

ANALYSIS OF VECTOR PRODUCTS AND COMMUTATIVE ALGEBRAS OF HYPERCOMPLEX NUMBERS

A.T. Ibrayev

February 2023

UDK 511.84, 512.642

MSC 11R52, 15A03

Abstract. An analysis of the content of scientific papers and books devoted to theoretical problems of the algebra of hypercomplex numbers shows that the construction of a commutative algebra of quaternions with division over the field of real numbers is considered impossible. There is practically no doubt about the existing theoretical statement that, unlike real and complex numbers, quaternions and other hypercomplex numbers cannot have the commutativity property. In this paper, we prove that the commutativity of the orthogonal product of two two-dimensional vectors and the commutativity of the quaternion algebra can be ensured by taking into account the directions of the orthogonal and collinear components of one vector with respect to the linear direction of another vector.

Keywords: hypercomplex number, quaternion, algebra, multiplication, division, commutativity, vector.

1 Introduction

After the creation of the algebra of complex numbers and the successful application of functions of complex variables to solve a number of theoretical problems, many great mathematicians of the world spent a lot of effort on solving the problem of constructing, first of all, an algebra of three-dimensional variables in order to subsequently construct algebras of other multidimensional variables. Therefore, the quaternions proposed by Hamilton in 1843 gave rise to the rapid development of vector algebra and a number

of other important sections of modern mathematics, which are an effective basis for building the fundamental foundations of theories and research methods in many areas of science and technology [1, 2, 3, 4, 5, 6, 7, 8]. Algebras of quaternions, octonions, and other hypercomplex numbers are widely used to solve various fundamental and practical scientific problems, for example, they are used to solve a number of complex specific problems in the fields of theoretical and applied physics, robotics, cryptography, and digital processing of multidimensional signals [9, 10, 11, 12, 13, 14].

An analysis of the content of scientific papers and books devoted to theoretical problems of the algebra of hypercomplex numbers shows that the construction of a commutative algebra of quaternions with division over the field of real numbers is considered impossible. There is practically no doubt about the existing theoretical statement that, unlike real and complex numbers, quaternions and hypercomplex numbers cannot have the commutativity property. Doubts about this problem are also eliminated by the fact that the well-known theorems of Frobenius and Hurwitz also imply the impossibility of constructing a commutative algebra of quaternions, octonions, and other hypercomplex numbers [15].

The possibility of constructing a commutative division algebra of quaternions and other hypercomplex numbers over the field of real numbers was first considered in the recently published paper [16]. This article provides a proof of the theorem that the commutativity of quaternions can be ensured by specifying a set of sign coefficients of the angles between the radius vectors in the coordinate planes of the vector part of the quaternion space coordinate system.

2 Properties of product of two two-dimensional vectors

In this paper, we continue the discussion of the theorem given in [16] and consider in more detail the condition for ensuring the commutativity of the orthogonal product of vectors and the product of hypercomplex numbers. We start with the formulation and proof of the following theorems.

Theorem 1. A complex number is a mapping of a two-dimensional vector with respect to the scalar direction.

Theorem 2. The commutativity of the orthogonal product of two two-dimensional vectors can be ensured by taking into account the directions of the orthogonal and collinear components of one vector relative to the linear direction of another vector.

Proof. An arbitrary vector \mathbf{r}_n in space can be represented by the follow-

ing formula:

$$\mathbf{r}_n = r_n \mathbf{e}_n. \quad (1)$$

Here r_n denotes the value of the module of the vector, \mathbf{e}_n - is a unit vector indicating the direction of the vector in the space under consideration, the index n - is the number of a specific vector. Note that the unit vector \mathbf{e}_n satisfies the condition $\mathbf{e}_n^2 = 1$.

