AMGROUPS

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Abstract. We redefine the concept of multigroups (mgroups, for short) using the definition of multigroups introduced by Nazmul et al., [8] to allow the flexibility of the identity element from a group X in delineating the mgroups and prove some related results.

1. INTRODUCTION

The theory of multisets is an extension of the set theory. Since inception, it has evoked a lot of research. For more details, the reader is referred to ([2],[3],[4],[5],[6],[7],[13],[14]). Theoretic study has included algebra aspect of fuzzy sets and multisets. Rosenfeld [1] iniated the theory of fuzzy groups in order to estabish the algebraic structures of fuzzy sets. Onasanya [11, 12] critically studied the work of Rosenfeld and carried out some reviews on fuzzy groups and anti fuzzy groups thereby proving some of the known results differently.

Nazmul et al., [8] replaced the underlying structure in group theory with multisets and some fundamental properties were presented. As a suitable generalization of group theory, Awolola and Ibrahim [9], Awolola and Ejegwa [10] discussed the concept further and investigated some properties. In this paper, we introduce a new concept of mgroups called amgroups ($EMGs$) by redefining the concept of mgroups from a multiset space $[X]^\infty$ and obtain some related results.

2. PRELIMINARIES

Definition 2.1. Let X be a set. A multiset (mset, for short) M drawn from X is represented by a count function C_M defined as $C_M : X \longrightarrow N_0 = \{0, 1, 2, ...\}$.

For each $x \in X$, $C_M(x)$ denotes the number of occurrences of the element x in the mset M. The representation of the mset M drawn from $X = \{x_1, x_2, ..., x_n\}$ will be as $M = [x_1, x_2, ..., x_n]_{m_1,m_2,...,m_n}$ such that x_i appears m_i times $(i = 1, 2, , n)$ in M.

Let for any positive integer n, $[X]^n$ be the set of all msets drawn from X such that no element in the mset occurs more than n times and $[X]^\infty$ be the set of all

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msets drawn from X such that there is no limit on the number of occurrences of an object in an mset. Therefore, $[X]^n$ and $[X]^\infty$ are referred to as mset spaces.

Definition 2.2. Let $M_1, M_2, M_i \in [X]^n, i \in I$. Then

(i) $M_1 \subseteq M_2 \Longleftrightarrow C_{M_1}(x) \leqslant C_{M_2}(x),\ \forall\ x \in X.$
(ii) $M_1 = M_2 \Longleftrightarrow C_{M_1}(x) = C_{M_2}(x),\ \forall x \in X.$
(iii) $\bigcap_{i\in I} M_i = \bigwedge_{i\in I} C_{M_i}(x),\ \forall\ x \in X$ (where $\bigwedge$ is the minimum operation).
(iv) $\bigcup_{i\in I} M_i = \bigvee_{i\in I} C_{M_i}(x),\ \forall\ x \in X$ (where $\bigvee$ is the maximum operation).
(v) $M_i^c = n - C_{M_i}(x),\ \forall\ x \in X,\ n \in \mathbb{Z}^+.$

Definition 2.3. Let X be a group. A multiset A over X is called *amgroup* if the count function of the elements of A or $C_A(x)$ satisfies the following conditions:

(i) $C_A(xy) \leqslant C_A(x) \vee C_A(y),\ \forall\ x, y \in X,$
(ii) $C_A(x^{-1}) = C_A(x),\ \forall\ x \in X.$

Example 2.1. Let $E = \langle a_1 | a_1^2 = 1\rangle \times \langle a_2 | a_2^2 = 1\rangle \times ...$ be an infinite elementary abelian 2-group, $\mu \in MG(E)^\infty$ an mgroup. Then in fact $\mu \in MG(E)^{\mu(1)}$, so μ is constant on some infinite $F \subseteq E$. On the other hand, setting $E_0 = \emptyset,\ E_i = \langle a_1\rangle \times ... \times \langle a_i\rangle,\ \mu(a) = i + 1$ provided $a \in E_{i+1}\backslash E_i$ then $\mu \in EMG(E)^\infty$ is an amgroup with $\mu \notin MG(E)^k$ for any positive integer K. Hence, amgroups may provide valuable new tools in infinite group theory.

