  is soluble,  $G$  has a unique normal Sylow  $p$ -subgroup  $P$  and  $P = G^\Omega$ ,  $\overline{P} = P/\Phi(P)$  is a minimal normal subgroup of  $\overline{G} = G/\Phi(P)$  and  $|P/\Phi(P)| > p$ . Besides,  $P$  has exponent  $p$  if  $p \neq 2$  and exponent at most 4 if  $p = 2$ .

If  $p \neq q$ , then  $G \simeq G/Q \cap P$  is supersoluble by Lemma 1, because  $G/Q$  and  $G/P$  are supersoluble. So  $p = q$  and  $Q \leq P$ . Since  $Q\Phi(P)/\Phi(P) \leq \overline{P}$  and  $\overline{P}$  is a minimal normal subgroup of  $\overline{G}$ , we have  $Q \leq \Phi(P)$  or  $Q\Phi(P) = P$ . If  $Q \leq \Phi(P)$ , then  $G$  is supersoluble by Lemma 1, because  $Q \leq \Phi(G)$  and  $G/Q$  is supersoluble, a contradiction. If  $Q\Phi(P) = P$ , then  $Q = P$ .

Suppose that  $p = 2$ . Let  $x \in P$  and  $P_1 = \langle x \rangle$ . Then  $|P_1| = 2$  or  $|P_1| = 4$ . By the hypothesis,  $P_1$  is a wtcc-subgroup in  $G$ . By Lemma 6 (3),  $G$  has a Hall  $2'$ -subgroup  $S$  such that  $P_1S \leq G$ . By [1, IV.2.8],  $P_1 \leq N_G(S)$  and consequently,  $P \leq N_G(S)$  and  $S$  is normal in  $G$ , a contradiction.

Assume that  $p > 2$ . Let  $\overline{K} = K/\Phi(P)$  be a subgroup of order  $p$  in  $\overline{P}$ . Then

$$\overline{K} = \langle x\Phi(P) \rangle = \langle x \rangle \Phi(P) / \Phi(P).$$

Since  $x \in P$ , it follows that  $|\langle x \rangle| = p$  and hence by Lemma 6 (2),  $\overline{K}$  is a wtcc-subgroup in  $\overline{G}$  and  $\overline{T} = T/\Phi(P)$  is a wtcc-supplement to  $\overline{K}$  in  $\overline{G}$ . Hence by Lemma 6 (3), for every  $r \in \pi(\overline{T})$ ,  $r \neq p$  there exists a Sylow  $r$ -subgroup  $\overline{R}$  of  $\overline{T}$

such that  $\overline{KR} \leq \overline{G}$ . It is clear that  $\overline{R}$  is a Sylow  $r$ -subgroup in  $\overline{G}$ . We have that

$$\overline{P} \cap \overline{KR} = \overline{K}(\overline{P} \cap \overline{R}) = \overline{K}$$

is normal in  $\overline{KR}$  and  $\overline{R} \leq N_{\overline{G}}(\overline{K})$ . Since  $\overline{P}$  is abelian,  $\overline{K}$  is normal in  $\overline{G}$ . Therefore  $\overline{K} = \overline{P}$ , a contradiction.

(1.4) Assume that the claim is false and let  $G$  be a minimal counterexample. By Lemma 6(1) and by (1.1), every maximal subgroup  $M$  of  $G$  is supersoluble. Hence  $G$  is a minimal non-supersoluble group and we can apply Lemma 4. Then  $G$  is soluble,  $|\pi(G)| \leq 3$  and  $G$  has a unique normal subgroup  $P = G^{\text{ul}}$ . It is clear that  $\Phi(G) = 1$ . Hence  $P$  is a minimal normal subgroup of order  $p^n$ ,  $n > 1$  and  $G = P \rtimes M$  for some maximal subgroup  $M$  of  $G$ .

If  $|\pi(G)| = 3$ , then  $G$  has an ordered Sylow tower of supersoluble type and  $M = T \rtimes R$ , where  $|T| = t$ ,  $|R| = r$  and  $t, r \in \pi(G)$ . The subgroups  $T$  and  $R$  are 2-maximal subgroups of  $G$ . Then by hypothesis,  $TY_1 = G = RY_2$ , where  $Y_1$  and  $Y_2$  are wtcc-supplements in  $G$ . Besides,  $P \leq Y_1$  and  $P \leq Y_2$ . Let  $P_1$  be a minimal normal subgroup of  $P$ . Then by Lemma 6(4),  $T \leq N_G(P_1)$  and  $R \leq N_G(P_1)$ . Then  $P_1$  is normal in  $G = PM = PTR$ , a contradiction.

