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## NEW GENERALIZED WEIGHTED FRACTIONAL VARIANTS OF HERMITE-HADAMARD INEQUALITIES WITH APPLICATIONS

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**ABSTRACT.** Integral inequalities play a fundamental role in various mathematical fields, which have led to new methods and theoretical developments, both pure and applied. The need for searching for precise inequalities, in which the notion of convexity plays an important role, has a high impact on approximation theory calculus. In this paper, we first obtained a new version of the weighted fractional identity that led us to obtain new variants of weighted Trapezoid, Simpson and Midpoint inequalities. We then presented several refinements of it in the framework of weighted integrals for the modified second type  $(h, m)$ -convex functions.

**Keywords:** Hermite–Hadamard integral inequality, Hölder’s inequality,  $(h, m)$ -convex modified functions, weighted fractional integrals operators.

### 1. INTRODUCTION

Let  $\varsigma_1, \varsigma_2 \in \mathbb{R}$  with  $\varsigma_1 < \varsigma_2$ . A function  $\psi : I = [\varsigma_1, \varsigma_2] \rightarrow \mathbb{R}$  is said to be convex if

$$\psi(\zeta\theta + (1 - \zeta)\vartheta) \leq \zeta\psi(\theta) + (1 - \zeta)\psi(\vartheta)$$

holds for all  $\zeta \in [0, 1]$  and  $\theta, \vartheta \in I$ . If the function is concave, then the inequalities will be in reverse order.

In the last few decades, convex functions have become the object of extensive research by scientists from different countries. Today, this concept in the literature has a lot of generalizations and extensions. In [22], the authors presented a fairly wide range of different classes convexity.

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Of the theory of inequalities, the most important one is the Hermite–Hadamard inequality:

$$(1) \quad \psi\left(\frac{\varsigma_1 + \varsigma_2}{2}\right) \leq \frac{1}{\varsigma_2 - \varsigma_1} \int_{\varsigma_1}^{\varsigma_2} \psi(x) dx \leq \frac{\psi(\varsigma_1) + \psi(\varsigma_2)}{2},$$

holds for convex functions on an interval.

The ground breaking inequality helps to predict estimations of the mean values of a convex function and it is pertinent mention that it also provides a refine estimation of Jensen gap.

Interesting results for integral inequalities by using fractional operators can be found, for example, [4–13, 29, 34] and the references therein.

For example, the authors in [4–7] obtained new results in terms of fractional integral operators of the Riemann–Liouville for  $(s, m)$ ,  $(\alpha, m)$  and  $(h, m)$ –convex functions. In [8], Bermudo et al. presented some variants of the Hermite–Hadamard inequality general convex functions in the context of  $q$ –calculus. In [9], Butt and Pečarić obtained some generalized Hermite–Hadamard’s integral inequalities for the monotone functions of the form  $f/h$ . In [11], the authors presented an article in which they obtained new Hermite–Hadamard inequalities of the Jensen–Mercer type for a harmonically convex function through fractional integrals. In [16], Guzmán et al. studied the most important properties of the generalized integral operators and obtained different inequalities of the Hermite–Hadamard type for the weighted symmetric convex and  $h$ –symmetrized convex functions. In [21], the authors have defined new integral operators to obtain new Hermite–Hadamard inequalities for  $h$ –convex functions.

Dragomir and Agarwal in [12], by using the convexity of the first derivatives of the function, obtained the following estimate for the trapezoid inequality:

**Theorem 1.** *Let  $\varsigma_1, \varsigma_2 \in \mathbb{R}$  with  $\varsigma_1 < \varsigma_2$ ,  $\psi : I = [\varsigma_1, \varsigma_2] \rightarrow \mathbb{R}$  and  $\psi \in C^1(\varsigma_1, \varsigma_2)$ . If  $|\psi'|$  is convex on  $[\varsigma_1, \varsigma_2]$ , then the inequality holds:*

$$(2) \quad \left| \frac{\psi(\varsigma_1) + \psi(\varsigma_2)}{2} - \frac{1}{\varsigma_2 - \varsigma_1} \int_{\varsigma_1}^{\varsigma_2} \psi(\zeta) d\zeta \right| \leq \frac{\varsigma_2 - \varsigma_1}{8} (|\psi'(\varsigma_1)| + |\psi'(\varsigma_2)|).$$

The definitions of sequentially expanding the concept of convex functions, which form the basis of our article, are given below [6, 7].

**Definition 1.** *Let  $h : [0, 1] \rightarrow (0, 1]$ ,  $\psi : [0, +\infty) \rightarrow [0, +\infty)$ . If inequality*

$$\psi(\zeta\theta + m(1 - \zeta)\vartheta) \leq h^s(\zeta)\psi(\theta) + m(1 - h^s(\zeta))\psi(\vartheta)$$

*is fulfilled for all  $\zeta \in [0, 1]$  and  $\theta, \vartheta \in [0, +\infty)$ , then the function  $\psi$  is called the  $(h, m)$ –convex modified of the first type on  $[0, +\infty)$ . Here  $m \in [0, 1]$  and  $s \in [-1, 1]$ .*

**Definition 2.** *et  $h : [0, 1] \rightarrow (0, 1]$ ,  $\psi : [0, +\infty) \rightarrow [0, +\infty)$ . If inequality*

$$\psi(\zeta\theta + m(1 - \zeta)\vartheta) \leq h^s(\zeta)\psi(\theta) + m(1 - h(\zeta))^s\psi(\vartheta)$$

*is fulfilled for all  $\zeta \in [0, 1]$  and  $\theta, \vartheta \in [0, +\infty)$ , then the function  $\psi$  is called the  $(h, m)$ –convex modified of the second type on  $[0, +\infty)$ . Here  $m \in [0, 1]$  and  $s \in [-1, 1]$ .*

We denote by  $N_{h,m}^{s,1}[0, +\infty)$  and  $N_{h,m}^{s,2}[0, +\infty)$  the class of all  $(h, m)$ –convex modified functions of the first and second type respectively on the  $[0, +\infty)$ .