Consider a pair of vectors and write them in the form proposed above by the formula (1)

$$\mathbf{r}_1 = r_1 \mathbf{e}_1, \mathbf{r}_2 = r_2 \mathbf{e}_2. \quad (2)$$

A vector \mathbf{r}_2 relative to a vector \mathbf{r}_1 can be represented as a sum of mutually orthogonal vectors, one of which is the projection of the vector \mathbf{r}_2 onto the vector direction \mathbf{r}_1 , and the other is perpendicular to this direction, that is, in the form

$$\mathbf{r}_2 = r_2 [\mathbf{e}_1^\rho \cos(\mathbf{e}_1, \hat{\mathbf{e}}_2) + \mathbf{e}_1^\tau \sin(\mathbf{e}_1, \hat{\mathbf{e}}_2)]. \quad (3)$$

Here $\mathbf{e}_1^\rho = \mathbf{e}_1$ is a radial unit vector, the direction of which is determined by the direction of the vector \mathbf{r}_1 ; \mathbf{e}_1^τ is a tangential unit vector whose direction is orthogonally flat. In addition, from (3) it follows

$$\mathbf{e}_2 = \mathbf{e}_1^\rho \cos(\mathbf{e}_1, \hat{\mathbf{e}}_2) + \mathbf{e}_1^\tau \sin(\mathbf{e}_1, \hat{\mathbf{e}}_2). \quad (4)$$

Expression (4) is squared

$$\begin{aligned} (\mathbf{e}_2)^2 &= [\mathbf{e}_1^\rho \cos(\mathbf{e}_1, \hat{\mathbf{e}}_2) + \mathbf{e}_1^\tau \sin(\mathbf{e}_1, \hat{\mathbf{e}}_2)] [\mathbf{e}_1^\rho \cos(\mathbf{e}_1, \hat{\mathbf{e}}_2) + \mathbf{e}_1^\tau \sin(\mathbf{e}_1, \hat{\mathbf{e}}_2)] = \\ &= \cos^2(\mathbf{e}_1, \hat{\mathbf{e}}_2) + \sin^2(\mathbf{e}_1, \hat{\mathbf{e}}_2) + (\mathbf{e}_1^\rho \times \mathbf{e}_1^\tau + \mathbf{e}_1^\tau \times \mathbf{e}_1^\rho) \cos(\mathbf{e}_1, \hat{\mathbf{e}}_2) \sin(\mathbf{e}_1, \hat{\mathbf{e}}_2). \end{aligned}$$

Considering that here

$$(\mathbf{e}_1^\rho)^2 = (\mathbf{e}_1^\tau)^2 = 1, \cos^2(\mathbf{e}_1, \hat{\mathbf{e}}_2) + \sin^2(\mathbf{e}_1, \hat{\mathbf{e}}_2) = 1,$$

for orthogonal product of unit radial and tangential vectors we get

$$\mathbf{e}_1^\rho \times \mathbf{e}_1^\tau + \mathbf{e}_1^\tau \times \mathbf{e}_1^\rho = 0, \mathbf{e}_1^\rho \times \mathbf{e}_1^\tau = -\mathbf{e}_1^\tau \mathbf{e}_1^\rho. \quad (5)$$

From (5) it follows

$$(\mathbf{e}_1^\rho \times \mathbf{e}_1^\tau)^2 = \mathbf{e}_1^\rho \mathbf{e}_1^\tau \mathbf{e}_1^\rho \mathbf{e}_1^\tau = -\mathbf{e}_1^\tau \mathbf{e}_1^\rho \mathbf{e}_1^\rho \mathbf{e}_1^\tau = -\mathbf{e}_1^\tau (\mathbf{e}_1^\rho)^2 \mathbf{e}_1^\tau = -1. \quad (6)$$