Definition 2.4. Let $A, B \in [X]^n$, we have the following definitions:

(i) $C_{A\circ B}(x) = \bigwedge\{C_A(y) \bigvee C_B(z) : y, z \in X,\ yz = x\},$
(ii) $C_{A^{-1}}(x) = C_A(x^{-1}).$

We call $A \circ B$ the product of A and B and A^{-1} the inverse of A.

Definition 2.5. Let $A, B \in EMG(X)$. Then A is said to be a subamgroup of B if $A \subseteq B$.

Example 2.2. Let $X = \langle a, b | a^2 = b^2 = 1, ba = ab\rangle$, $A = [1, a, b, ab]_{1,2,4,4}$ and $B = [1, a, b, ab]_{2,3,4,4}$. Clearly, $A, B \in EMG(X)$ and $A \subseteq B$. Thus, A is a subamgroup of B.

Definition 2.6. Let $A \in EMG(X)$. Then A is called an abelian amgroup over X if $C_A(xy) = C_A(yx),\ \forall\ x, y \in X$. The set of all abelian amgroups over X is denoted by $AEMG(X)$.

3. MAIN RESULTS

Proposition 3.1. *Let $A \in EMG(X)$.*

(i) $C_A(x^n) \leqslant C_A(x),\ \forall\ x \in X.$
(ii) *If $C_A(x^{-1}) \leqslant C_A(x)$, then $C_A(x^{-1}) = C_A(x)$.*
(iii) *If $C_A(x) < C_A(y)$, for some $x, y \in X$, then $C_A(xy) = C_A(y) = C_A(yx)$.*
(iv) *$C_A(xy^{-1}) = C_A(e)$ implies $C_A(x) = C_A(y)$.*

Proof. (i)-(ii) immediate
(iii) Given that $C_A(x) < C_A(y)$ for some $x, y \in X$. Since $A \in EMG(X)$, then $C_A(xy) \leqslant C_A(x) \bigvee C_A(y) = C_A(y)$.
Now, $C_A(y) = C_A(xyx^{-1}) \leqslant C_A(xy) \bigvee C_A(x) = C_A(xy)$, since $C_A(x) < C_A(y)$, $C_A(x) < C_A(xy)$
Therefore, $C_A(xy) = C_A(y)$.
Similarly, $C_A(yx) = C_A(y)$.
Hence, the proof.

(iv) Given $A \in EMG(X)$ and $C_A(xy^{-1}) = C_A(e)\ \forall\ x, y \in X$. Then

$$\begin{aligned}
C_A(x) = C_A(x(y^{-})y)) &= C_A((xy^{-1})y) \\
&\leqslant C_A(xy^{-1}) \bigvee C_A(y) = C_A(e) \bigvee C_A(y) = C_A(y) \\
\Longrightarrow C_A(x) \leqslant C_A(y) & \\
Now, C_A(y) = C_A(y^{-1}) = C_A(ey^{-1}) &= C_A((x^{-1}x)y^{-1}) \\
&\leqslant C_A(x^{-1}) \bigvee C_A(xy^{-1}) = C_A(x) \bigvee C_A(e) = C_A(x) \\
\Longrightarrow C_A(y) \leqslant C_A(x) &
\end{aligned}$$

Hence, $C_A(x) = C_A(y)$.