**Remark 1.** From Definitions 1 and 2, we have the following classifications of functions on  $[0, +\infty)$ .

- (i01) If  $h(\zeta) = \zeta$ , then  $\psi$  recaptures the class of  $m$ -convex functions, see [30].
- (i02) If  $h(\zeta) = \zeta^\alpha$  with  $\alpha \in (0, 1]$ , then  $\psi$  recaptures the class of  $(\alpha, s, m)$ -convex functions, see [36].
- (i03) If  $s = 1$ , then  $\psi$  recaptures the class of  $(h, m)$ -convex function.
- (i04) If  $h(\zeta) = \zeta$ ,  $s \in (0, 1]$  and  $m = 1$ , then  $\psi$  recaptures the class of  $s$ -convex functions, see [17].
- (i05) If  $h(\zeta) = \zeta$ ,  $m = 1$  and  $s \in [-1, 1]$ , then  $\psi$  recaptures the extended class of  $s$ -convex functions, see [35].
- (i06) If  $h(\zeta) = \zeta$ ,  $m = s = 1$ , then  $\psi$  is a convex function.

To begin with, we will give a definition of Riemann–Liouville fractional integrals, and then, for a better understanding of the topic, we will give a definition of weighted integral operators (with  $0 \leq \varsigma_1 < \zeta < \varsigma_2 \leq \infty$  and real number  $\kappa \geq 0$ ), which will be the basis for our article.

**Definition 3.** Let  $\psi \in L_1[\varsigma_1, \varsigma_2]$ ,  $\alpha \in \mathbb{C}$  and  $\Re(\alpha) > 0$ . Right and left fractional integrals defined below

$${}^\alpha I_{\varsigma_1^+} \psi(x) = \frac{1}{\Gamma(\alpha)} \int_{\varsigma_1}^x (x - \zeta)^{\alpha-1} \psi(\zeta) d\zeta, \quad x > \varsigma_1,$$

$${}^\alpha I_{\varsigma_2^-} \psi(x) = \frac{1}{\Gamma(\alpha)} \int_x^{\varsigma_2} (\zeta - x)^{\alpha-1} \psi(\zeta) d\zeta, \quad x < \varsigma_2,$$

are called right and left Riemann–Liouville fractional integrals, respectively. Here  $\Gamma(\alpha)$  is the gamma function.

**Definition 4.** Let  $\psi \in L_1(\varsigma_1, \varsigma_2)$ ,  $w : [0, \infty) \rightarrow [0, \infty)$  and  $w \in C[0, \infty)$  with derivatives piecewise continuous on  $[0, \infty)$  up to the second order inclusive. Integral operators defined below

$$J_{\varsigma_1^+}^w \psi(x) = \int_{\varsigma_1}^x w' \left( \frac{x - \zeta}{\frac{\varsigma_2 - \varsigma_1}{1 + \kappa}} \right) \psi(\zeta) d\zeta, \quad x > \varsigma_1,$$

$$J_{\varsigma_2^-}^w \psi(x) = \int_x^{\varsigma_2} w' \left( \frac{\zeta - x}{\frac{\varsigma_2 - \varsigma_1}{1 + \kappa}} \right) \psi(\zeta) d\zeta, \quad x < \varsigma_2.$$

called weighted fractional integrals (right and left, respectively).

**Remark 2.** If we take  $w'(\zeta) = \frac{(\varsigma_2 - \varsigma_1) \zeta^{\alpha-1}}{\Gamma(\alpha)}$  and  $\kappa = 0$ , we get the Riemann–Liouville fractional integral, but if we take  $w'(\zeta) \equiv 1$  it turns into a classical Riemann integral.

In the present article, we presented some new unified variants of weighted Hadamard inequality (1) by employing generalized convexity for generalized fractional integral operators given in Definition 4. Several special cases are discussed as well.

## 2. MAIN RESULTS

The results are obtained with the help of the following lemma.

**Lemma 1.** Let  $\psi : [\varsigma_1, \varsigma_2] \subset \mathbb{R} \rightarrow \mathbb{R}$  with  $\psi \in C^1(\varsigma_1, \varsigma_2)$  and let  $w : [0, \infty) \rightarrow [0, \infty)$ ,  $w \in C[0, \infty)$  with the first and second order derivatives piecewise continuous on  $[0, \infty)$ . If  $\psi' \in L_1(\varsigma_1, \varsigma_2)$ , then equality holds:

$$\begin{aligned}
 (3) \quad & [\psi(\varsigma_1) + \psi(\varsigma_2)] w(0) - \left[ \psi \left( \frac{\kappa\varsigma_1 + \varsigma_2}{1 + \kappa} \right) + \psi \left( \frac{\varsigma_1 + \kappa\varsigma_2}{1 + \kappa} \right) \right] w(1) \\
 & + \frac{1 + \kappa}{\varsigma_2 - \varsigma_1} \left[ J_{\left( \frac{\varsigma_1 + \kappa\varsigma_2}{1 + \kappa} \right)^+}^w \psi(\varsigma_2) + J_{\left( \frac{\kappa\varsigma_1 + \varsigma_2}{1 + \kappa} \right)^-}^w \psi(\varsigma_1) \right] \\
 & = \frac{\varsigma_2 - \varsigma_1}{1 + \kappa} \left[ \int_0^1 w(\zeta) \psi' \left( \frac{\zeta}{1 + \kappa} \varsigma_1 + \frac{1 + \kappa - \zeta}{1 + \kappa} \varsigma_2 \right) d\zeta \right. \\
 & \quad \left. - \int_0^1 w(\zeta) \psi' \left( \frac{1 + \kappa - \zeta}{1 + \kappa} \varsigma_1 + \frac{\zeta}{1 + \kappa} \varsigma_2 \right) d\zeta \right].
 \end{aligned}$$