Next, consider equation (3) in more detail

$$\begin{aligned} \mathbf{r}_2 &= r_2 \mathbf{e}_2 = r_2 [\mathbf{e}_1^\rho \cos(\mathbf{e}_1, \hat{\mathbf{e}}_2) + \mathbf{e}_1^\tau \sin(\mathbf{e}_1, \hat{\mathbf{e}}_2)] = \\ &= r_2 \mathbf{e}_1^\rho [\cos(\mathbf{e}_1, \hat{\mathbf{e}}_2) + \mathbf{e}_1^\tau \times \mathbf{e}_1^\rho \sin(\mathbf{e}_1, \hat{\mathbf{e}}_2)]. \end{aligned}$$

Here from (6) follows $\mathbf{e}_1^\rho \times \mathbf{e}_1^\tau = i$. Therefore, the last equation can be written as

$$\mathbf{r}_2 = r_2 \mathbf{e}_1^\rho [\cos(\mathbf{e}_1, \hat{\mathbf{e}}_2) + i \sin(\mathbf{e}_1, \hat{\mathbf{e}}_2)]. \quad (7)$$

From (7) we can conclude that the complex number is a mapping of a two-dimensional vector with respect to the scalar direction. In this case, the direction of the unit vector \mathbf{e}_1 is scalar. **Theorem 1 is proved.**

The angle $(\mathbf{e}_1, \hat{\mathbf{e}}_2)$ in equation (7) is measured around the origin point in the direction from the unit vector \mathbf{e}_1 to the unit vector \mathbf{e}_2 . When counting this angle in the opposite direction, we have

$$(\mathbf{e}_2, \hat{\mathbf{e}}_1) = -(\mathbf{e}_1, \hat{\mathbf{e}}_2). \quad (8)$$

Taking into account (8), equation (7) can be represented as

$$\mathbf{r}_2 = r_2 \mathbf{e}_1 [\cos(\mathbf{e}_2, \hat{\mathbf{e}}_1) - i \sin(\mathbf{e}_2, \hat{\mathbf{e}}_1)]. \quad (9)$$

Remark 1. It is easy to see from (7) and (9) that the complex conjugate numbers differ only in the direction of the angle between the same vectors.

In general, the product of two coplanar vectors relative to the same common direction is a complex number. For example,

$$\begin{aligned} \mathbf{r}_1 \mathbf{r}_2 &= r_1 \mathbf{e}_1 r_2 \mathbf{e}_1 [\cos(\mathbf{e}_1, \hat{\mathbf{e}}_2) + i \sin(\mathbf{e}_1, \hat{\mathbf{e}}_2)] = \\ &= r_1 r_2 [\cos(\mathbf{e}_1, \hat{\mathbf{e}}_2) + i \sin(\mathbf{e}_1, \hat{\mathbf{e}}_2)]. \end{aligned} \quad (10)$$

Here both vectors are referenced and are presented with respect to the direction of the unit vector \mathbf{e}_1 .

Let's move on from the polar coordinate system to the two-dimensional Cartesian coordinate system x, y , for this we accept

$$\mathbf{e}_1^\rho = \mathbf{e}_x, \mathbf{e}_1^\tau = \mathbf{e}_y, \mathbf{e}_2 = \mathbf{e}_n. \quad (11)$$

Taking into account (11), equation (4) takes the form

$$\mathbf{e}_n = \mathbf{e}_x \cos(\mathbf{e}_x, \hat{\mathbf{e}}_n) + \mathbf{e}_y \sin(\mathbf{e}_x, \hat{\mathbf{e}}_n). \quad (12)$$

Additionally designating

$$(\mathbf{e}_x, \hat{\mathbf{e}}_n) = \varphi_{xy}^n, \quad (13)$$

from (12) we get

$$\mathbf{e}_n = \mathbf{e}_x \cos \varphi_{xy}^n + \mathbf{e}_y \sin \varphi_{xy}^n. \quad (14)$$