So,  $|\pi(G)| = 2$ . Then  $M$  is a  $q$ -subgroup. If  $|M| > q$ , then  $M$  has a maximal subgroup  $M_1$  such that  $M_1 \neq 1$ . It is clear that  $H = P \rtimes M_1$  is a maximal subgroup of  $G$ . Since  $H$  is supersoluble, it follows that  $H$  has a maximal subgroup  $H_1$  such that  $M_1 \leq H_1$  and  $|H : H_1| = p$ . By hypothesis,  $H_1$  is a wtcc-subgroup in  $G$ . Then  $H_1V = G$ , where  $V$  is a wtcc-supplement to  $H_1$  in  $G$ . Let

$$1 = K_0 \leq K_1 \leq \dots \leq K_{s-1} \leq K_s = H_1 -$$

be a wtcc-series of  $H_1$ . Since  $G$  is  $p$ -closed,  $H_1$  is  $p$ -closed and  $|K_1| = p$ . By Lemma 6(3),  $V$  has a Sylow  $q$ -subgroup  $V_q$  such that  $K_1V_q \leq G$ . Hence  $V_q \leq N_G(K_1)$  and  $K_1$  is normal in  $G = H_1V = H_1PV_q$ , a contradiction.

Therefore,  $|M| = q$  and  $P$  is a maximal subgroup of  $G$ . Let  $P_1$  be a maximal subgroup of  $P$ . Then by hypothesis,  $P_1K = G$ , where  $K$  is a wtcc-supplement to  $P_1$  in  $G$ . By Lemma 6(3),  $K$  has a Sylow  $q$ -subgroup  $K_1$  such that  $P_1K_1 \leq G$  and  $K_1 \leq N_G(P_1)$ . Hence  $P_1$  is normal in  $G = P_1K = PK_1$ . Hence  $|P| = p$ , a contradiction.

(1.5) Let  $P$  be a Sylow  $p$ -subgroup of  $G$ . If  $P$  is cyclic, then  $G$  is  $p$ -supersoluble. Let  $P$  be non-cyclic. Then by Lemma 6(3), for every maximal subgroup  $P_i$  of  $P$  and every  $q \in \pi(G) \setminus \{p\}$  there exists a Sylow  $q$ -subgroup  $Q$  of  $G$  such that  $P_iQ \leq G$ . By [7, Theorem 3.4],  $G$  is  $p$ -supersoluble. Since it is true for any  $p \in \pi(G)$ , we have  $G$  is supersoluble.

2. Assume that the claim is false and let  $G$  be a minimal counterexample. Let  $N$  be a non-trivial normal subgroup of  $G$ . The subgroups  $AN/N \simeq A/A \cap N$  and  $BN/N \simeq B/B \cap N$  are wtcc-subgroups in  $G/N$  by Lemma 6(2),  $AN/N \simeq A/A \cap N$  and  $BN/N \simeq B/B \cap N$  are supersoluble. Hence  $G/N = (AN/N)(BN/N)$  is supersoluble by induction.

Since by hypothesis,  $A$  and  $B$  are supersoluble wtcc-subgroups of soluble group  $G$ , by Lemma 6(6),  $A_p$  and  $B_p$  are subnormal in  $G$  for the greatest prime  $p \in \pi(G)$ . Because  $P = A_pB_p$  is a Sylow  $p$ -subgroup of  $G$ , we have  $G$  is  $p$ -closed. By Lemma 3,  $G$  has a unique minimal normal subgroup  $N$  such that  $N = C_G(N) = O_p(G) = F(G) = P$  and  $N$  is an elementary abelian subgroup of order  $p^n$ ,  $n > 1$ .

Without loss of generality, we assume that  $A_p \neq 1$ . Let

$$1 = A_0 \leq A_1 \leq \dots \leq A_{s-1} \leq A_s = A$$

be a wtcc-series of  $A$ . Since  $A$  is  $p$ -closed,  $|A_1| = p$ . By Lemma 6(3),  $A_1 Y_1 \leq G$  for some Hall  $p'$ -subgroup  $Y_1$  of  $Y$ . Then

$$A_1 = P \cap A_1 Y_1 = A_1 (P \cap Y_1)$$

is normal in  $A_1 Y_1$ . Hence  $Y_1 \leq N_G(A_1)$ . Since  $P$  is abelian, a Sylow  $p$ -subgroup  $Y_p$  of  $Y$  centralizes  $A_1$  and  $A_1$  is normal in  $G = A Y_p Y_1$ , a contradiction.

The theorem is proved.

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