*Proof.* For simplicity,

$$\begin{aligned}
 & \int_0^1 w(\zeta) \psi' \left( \frac{\zeta}{1 + \kappa} \varsigma_1 + \frac{1 + \kappa - \zeta}{1 + \kappa} \varsigma_2 \right) d\zeta \\
 & \quad - \int_0^1 w(\zeta) \psi' \left( \frac{1 + \kappa - \zeta}{1 + \kappa} \varsigma_1 + \frac{\zeta}{1 + \kappa} \varsigma_2 \right) d\zeta := I_1 - I_2.
 \end{aligned}$$

By integrating by parts in  $I_1$ , we have

$$\begin{aligned}
 I_1 &= \frac{1 + \kappa}{\varsigma_2 - \varsigma_1} \left[ w(0) \psi(\varsigma_2) - w(1) \psi \left( \frac{\varsigma_1 + \kappa\varsigma_2}{1 + \kappa} \right) \right] \\
 & \quad + \frac{(1 + \kappa)}{(\varsigma_2 - \varsigma_1)} \int_0^1 w'(\zeta) \psi \left( \frac{\zeta}{1 + \kappa} \varsigma_1 + \frac{1 + \kappa - \zeta}{1 + \kappa} \varsigma_2 \right) d\zeta \\
 &= \frac{1 + \kappa}{\varsigma_2 - \varsigma_1} \left[ w(0) \psi(\varsigma_2) - w(1) \psi \left( \frac{\varsigma_1 + \kappa\varsigma_2}{1 + \kappa} \right) \right] \\
 & \quad + \frac{(1 + \kappa)^2}{(\varsigma_2 - \varsigma_1)^2} J_{\left( \frac{\varsigma_1 + \kappa\varsigma_2}{1 + \kappa} \right)^+}^w \psi(\varsigma_2),
 \end{aligned}$$

since

$$\int_0^1 w'(\zeta) \psi \left( \frac{\zeta}{1 + \kappa} \varsigma_1 + \frac{1 + \kappa - \zeta}{1 + \kappa} \varsigma_2 \right) d\zeta = \frac{1 + \kappa}{\varsigma_2 - \varsigma_1} \int_{\frac{\varsigma_1 + \kappa\varsigma_2}{1 + \kappa}}^{\varsigma_2} w' \left[ \frac{\varsigma_2 - z}{1 + \kappa} \right] \psi(z) dz.$$

Analogously for  $I_2$ , we have

$$I_2 = \frac{1 + \kappa}{\varsigma_2 - \varsigma_1} \left[ w(1) \psi \left( \frac{\kappa\varsigma_1 + \varsigma_2}{1 + \kappa} \right) - w(0) \psi(\varsigma_1) \right] - \frac{(1 + \kappa)^2}{(\varsigma_2 - \varsigma_1)^2} J_{\left( \frac{\kappa\varsigma_1 + \varsigma_2}{1 + \kappa} \right)^-}^w \psi(\varsigma_1).$$

Substituting in  $I_1 - I_2$ , grouping appropriately and multiplying by  $\frac{\varsigma_2 - \varsigma_1}{1 + \kappa}$ , we obtain the inequality (3).  $\square$

The following remarks have the mission of illustrating the scope and strength of the previous result. The choice of special weight functions in the Lemma 1 leads us to many interesting results published in the literature.

**Remark 3.** *Special cases of the Lemma 1:*

(i01) If we take  $\kappa = 0$  in (3), then we get the following new general weighted identity.

$$(4) \quad [w(0) - w(1)] [\psi(\varsigma_2) + \psi(\varsigma_1)] + \frac{1}{\varsigma_2 - \varsigma_1} \left( J_{\varsigma_1^+}^w \psi(\varsigma_2) + J_{\varsigma_2^-}^w \psi(\varsigma_1) \right) \\ = (\varsigma_2 - \varsigma_1) \left[ \int_0^1 w(\zeta) \psi'(\zeta \varsigma_1 + (1 - \zeta) \varsigma_2) d\zeta \right. \\ \left. - \int_0^1 w(\zeta) \psi'((1 - \zeta) \varsigma_1 + \zeta \varsigma_2) d\zeta \right].$$

(i02) If we take  $\kappa = 0, w(\zeta) = 1 - \zeta^\alpha$  in (3), substituting  $x$  for  $\varsigma_1$  in  $I_1$  and  $x$  for  $\varsigma_2$  in  $I_2$ , then we will have [23, Lemma 2].

(i03) If we take  $\kappa = 0, w(\zeta) = (1 - \zeta)^\alpha - \zeta^\alpha$  and work only with  $I_1$ , then we get [28, Lemma 2]; furthermore, if we substitute  $\varsigma_2$  for  $r \in (\varsigma_1, \varsigma_2]$ , then we will have [24, Lemma 3].

**Remark 4.** Other special cases of the Lemma 1:

(i04) If we take  $\kappa = 1, w(\zeta) = \zeta$  in (3), then we get

$$(5) \quad [w(0) - w(1)] [\psi(\varsigma_2) + \psi(\varsigma_1)] \\ + \frac{1}{\varsigma_2 - \varsigma_1} \left( J_{\left(\frac{\varsigma_1 + \varsigma_2}{2}\right)^+}^w \psi(\varsigma_2) + J_{\left(\frac{\varsigma_1 + \varsigma_2}{2}\right)^-}^w \psi(\varsigma_1) \right) \\ = \frac{\varsigma_2 - \varsigma_1}{4} \left[ \int_0^1 \zeta \psi' \left( \frac{\zeta}{2} \varsigma_1 + \frac{2 - \zeta}{2} \varsigma_2 \right) d\zeta - \int_0^1 \zeta \psi' \left( \frac{2 - \zeta}{2} \varsigma_1 + \frac{\zeta}{2} \varsigma_2 \right) d\zeta \right].$$