Using (11) - (14) an arbitrary vector in a two-dimensional Cartesian coordinate system in canonical form can be represented as

$$\begin{aligned} \mathbf{r}_n &= r_n \mathbf{e}_n = r_n [\mathbf{e}_x \cos(\mathbf{e}_x, \hat{\mathbf{e}}_n) + \mathbf{e}_y \sin(\mathbf{e}_x, \hat{\mathbf{e}}_n)] = \\ &= r_n (\mathbf{e}_x \cos \varphi_{xy}^n + \mathbf{e}_y \sin \varphi_{xy}^n) = x_n \mathbf{e}_x + y_n \mathbf{e}_y. \end{aligned} \quad (15)$$

As noted above, the vector (15) is related to the complex number by the following equation

$$x_n + iy_n = \mathbf{e}_x(x_n\mathbf{e}_x + y_n\mathbf{e}_y).$$

From expressions (11) and (14) it follows

$$\mathbf{e}_x \times \mathbf{e}_y = -\mathbf{e}_y \times \mathbf{e}_x. \quad (16)$$

Formula (16) is practically a copy of formula (5) in the considered two-dimensional Cartesian coordinate system.

When multiplying two vectors \mathbf{r}_a and \mathbf{r}_b the product of unit vectors, instead of the canonical form of equation (16), it should be represented in a more general form

$$\mathbf{e}_x^a \times \mathbf{e}_y^b = -\mathbf{e}_y^a \times \mathbf{e}_x^b = -\mathbf{e}_x^b \times \mathbf{e}_y^a = \mathbf{e}_y^b \times \mathbf{e}_x^a. \quad (17)$$

In (17), the subscripts indicate the direction of the angles in the coordinate system used, and the superscripts show from the direction of which vector the angle between the vectors is actually measured when they are multiplied. Note that the indices of the first multiplier indicate the directions from which the angle between the vectors is counted.

The product of unit vectors (17) to the second degree has the form

$$(\mathbf{e}_x^a \times \mathbf{e}_y^b)^2 = -1. \quad (18)$$

Taking into account (17) and (18), the product of unit vectors can be represented as

$$\mathbf{e}_x^a \times \mathbf{e}_y^b = ie_{xy}^{ab}. \quad (19)$$

Here $i^2 = -1$, $(e_{xy}^{ab})^2 = 1$. Besides,

$$e_{xy}^{ab} = -e_{yx}^{ab} = -e_{xy}^{ba} = e_{yx}^{ba}. \quad (20)$$

Equation (20) follows from (17) and (19).

Let us now consider in more detail the product of two vectors \mathbf{r}_a and \mathbf{r}_b

$$\begin{aligned} \mathbf{r}_a \mathbf{r}_b &= r_a r_b \mathbf{e}^a \mathbf{e}^b = r_a r_b (\mathbf{e}_x^a \cos \varphi_{xy}^a + \mathbf{e}_y^a \sin \varphi_{xy}^a) (\mathbf{e}_x^b \cos \varphi_{xy}^b + \mathbf{e}_y^b \sin \varphi_{xy}^b) = \\ &= r_a r_b [\cos(\varphi_{xy}^b - \varphi_{xy}^a) + ie_{xy}^{ab} \sin(\varphi_{xy}^b - \varphi_{xy}^a)]. \end{aligned}$$

Here the first multiplier is the vector \mathbf{r}_a . Let us further consider the product of the same vectors in the case when the first multiplier is the vector \mathbf{r}_b

$$\begin{aligned}\mathbf{r}_b\mathbf{r}_a &= r_ar_b [\cos(\varphi_{xy}^a - \varphi_{xy}^b) + ie_{xy}^{ba}\sin(\varphi_{xy}^a - \varphi_{xy}^b)] = \\ &= r_ar_b [\cos(\varphi_{xy}^b - \varphi_{xy}^a) + i(-e_{xy}^{ab})\sin(-1)(\varphi_{xy}^b - \varphi_{xy}^a)] = \\ &= r_ar_b [\cos(\varphi_{xy}^b - \varphi_{xy}^a) + ie_{xy}^{ab}\sin(\varphi_{xy}^b - \varphi_{xy}^a)].\end{aligned}$$