□

Proposition 3.2. *Let $A \in [X]^n$. Then $A \in EMG(X)$ if and only if $C_A(xy^{-1}) \leqslant C_A(x) \bigvee C_A(y),\ \forall\ x, y \in X$.*

Proof. Let $A \in EMG(X)$. Then
$C_A(xy^{-1}) \leqslant C_A(x) \bigvee C_A(y^{-1}) = C_A(x) \bigvee C_A(y),\ \forall\ x, y \in X$
Conversely, let the given condition be satisfied.
i.e., $C_A(xy^{-1}) \leqslant C_A(x) \bigvee C_A(y)$
Now, $C_A(e) = C_A(xx^{-1}) \leqslant C_A(x) \bigvee C_A(x) = C_A(x)$
Also, $C_A(x^{-1}) = C_A(ex^{-1}) \leqslant C_A(e) \bigvee C_A(x) = C_A(x)$
Now, $C_A(xy) = C_A(x(y^{-1})^{-1}) \leqslant C_A(x) \bigvee C_A(y^{-1}) = C_A(x) \bigvee C_A(y)$
Hence, the proof. □

Proposition 3.3. *Let $A \in [X]^n$. Then $A \in EMG(X)$ if and only if $A \leqslant A \circ A$ and $A^{-1} = A$.*

Proof. Let $x, y \in X$. Since $A \in EMG(X)$, then $C_A(xy) \leqslant C_A(x) \bigvee C_A(y)$.
$\Longrightarrow C_{A \circ A}(z) = \bigwedge_{z=xy} \{C_A(x) \bigvee C_A(y)\} \geqslant \bigwedge_{z=xy} C_A(xy) = C_A(z)$
Therefore, $A \leqslant A \circ A$.
On the other hand, $A \in EMG(X) \Longrightarrow C_A(x^{-1}) = C_A(x),\ \forall x \in X$.
But by definition, $C_A(x^{-1}) = C_{A^{-1}}(x)$. Therefore, $A^{-1} = A$.
Conversely, let the given conditions be satisfied. If $A = A \circ A$ and $A^{-1} = A$, then it is sufficient to prove $A \in EMG(X)$.
Now, $C_{A \circ A}(z) = \bigwedge_{z=xy} \{C_A(x) \bigvee C_A(y)\} \leqslant C_A(x) \bigvee C_A(y),\ \forall\ x, y \in X$

$\Longrightarrow C_A(xy) \leqslant C_A(x) \bigvee C_A(y),\ xy = z$
Since $C_A(x) = C_{A^{-1}}(x)$ and $C_{A^{-1}}(x) = C_A(x^{-1})$, then it follows that $C_A(x^{-1}) = C_A(x),\ \forall\ x \in X$.
Therefore, $A \in EMG(X)$. □

Proposition 3.4. *Let $A, B \in EMG(X)$. Then $A \cup B \in EMG(X)$.*

Proof. Let $x, y \in A \cup B \in EMG(X)$.
$\Longrightarrow x, y \in A$ or $x, y \in B$
$\Longrightarrow C_A(xy) \leqslant C_A(x) \vee C_A(y)$ or $C_B(xy) \leqslant C_B(x) \vee C_B(y)$
Now,

$$\begin{aligned} C_{A\cup B}(xy) = C_A(xy) \bigvee C_B(xy) &\leqslant \Big[C_A(x) \bigvee C_A(y)\Big] \bigvee \Big[C_B(x) \bigvee C_B(y)\Big] \\ &= \Big[C_A(x) \bigvee C_B(x)\Big] \bigvee \Big[C_A(y) \bigvee C_B(y)\Big] \\ &= C_{A\cup B}(x) \bigvee C_{A\cup B}(y) \end{aligned}$$

and $C_{A\cup B}(x^{-1}) = C_A(x^{-1}) \bigvee C_B(x^{-1}) = C_A(x) \bigvee C_B(x) = C_{A\cup B}(x)$
Therefore, $A \cup B \in EMG(X)$. □

Remark 3.1. If $\{A_i\}_{i\in I}$ is a family of amgroups, then $\bigcap_{i\in I} A_i$ need not be an amgroup over X.

Remark 3.2. If $A \in EMG(X)$, then A^c need not be an $EMG(X)$. However, $A^c \in EMG(X)$ if and only if $C_A(x) = C_A(e),\ \forall x \in X$.