(i05) If we take  $\kappa = 1, w(\zeta) = 1 - \zeta$ , by replacing in (3)  $1 - \zeta$  by  $\zeta$ , then we obtain

$$(6) \quad \frac{\psi(\varsigma_1) + \psi(\varsigma_2)}{2} - \frac{1}{\varsigma_2 - \varsigma_1} \int_{\varsigma_1}^{\varsigma_2} \psi(\zeta) d\zeta \\ = \frac{\varsigma_2 - \varsigma_1}{4} \left[ \int_0^1 \zeta \psi' \left( \frac{1 - \zeta}{2} \varsigma_1 + \frac{1 + \zeta}{2} \varsigma_2 \right) d\zeta \right. \\ \left. - \int_0^1 \zeta \psi' \left( \frac{1 + \zeta}{2} \varsigma_1 + \frac{1 - \zeta}{2} \varsigma_2 \right) d\zeta \right],$$

which is given as in [1, Lemma 2.1].

Our first and main result was obtained from Lemma 1 in the following theorem.

**Theorem 2.** Let  $\psi : [\varsigma_1, \frac{\varsigma_2}{m}] \rightarrow \mathbb{R}$  be a  $L^1$  function. If  $|\psi'| \in N_{h,m}^{s,2} [\varsigma_1, \frac{\varsigma_2}{m}]$ , we have the following inequality:

$$(7) \quad \left| \mathbb{D} + \frac{1 + \kappa}{\varsigma_2 - \varsigma_1} \left[ J_{\left(\frac{\varsigma_1 + \kappa \varsigma_2}{1 + \kappa}\right)^+}^w \psi(\varsigma_2) + J_{\left(\frac{\kappa \varsigma_1 + \varsigma_2}{1 + \kappa}\right)^-}^w \psi(\varsigma_1) \right] \right| \\ \leq \frac{\varsigma_2 - \varsigma_1}{1 + \kappa} \left[ |\psi'(\varsigma_1)| \mathbb{B} + m \left| \psi' \left( \frac{\varsigma_2}{m} \right) \right| \mathbb{C} \right],$$

where

$$\mathbb{D} = [\psi(\varsigma_1) + \psi(\varsigma_2)] w(0) - \left[ \psi \left( \frac{\kappa \varsigma_1 + \varsigma_2}{1 + \kappa} \right) + \psi \left( \frac{\varsigma_1 + \kappa \varsigma_2}{1 + \kappa} \right) \right] w(1), \\ \mathbb{B} = \int_0^1 |w(\zeta)| \left[ h^s \left( \frac{\zeta}{1 + \kappa} \right) + h^s \left( \frac{1 + \kappa - \zeta}{1 + \kappa} \right) \right] d\zeta,$$

$$\mathbb{C} = \int_0^1 |w(\zeta)| \left[ \left( 1 - h \left( \frac{1 + \kappa - \zeta}{1 + \kappa} \right) \right)^s + \left( 1 - h \left( \frac{\zeta}{1 + \kappa} \right) \right)^s \right] d\zeta.$$

*Proof.* From Lemma 1, we obtain

$$\begin{aligned} (8) \quad & \left| \int_0^1 w(\zeta) \psi' \left( \frac{\zeta}{1 + \kappa} \varsigma_1 + \frac{1 + \kappa - \zeta}{1 + \kappa} \varsigma_2 \right) d\zeta \right. \\ & \quad \left. - \int_0^1 w(\zeta) \psi' \left( \frac{1 + \kappa - \zeta}{1 + \kappa} \varsigma_1 + \frac{\zeta}{1 + \kappa} \varsigma_2 \right) d\zeta \right| \\ & \leq \int_0^1 |w(\zeta)| \left| \psi' \left( \frac{\zeta}{1 + \kappa} \varsigma_1 + \frac{1 + \kappa - \zeta}{1 + \kappa} \varsigma_2 \right) \right| d\zeta \\ & \quad + \int_0^1 |w(\zeta)| \left| \psi' \left( \frac{1 + \kappa - \zeta}{1 + \kappa} \varsigma_1 + \frac{\zeta}{1 + \kappa} \varsigma_2 \right) \right| d\zeta. \end{aligned}$$

Taking into account modified  $(h, m)$ -convexity of  $|\psi'|$ , we get

$$\begin{aligned} (9) \quad & \left| \psi' \left( \frac{\zeta}{1 + \kappa} \varsigma_1 + \frac{1 + \kappa - \zeta}{1 + \kappa} \varsigma_2 \right) \right| \leq |\psi'(\varsigma_1)| h^s \left( \frac{\zeta}{1 + \kappa} \right) \\ & \quad + m \left| \psi' \left( \frac{\varsigma_2}{m} \right) \right| \left( 1 - h \left( \frac{\zeta}{1 + \kappa} \right) \right)^s \end{aligned}$$

and

$$\begin{aligned} (10) \quad & \left| \psi' \left( \frac{1 + \kappa - \zeta}{1 + \kappa} \varsigma_1 + \frac{\zeta}{1 + \kappa} \varsigma_2 \right) \right| \leq |\psi'(\varsigma_1)| h^s \left( \frac{1 + \kappa - \zeta}{1 + \kappa} \right) \\ & \quad + m \left| \psi' \left( \frac{\varsigma_2}{m} \right) \right| \left( 1 - h \left( \frac{1 + \kappa - \zeta}{1 + \kappa} \right) \right)^s. \end{aligned}$$