Comparison of the two obtained results shows that the complete product of two vectors has the commutativity property

$$\mathbf{r}_a\mathbf{r}_b = \mathbf{r}_b\mathbf{r}_a. \quad (21)$$

Theorem 2 is proved. Note that the complete product of two vectors (21) can be represented as

$$\mathbf{r}_a\mathbf{r}_b = \mathbf{r}_a \cdot \mathbf{r}_b + \mathbf{r}_a \times \mathbf{r}_b, \quad (22)$$

where

$$\mathbf{r}_a \cdot \mathbf{r}_b = r_ar_b\cos(\varphi_{xy}^b - \varphi_{xy}^a) \quad (23)$$

is the scalar (collinear) product of vectors, and the orthogonal product

$$\mathbf{r}_a \times \mathbf{r}_b = ie_{xy}^{ab}r_ar_b\sin(\varphi_{xy}^b - \varphi_{xy}^a) \quad (24)$$

differs from the vector product formula accepted in modern vector algebra only by the coefficient ie_{xy}^{ab} , which confirms that the complete product of two-dimensional vectors is a complex number and in the vector product it is necessary to correctly take into account the directions of counting the angles between the multiplier vectors. In addition, equations (20) and (24) confirm that, with careful consideration of the directions of the angles between the multipliers, the vector product has the commutativity property.

Remark 2. In a vector product, the third direction for the product of unit vectors is not the third linear direction perpendicular to the directions of the multipliers, but the direction of counting the angles between the vectors.

3 The commutativity of the quaternion algebra

Theorem 3. The commutativity of the product of quaternions can be ensured by observing the correct rules for taking into account the directions of the angles between the coplanar vector components of the quaternions, which are orthogonal to the scalar direction.

Proof. In the Cartesian coordinate system, the complete product of vectors \mathbf{r}_a and \mathbf{r}_b has the form

$$\mathbf{r}_a \mathbf{r}_b = (x_a \mathbf{e}_x^a + y_a \mathbf{e}_y^a)(x_b \mathbf{e}_x^b + y_b \mathbf{e}_y^b) = x_a x_b + y_a y_b + i e_{xy}^{ab} (x_a y_b - x_b y_a), \quad (25)$$

where the scalar and cross products are expressed by the following equations

$$\begin{aligned} \mathbf{r}_a \cdot \mathbf{r}_b &= x_a x_b + y_a y_b, \\ \mathbf{r}_a \times \mathbf{r}_b &= i e_{xy}^{ab} (x_a y_b - x_b y_a). \end{aligned}$$

Now let's analyze what quantitative numerical parameters and signs of directions characterize the two-dimensional vectors $\mathbf{r}_n = x_n \mathbf{e}_x^n + y_n \mathbf{e}_y^n$ considered above and their full product.

From the given vector formula and equation (25), it can be seen that two-dimensional vectors and their product contain the following components:

- scalar component, represented by real numbers, which can be associated with the direction in space, which is chosen as the base (scalar) direction;
- vector components characterizing the values of projections of vectors in the coordinate directions x and y , orthogonal to the base (scalar) direction;
- a planar-vector component that allows you to determine the reference directions and the angles between the vectors in the plane xy , that is, in the same coordinate directions x and y orthogonal to the base (scalar).

Taking into account the components listed above for two-dimensional vectors and their product, and considering a scalar number as a quantitative characteristic of a separate (scalar) direction, we can expand the dimension of the mapped vector space and represent the extended analog of the complex number for this vector space in the following form

$$q = x_0 + i(x_1 e_1 + x_2 e_2 + x_{12} e_{12}), \quad (26)$$

where x_0, x_1, x_2, x_{12} – real numbers,

$$i^2 = -1, e_1^2 = e_2^2 = e_{12}^2 = 1, e_{12} = e_1 e_2 = -e_2 e_1 = -e_{21}.$$

In essence, the four-component number given in (26) is a quaternion whose algebra, as will be shown below, has the commutativity property. Here, the quaternion, being a four-component number, characterizes the three-dimensional space, since the component $x_{12} e_{12}$ in equation (26) belongs to the plane $x_1 x_2$ and linearly describes the directions and angles between pairs of vectors on this plane.