Proposition 3.5. *Let $A \in EMG(X)$ and $x \in X$. Then $C_A(xy) = C_A(y)\ \forall\ y \in X$ if and only if $C_A(x) = C_A(e)$.*

Proof. If $C_A(xy) = C_A(y)\ \forall\ y \in X$, then $y = e$.
Conversely, assume $C_A(x) = C_A(e)$. Then $C_A(xy) \leqslant C_A(x) \bigvee C_A(y) = C_A(y)$, and, on the other hand, $C_A(y) \leqslant C_A(x^{-1}) \bigvee C_A(xy) = C_A(xy)$ □

Proposition 3.6. *Let $A \in EMG(X)$. Then the non-empty sets defined as $A^n = \{x \in X : C_A(x) \leqslant n,\ n \in \mathbb{Z}^+\}$ and $A_* = \{x \in X : C_A(x) = C_A(e)\}$ are subgroups of X.*

Proof. Let $x, y \in A^n$. It implies that $C_A(x) \leqslant n$ and $C_A(y) \leqslant n$. Then
$C_A(xy^{-1}) \leqslant [C_A(x) \bigvee C_A(y)] \leqslant n$
$\Longrightarrow$ if $x, y \in A^n$, then $xy^{-1} \in A^n$.
Hence, $A^n,\ n \in \mathbb{Z}^+$ are subgroups of X.
Again, let $x, y \in A_*$. Then $C_A(x) = C_A(y) = C_A(e)$
Now, $C_A(xy^{-1}) \leqslant [C_A(x) \bigvee C_A(y)] = [C_A(e) \bigvee C_A(e)] = C_A(e)$
But $C_A(e) \leqslant C_A(xy^{-1})$ i.e., $C_A(xy^{-1}) = C_A(e)$
$\Longrightarrow xy^{-1} \in A_*$. Hence, A_* is a subgroup of X. □

Proposition 3.7. *Let $A \in EMG(X)$. Then the following assertions are equivalent:*

(i) $C_A(xy) = C_A(yx),\ \forall\ x, y \in X$,

(ii) $C_A(xyx^{-1}) = C_A(y), \ \forall \ x, y \in X,$
(iii) $C_A(xyx^{-1}) \leqslant C_A(y), \ \forall \ x, y \in X,$
(iv) $C_A(xyx^{-1}) \geqslant C_A(y), \ \forall \ x, y \in X.$

Proof. $(i) \Longrightarrow (ii)$ Let $x, y \in X$. Then $C_A(x^{-1}xy) = C_A(ey) = C_A(y)$.
$(ii) \Longrightarrow (iii)$ Trivial.
$(iii) \Longrightarrow (iv)$ $C_A(xyx^{-1}) \geqslant C_A\left(x^{-1}\left[xyx^{-1}\right](x^{-1})^{-1}\right) = C_A(y)$.
$(iv) \Longrightarrow (i)$ Let $x, y \in X$. Then $C_A(xy) = C_A(x\,[yx]\,x^{-1}) \geqslant C_A(yx) = C_A(y\,[xy]\,y^{-1}) \geqslant C_A(xy)$. Hence, $C_A(xy) = C_A(yx)$. Thus the above assertions are equivalent. □

Proposition 3.8. *Let $A \in AEMG(X)$. Then A_*, A^n, $n \in \mathbb{Z}^+$ are normal subgroups of X.*

Proof. (i) Assume $C_A(e) = 1$. Then $A_* = A^1$. Hence, it is not difficulty to see that A_* is a normal subgroup of X.
(ii) Let $x \in X$ and $y \in A^n$, then $C_A(y) \leqslant n$. Since $A \in AEMG(X)$, then $C_A(xy) = C_A(yx) \ \forall \ x, y \in X$. By proposition 3.7, $C_A(xyx^{-1}) = C_A(y)$ and this implies $C_A(xyx^{-1}) = C_A(y) \leqslant n$. Thus, $xyx^{-1} \in A^n$ is a normal subgroup of X. □

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