By substituting (9) and (10) in (8), we have the required result.  $\square$

**Remark 5.** If we take  $\kappa = 0$ , we get some special cases of the Theorem 2:

- (i01) Theorem 2.1 from [3] (case  $q = 1$ ), using  $w(\zeta) = 1 - 2\zeta$  by working only with  $I_1$  for  $s = 1$ ,  $h(\zeta) = \zeta$  and  $m$ -convex functions.
- (i02) Theorem 2.2 from [12], by using  $I_2$  with  $w(\zeta) = 1 - 2\zeta$ .
- (i03) Theorem 2.4 of [15] for  $h(\zeta) = \zeta$  and  $m = s = 1$  with  $I_1$  and  $w(\zeta) = (1 - \zeta)^\alpha - \zeta^\alpha$ .
- (i04) Theorem 2.2 of [19], for an appropriate  $w(\zeta)$  function.
- (i05) Theorem 2.3 of [20], with  $I_2$  and  $w(\zeta) = 1 - 2\zeta$ ,
- (i06) Theorem 7 from [23] (see Remark 3(ii)).
- (i07) Theorem 3.1 from [26], where  $I_2$  was used and the interval  $[0, 1]$  was divided by using  $w_1 = \lambda - \zeta$  for  $[0, \frac{1}{2}]$  and  $w_2 = \mu - \zeta$  for  $[\frac{1}{2}, 1]$ , where  $\lambda$  and  $\mu$  real numbers such that  $0 \leq \lambda \leq \frac{1}{2} \leq \mu \leq 1$ .
- (i08) Theorem 3 of [28] by taking  $w(\zeta) = (1 - \zeta)^\alpha - \zeta^\alpha$  and  $I_2$ .
- (i09) Theorem 5 of [37] by working with  $w(\zeta) = (1 - \zeta)^\alpha - \zeta^\alpha$  and by using only  $I_2$ .

**Corollary 1.** If we choose  $w(\zeta) = 1 - 2\zeta$ ,  $h(\zeta) = \zeta$ ,  $m = s = 1$  and  $\kappa = 1$  in Theorem 2, then we get the Simpson type inequality:

$$\begin{aligned} (11) \quad & \left| \frac{1}{2} \left[ \frac{\psi(\varsigma_1) + \psi(\varsigma_2)}{2} + \psi \left( \frac{\varsigma_1 + \varsigma_2}{2} \right) \right] - \frac{1}{\varsigma_2 - \varsigma_1} \int_{\varsigma_1}^{\varsigma_2} \psi(\zeta) d\zeta \right| \\ & \leq \frac{\varsigma_2 - \varsigma_1}{16} (|\psi'(\varsigma_1)| + |\psi'(\varsigma_2)|). \end{aligned}$$

*Proof.* From (7), we get

$$\begin{aligned}\mathbb{D} &= [\psi(\varsigma_1) + \psi(\varsigma_2)] w(0) - \left[ \psi \left( \frac{\kappa\varsigma_1 + \varsigma_2}{1 + \kappa} \right) + \psi \left( \frac{\varsigma_1 + \kappa\varsigma_2}{1 + \kappa} \right) \right] w(1) \\ &= [\psi(\varsigma_1) + \psi(\varsigma_2)] + 2\psi \left( \frac{\varsigma_1 + \varsigma_2}{2} \right),\end{aligned}$$

since  $w'(\zeta) = -2$ ,

$$\begin{aligned}\left| \mathbb{D} + \frac{2}{\varsigma_2 - \varsigma_1} \left[ J_{\left(\frac{\varsigma_1 + \varsigma_2}{2}\right)^+}^w \psi(\varsigma_2) + J_{\left(\frac{\varsigma_1 + \varsigma_2}{2}\right)^-}^w \psi(\varsigma_1) \right] \right| \\ = \left| \psi(\varsigma_1) + \psi(\varsigma_2) + 2\psi \left( \frac{\varsigma_1 + \varsigma_2}{2} \right) - \frac{4}{\varsigma_2 - \varsigma_1} \int_{\varsigma_1}^{\varsigma_2} \psi(\zeta) d\zeta \right|,\end{aligned}$$

$$\mathbb{B} = \int_0^1 |1 - 2\zeta| \left[ h \left( \frac{\zeta}{2} \right) + h \left( \frac{2 - \zeta}{2} \right) \right] d\zeta = \int_0^1 |1 - 2\zeta| \left[ \frac{\zeta}{2} + \frac{2 - \zeta}{2} \right] d\zeta = \frac{1}{2},$$

$$\begin{aligned}\mathbb{C} &= \int_0^1 |1 - 2\zeta| \left[ \left( 1 - h \left( \frac{2 - \zeta}{2} \right) \right) + \left( 1 - h \left( \frac{\zeta}{2} \right) \right) \right] d\zeta \\ &= \int_0^1 |1 - 2\zeta| \left[ 1 - \frac{2 - \zeta}{2} + 1 - \frac{\zeta}{2} \right] d\zeta = \int_0^1 |1 - 2\zeta| d\zeta = \frac{1}{2},\end{aligned}$$

and

$$\begin{aligned}\frac{\varsigma_2 - \varsigma_1}{1 + \kappa} \left\{ |\psi'(\varsigma_1)| \mathbb{B} + m \left| \psi' \left( \frac{\varsigma_2}{m} \right) \right| \mathbb{C} \right\} &= \frac{\varsigma_2 - \varsigma_1}{2} \left[ \frac{|\psi'(\varsigma_1)|}{2} + \frac{|\psi'(\varsigma_2)|}{2} \right] \\ &= \frac{\varsigma_2 - \varsigma_1}{4} (|\psi'(\varsigma_1)| + |\psi'(\varsigma_2)|).\end{aligned}$$

Thus, we have

$$\begin{aligned}\left| \psi(\varsigma_1) + \psi(\varsigma_2) + 2\psi \left( \frac{\varsigma_1 + \varsigma_2}{2} \right) - \frac{4}{\varsigma_2 - \varsigma_1} \int_{\varsigma_1}^{\varsigma_2} \psi(\zeta) d\zeta \right| \\ \leq \frac{\varsigma_2 - \varsigma_1}{4} (|\psi'(\varsigma_1)| + |\psi'(\varsigma_2)|).\end{aligned}$$

By dividing both parts of the last inequality by 4, we obtain (11). The proof is completed.  $\square$