Note that a quaternion in three-dimensional space is associated with a four-component composite vector

$$\mathbf{r} = x_0 \mathbf{e}_0 + x_1 \mathbf{e}_1 + x_2 \mathbf{e}_2 + x_{12} \mathbf{e}_{12}, \quad (27)$$

where \mathbf{e}_0 corresponds to the scalar direction; \mathbf{e}_1 and \mathbf{e}_2 are orthogonal to each other radial unit direction vectors in the plane x_1x_2 , which is orthogonal to the scalar direction; \mathbf{e}_{12} displays the tangential direction in the plane x_1x_2 , that is, the direction of the unit vector \mathbf{e}_{12} is planar orthogonal to the radial direction in the specified plane.

It is easy to see that quaternion (26) can be obtained from (27) by multiplying both parts of this equation by \mathbf{e}_0 , that is $q = \mathbf{e}_0\mathbf{r}$. When multiplying the right side of equation (27) on \mathbf{e}_0 , it is taken into account that

$$\mathbf{e}_0^2 = 1, \mathbf{e}_0\mathbf{e}_1 = ie_1, \mathbf{e}_0\mathbf{e}_2 = ie_2, \mathbf{e}_0\mathbf{e}_{12} = ie_{12}.$$

Consider now the product of two quaternions

$$\begin{aligned} q_a q_b = & x_0^a x_0^b - (x_1^a x_1^b + x_2^a x_2^b + x_{12}^a x_{12}^b) + \\ & + i[(x_0^a x_1^b + x_0^b x_1^a + x_{12}^a x_2^b - x_2^a x_{12}^b)e_1 + \\ & + (x_0^a x_2^b + x_0^b x_2^a + x_{12}^a x_{12}^b - x_{12}^a x_1^b)e_2 + \\ & + (x_0^a x_{12}^b + x_0^b x_{12}^a + x_1^a x_2^b - x_2^a x_1^b)e_{12}]. \end{aligned} \quad (28)$$

When performing the multiplication in (28), it was taken into account that

$$\begin{aligned} e_1^a = e_1^b = e_1, e_2^a = e_2^b = e_2, e_{12}^a = e_{12}^b = e_{12}, \\ e_1^a e_2^b = -e_1^b e_2^a = -e_{12}^{ba} = e_{12}^{ab} = e_{12}, \quad (29) \\ e_1^a e_{12}^b = -e_{12}^a e_1^b = e_2, e_{12}^a e_2^b = -e_2^a e_{12}^b = e_1, e_1^2 = e_2^2 = e_{12}^2 = 1. \end{aligned}$$

It is easy to see from (28) and (29) that the product of the considered quaternions has the commutativity property. **Theorem 3 is proved.**

Remark 3. As in the theory of complex variables, quaternion (26) has a conjugate four-dimensional number in the form

$$\bar{q} = x_0 - i(x_1 e_1 + x_2 e_2 + x_{12} e_{12}). \quad (30)$$

The product of conjugate quaternions (26) and (30) is

$$q\bar{q} = x_0^2 + x_1^2 + x_2^2 + x_{12}^2. \quad (31)$$

Remark 4. Equations (28) - (31) confirm that the quaternion algebra (26) is a commutative division algebra over the field of real numbers.