**Corollary 2.** *If we choose  $w(\zeta) = 1 - 2\zeta$ ,  $h(\zeta) = \zeta$ ,  $m = s = 1$  and  $\kappa = 0$  in Theorem 2, then we get Trapezoidal inequality (2).*

*Proof.* From (7), we get

$$\begin{aligned}\mathbb{D} &= [\psi(\varsigma_1) + \psi(\varsigma_2)] w(0) - \left[ \psi \left( \frac{\kappa\varsigma_1 + \varsigma_2}{1 + \kappa} \right) + \psi \left( \frac{\varsigma_1 + \kappa\varsigma_2}{1 + \kappa} \right) \right] w(1) \\ &= [\psi(\varsigma_1) + \psi(\varsigma_2)] w(0) - [\psi(\varsigma_2) + \psi(\varsigma_1)] \\ &= 2 [\psi(\varsigma_1) + \psi(\varsigma_2)] w(1),\end{aligned}$$

since  $w'(\zeta) = -2$ ,

$$\begin{aligned}\left| \mathbb{D} + \frac{1}{\varsigma_2 - \varsigma_1} \left( J_{\varsigma_1^+}^w \psi(\varsigma_2) + J_{\varsigma_2^-}^w \psi(\varsigma_1) \right) \right| \\ = \left| 2 [\psi(\varsigma_1) + \psi(\varsigma_2)] - \frac{4}{\varsigma_2 - \varsigma_1} \int_{\varsigma_1}^{\varsigma_2} \psi(\zeta) d\zeta \right|,\end{aligned}$$

$$\begin{aligned}\mathbb{B} &= \int_0^1 |1-2\zeta| \left[ h\left(\frac{\zeta}{2}\right) + h\left(\frac{2-\zeta}{2}\right) \right] d\zeta = \int_0^1 |1-2\zeta| \left[ \frac{\zeta}{2} + \frac{2-\zeta}{2} \right] d\zeta = \frac{1}{2}, \\ \mathbb{C} &= \int_0^1 |1-2\zeta| \left[ \left(1-h\left(\frac{2-\zeta}{2}\right)\right) + \left(1-h\left(\frac{\zeta}{2}\right)\right) \right] d\zeta \\ &= \int_0^1 |1-2\zeta| \left[ 1 - \frac{2-\zeta}{2} + 1 - \frac{\zeta}{2} \right] d\zeta = \int_0^1 |1-2\zeta| d\zeta = \frac{1}{2},\end{aligned}$$

and

$$\begin{aligned}\frac{\varsigma_2 - \varsigma_1}{1 + \kappa} \left\{ |\psi'(\varsigma_1)| \mathbb{B} + m \left| \psi' \left( \frac{\varsigma_2}{m} \right) \right| \mathbb{C} \right\} &= (\varsigma_2 - \varsigma_1) \left[ \frac{|\psi'(\varsigma_1)|}{2} + \frac{|\psi'(\varsigma_2)|}{2} \right] \\ &= \frac{\varsigma_2 - \varsigma_1}{2} (|\psi'(\varsigma_1)| + |\psi'(\varsigma_2)|).\end{aligned}$$

Thus, we have

$$\left| 2[\psi(\varsigma_1) + \psi(\varsigma_2)] - \frac{4}{\varsigma_2 - \varsigma_1} \int_{\varsigma_1}^{\varsigma_2} \psi(\zeta) d\zeta \right| \leq \frac{\varsigma_2 - \varsigma_1}{2} (|\psi'(\varsigma_1)| + |\psi'(\varsigma_2)|).$$

By dividing both parts of the last inequality by 4, we obtain (2). The proof is completed.  $\square$

By imposing additional assumptions on  $|\psi'|$ , we can obtain new refinements of known results.

**Theorem 3.** *If  $\psi : [\varsigma_1, \frac{\varsigma_2}{m}] \rightarrow \mathbb{R}$  is a  $L^1$  function and if  $|\psi'|^q \in N_{h,m}^{s,2} [\varsigma_1, \frac{\varsigma_2}{m}]$ , then for  $q \geq 1$  the following inequality*

$$\begin{aligned}(12) \quad & \left| \mathbb{D} + \frac{1 + \kappa}{\varsigma_2 - \varsigma_1} \left[ J_{\left(\frac{\varsigma_1 + \kappa \varsigma_2}{1 + \kappa}\right)^+}^w \psi(\varsigma_2) + J_{\left(\frac{\kappa \varsigma_1 + \varsigma_2}{-1 + \kappa}\right)^-}^w \psi(\varsigma_1) \right] \right| \\ & \leq \frac{\varsigma_2 - \varsigma_1}{1 + \kappa} B_q \left\{ \left( |\psi'(\varsigma_1)|^q C_{11} + m \left| \psi' \left( \frac{\varsigma_2}{m} \right) \right|^q C_{12} \right)^{\frac{1}{q}} \right. \\ & \quad \left. + \left( |\psi'(\varsigma_1)|^q C_{21} + m \left| \psi' \left( \frac{\varsigma_2}{m} \right) \right|^q C_{22} \right)^{\frac{1}{q}} \right\}\end{aligned}$$

holds, with  $\mathbb{D}$  as before,

$$\begin{aligned}B_q &= \left( \int_0^1 |w(\zeta)|^p d\zeta \right)^{\frac{1}{p}} = \left( \int_0^1 |w(\zeta)|^{\frac{q}{q-1}} d\zeta \right)^{1-\frac{1}{q}}, \\ C_{11} &= \int_0^1 h^s \left( \frac{\zeta}{1 + \kappa} \right) d\zeta, \quad C_{12} = \int_0^1 \left( 1 - h \left( \frac{\zeta}{1 + \kappa} \right) \right)^s d\zeta, \\ C_{21} &= \int_0^1 h^s \left( \frac{1 + \kappa - \zeta}{1 + \kappa} \right) d\zeta \quad \text{and} \quad C_{22} = \int_0^1 \left( 1 - h \left( \frac{1 + \kappa - \zeta}{1 + \kappa} \right) \right)^s d\zeta.\end{aligned}$$

*Proof.* By proceeding as in the previous Theorem, from Lemma 1, we have

$$(13) \quad \left| \int_0^1 w(\zeta) \psi' \left( \frac{\zeta}{1+\kappa} \varsigma_1 + \frac{1+\kappa-\zeta}{1+\kappa} \varsigma_2 \right) d\zeta \right. \\ \left. - \int_0^1 w(\zeta) \psi' \left( \frac{1+\kappa-\zeta}{1+\kappa} \varsigma_1 + \frac{\zeta}{1+\kappa} \varsigma_2 \right) d\zeta \right| \\ \leq \int_0^1 |w(\zeta)| \left| \psi' \left( \frac{\zeta}{1+\kappa} \varsigma_1 + \frac{1+\kappa-\zeta}{1+\kappa} \varsigma_2 \right) \right| d\zeta \\ + \int_0^1 |w(\zeta)| \left| \psi' \left( \frac{1+\kappa-\zeta}{1+\kappa} \varsigma_1 + \frac{\zeta}{1+\kappa} \varsigma_2 \right) \right| d\zeta.$$

By using the Hölder's inequality in each of the integrals of (13) leads us to

$$(14) \quad \int_0^1 |w(\zeta)| \left| \psi' \left( \frac{\zeta}{1+\kappa} \varsigma_1 + \frac{1+\kappa-\zeta}{1+\kappa} \varsigma_2 \right) \right| d\zeta \\ \leq \left( \int_0^1 |w(\zeta)|^p d\zeta \right)^{\frac{1}{p}} \left( \int_0^1 \left| \psi' \left( \frac{\zeta}{1+\kappa} \varsigma_1 + \frac{1+\kappa-\zeta}{1+\kappa} \varsigma_2 \right) \right|^q d\zeta \right)^{\frac{1}{q}}$$

and

$$(15) \quad \int_0^1 |w(\zeta)| \left| \psi' \left( \frac{1+\kappa-\zeta}{1+\kappa} \varsigma_1 + \frac{\zeta}{1+\kappa} \varsigma_2 \right) \right| d\zeta \\ \leq \left( \int_0^1 |w(\zeta)|^p d\zeta \right)^{\frac{1}{p}} \left( \int_0^1 \left| \psi' \left( \frac{1+\kappa-\zeta}{1+\kappa} \varsigma_1 + \frac{\zeta}{1+\kappa} \varsigma_2 \right) \right|^q d\zeta \right)^{\frac{1}{q}}$$

with  $\frac{1}{p} + \frac{1}{q} = 1$ . By considering the  $(h, m)$ -convexity of the second type of  $|\psi'|^q$ , we obtain

$$(16) \quad \int_0^1 \left| \psi' \left( \frac{\zeta}{1+\kappa} \varsigma_1 + \frac{1+\kappa-\zeta}{1+\kappa} \varsigma_2 \right) \right|^q d\zeta \leq |\psi'(\varsigma_1)|^q \int_0^1 h^s \left( \frac{\zeta}{1+\kappa} \right) d\zeta \\ + m \left| \psi' \left( \frac{\varsigma_2}{m} \right) \right|^q \int_0^1 \left( 1 - h \left( \frac{\zeta}{1+\kappa} \right) \right)^s d\zeta$$

and

$$(17) \quad \int_0^1 \left| \psi' \left( \frac{1+\kappa-\zeta}{1+\kappa} \varsigma_1 + \frac{\zeta}{1+\kappa} \varsigma_2 \right) \right|^q d\zeta \leq |\psi'(\varsigma_1)|^q \int_0^1 h^s \left( \frac{1+\kappa-\zeta}{1+\kappa} \right) d\zeta \\ + m \left| \psi' \left( \frac{\varsigma_2}{m} \right) \right|^q \int_0^1 \left( 1 - h \left( \frac{1+\kappa-\zeta}{1+\kappa} \right) \right)^s d\zeta.$$

By denoting and required and substituting (16), (17) in (14) and (15), we obtain the desired result (12).  $\square$

**Remark 6.** If we take  $\kappa = 0$ , we get some special cases of the Theorem 3:

- (i.01) Theorem 11 of [33] by using  $I_1$ , with  $w(\zeta) = (1-\zeta)^\alpha - \zeta^\alpha$ .
- (i.02) Theorem 3.2 of [14] by using  $I_1$ , with  $w(\zeta) = (1-\zeta)^\alpha - \zeta^\alpha$ .
- (i.03) Theorem 6 of [37] by using  $I_1$  with  $w(\zeta) = (1-\zeta)^\alpha - \zeta^\alpha$ .
- (i.04) Theorem 2.1 (second part) and Theorem 2.2 of [3] by using  $I_1$  with  $w(\zeta) = 1 - 2\zeta$  and  $m$ -convex functions.
- (i.05) Theorem 2.3 [12] by using  $I_1$  with  $w(\zeta) = 1 - 2\zeta$ .
- (i.06) Theorem 8 of [23], for  $s$ -convex (see Remarks 3(ii)).
- (i.07) Theorem 1 of [25] by using  $I_1$  with  $w(\zeta) = 1 - 2\zeta$ .

(i.08) Theorem 2.1 of [31] (see Remarks 5(i09)).

(i.09) Theorem 2 of [2] with  $w(\zeta) = p(\zeta)$ .