4 Conclusion

Summing up, we note that in this paper we consider the problems of ensuring the commutativity of the orthogonal product of vectors and the

commutativity of the quaternion algebra. Proofs are given that the commutativity of the orthogonal product of two two-dimensional vectors can be ensured by taking into account the directions of the orthogonal and collinear components of one vector relative to the linear direction of the other vector, the commutativity of the quaternion algebra can be ensured by observing the correct rules for taking into account the directions of the angles between the coplanar vector components of quaternions that are orthogonal scalar direction. Using the method proposed in this paper, the author of this paper continued to consider the problems of developing hypercomplex numbers with a higher dimension, whose algebras have the property of commutativity of multiplication and the property of division over the field of real numbers. The results of these works will be presented by the author in future articles. The obtained results on the improvement of the algebra of hypercomplex numbers with division over the field of real numbers can be used to solve a number of scientific and technical problems in the fields of field theory, physical electronics, robotics, cryptography and digital processing of multidimensional signals.

References

- [1] Hamilton, William Rowan. (1844) On quaternions or on a new system of imaginaries in algebra // Philosophical Magazine. N 25 (3), P. 489–495.
- [2] Finkelstein, David; Jauch, Josef M., Schiminovich, Samuel; Speiser, David. (1962) Foundations of quaternion quantum mechanics // J. Math. Phys. N 3, P. 207–220. doi:10.1063/1.1703794.
- [3] Kochin N. E. (1965) Vector calculus and beginnings of tensor calculus. M. Nauka. 426 p.
- [4] Landau L. D., Lifshits E. M. (1968) Theoretical Physics: Textbook in 10 volumes. Volume 2. Field theory. M., Nauka,. 512 p.
- [5] Ibrayev A.T. (2015) Theory of Cathode Lens with Multipole Components of Electrostatic Field and the Space Charge // Microscopy and Microanalysis. V. 21, N6, P. 270-275. <http://dx.doi.org/10.1017/S1431927615013495>.
- [6] Zhang, Fuzhen. (1997) Quaternions and Matrices of Quaternions // Linear Algebra and its Applications, 251, P. 21–57. doi:10.1016/0024-3795(95)00543-9.
- [7] Vince, John A. (2008) Geometric Algebra for Computer Graphics. Springer. ISBN 978-1-84628-996-5.

- [8] Ron Goldman. (2010) Rethinking Quaternions: Theory and Computation // Morgan & Claypool. ISBN 978-1-60845-420-4.
- [9] Hanson, Andrew J. (2006) Visualizing Quaternions. Elsevier. ISBN 0-12-088400-3.
- [10] Kuipers, Jack. (2002) Quaternions and Rotation Sequences: A Primer with Applications to Orbits, Aerospace, and Virtual Reality. Princeton University Press. ISBN 0-691-10298-8.
- [11] Trifonov V. (2007) Natural Geometry of Nonzero Quaternions // International Journal of Theoretical Physics. 46 (2), P. 251–257. doi:10.1007/s10773-006-9234-9.
- [12] Branets V.N., Shmyglevsky I.P. (1973) Applying Quaternions to Rigid Body Orientation Problems. M. 320 p.
- [13] Berezin A.V., Kurochkin Yu.A., Tolkachev E.A. (2003) Quaternions in relativistic physics. M. 200 p.
- [14] Di Gennaro S. (2003) Passive Attitude Control of Flexible Spacecraft from Quaternion Measurements // Journal of Optimization Theory and Applications N1. P 41-60.
- [15] Kantor I.L., Solodovnikov A.S. (1973) Hypercomplex numbers. M. 144 p.
- [16] Ibrayev A.T. (2020) Method for constructing the commutative algebra of quaternion and octonion. - News of the National Academy of sciences of the Republic of Kazakhstan. Physico-mathematical series, V6, N 334, P. 5-12. <https://doi.org/10.32014/2020.2518-1726.91>.

Alpamys T. Ibrayev

Kazakhstan Academy of Information and Business

Almaty, 050052, Kazakhstan

E-mail: pok_rk@mail.ru