**Remark 7.** For  $\kappa = 1$  we have a new result for  $(h, m)$ -convex functions:

**Corollary 3.** If  $\phi : [\varsigma_1, \frac{\varsigma_2}{m}] \rightarrow \mathbb{R}$  is a  $L^1$  function and if  $|\psi'|^q \in N_{h,m}^{s,2} [\varsigma_1, \frac{\varsigma_2}{m}]$ , then for  $q \geq 1$  the following inequality

$$(18) \quad \left| \mathbb{D} + \frac{2}{\varsigma_2 - \varsigma_1} \left( J_{\varsigma_1+}^w \phi \left( \frac{\varsigma_1 + \varsigma_2}{2} \right) + J_{\varsigma_2-}^w \phi \left( \frac{\varsigma_1 + \varsigma_2}{2} \right) \right) \right| \\ \leq \frac{\varsigma_2 - \varsigma_1}{2} \cdot B_q \cdot \left\{ \left( |\phi'(\varsigma_1)|^q C_{11} + m \left| \phi' \left( \frac{\varsigma_2}{m} \right) \right|^q C_{12} \right)^{\frac{1}{q}} \right. \\ \left. + \left( |\phi'(\varsigma_1)|^q C_{21} + m \left| \phi' \left( \frac{\varsigma_2}{m} \right) \right|^q C_{22} \right)^{\frac{1}{q}} \right\}$$

holds, with  $\mathbb{D}$  as before,  $B_q = \left( \int_0^1 |w(t)|^p dt \right)^{\frac{1}{p}} = \left( \int_0^1 |w(t)|^{\frac{q}{q-1}} dt \right)^{1-\frac{1}{q}}$ ,

$$C_{11} = \int_0^1 h^s \left( \frac{\zeta}{2} \right) d\zeta, \quad C_{12} = \int_0^1 \left( 1 - h \left( \frac{\zeta}{2} \right) \right)^s d\zeta, \\ C_{21} = \int_0^1 h^s \left( \frac{2 - \zeta}{2} \right) d\zeta, \quad C_{22} = \int_0^1 \left( 1 - h \left( \frac{2 - \zeta}{2} \right) \right)^s d\zeta.$$

**Remark 8.** With  $\kappa = 1$  and the change of variables  $t \rightarrow (t+1)$ , we obtain Theorem 6 from [32] for convex functions (i.e.  $m = s = 1$  and  $h(t) = t$ ), with  $w_1(t) = \frac{tb+(1-t)a}{4}$  in  $I_1$  and  $w_2(t) = -\frac{ta+(1-t)b}{4}$  for  $I_2$ .

**Remark 9.** Under the above assumptions also we have the Theorem 11 of [33].

**Theorem 4.** If  $\psi : [\varsigma_1, \frac{\varsigma_2}{m}] \rightarrow \mathbb{R}$  is a  $L^1$  function and if  $|\psi'|^q \in N_{h,m}^{s,2} [\varsigma_1, \frac{\varsigma_2}{m}]$ , then for  $q > 1$  we have

$$(19) \quad \left| \mathbb{D} + \frac{1 + \kappa}{\varsigma_2 - \varsigma_1} \left( J_{\left(\frac{\varsigma_1 + \kappa \varsigma_2}{1 + \kappa}\right)+}^w \psi(\varsigma_2) + J_{\left(\frac{\kappa \varsigma_1 + \varsigma_2}{1 + \kappa}\right)-}^w \psi(\varsigma_1) \right) \right| \\ \leq \frac{\varsigma_2 - \varsigma_1}{1 + \kappa} D \left\{ \left( |\psi'(\varsigma_1)|^q D_{11} + m \left| \psi' \left( \frac{\varsigma_2}{m} \right) \right|^q D_{12} \right)^{\frac{1}{q}} \right. \\ \left. + \left( |\psi'(\varsigma_1)|^q D_{21} + m \left| \psi' \left( \frac{\varsigma_2}{m} \right) \right|^q D_{22} \right)^{\frac{1}{q}} \right\}$$

with  $\mathbb{D}$  as before,  $D = \left( \int_0^1 |w(\zeta)| d\zeta \right)^{1-\frac{1}{q}}$ ,

$$D_{11} = \int_0^1 |w(\zeta)| h^s \left( \frac{\zeta}{1 + \kappa} \right) d\zeta, \quad D_{12} = \int_0^1 |w(\zeta)| \left( 1 - h \left( \frac{\zeta}{1 + \kappa} \right) \right)^s d\zeta, \\ D_{21} = \int_0^1 |w(\zeta)| h^s \left( \frac{1 + \kappa - \zeta}{1 + \kappa} \right) d\zeta, \quad D_{22} = \int_0^1 |w(\zeta)| \left( 1 - h \left( \frac{1 + \kappa - \zeta}{1 + \kappa} \right) \right)^s d\zeta.$$

*Proof.* As before, taking into account Lemma 1, we have

$$(20) \quad \left| \int_0^1 |w(\zeta)| \psi' \left( \frac{\zeta}{1+\kappa} \varsigma_1 + \frac{1+\kappa-\zeta}{1+\kappa} \varsigma_2 \right) d\zeta \right. \\ \left. - \int_0^1 |w(\zeta)| \psi' \left( \frac{1+\kappa-\zeta}{1+\kappa} \varsigma_1 + \frac{\zeta}{1+\kappa} \varsigma_2 \right) d\zeta \right| \\ \leq \int_0^1 |w(\zeta)| \left| \psi' \left( \frac{\zeta}{1+\kappa} \varsigma_1 + \frac{1+\kappa-\zeta}{1+\kappa} \varsigma_2 \right) \right| d\zeta \\ + \int_0^1 |w(\zeta)| \left| \psi' \left( \frac{1+\kappa-\zeta}{1+\kappa} \varsigma_1 + \frac{\zeta}{1+\kappa} \varsigma_2 \right) \right| d\zeta$$

and by using well-known power mean inequality, we have

$$(21) \quad \int_0^1 |w(\zeta)| \left| \psi' \left( \frac{\zeta}{1+\kappa} \varsigma_1 + \frac{1+\kappa-\zeta}{1+\kappa} \varsigma_2 \right) \right| d\zeta \\ \leq \left( \int_0^1 |w(\zeta)| d\zeta \right)^{1-\frac{1}{q}} \left( \int_0^1 |w(\zeta)| \left| \psi' \left( \frac{\zeta}{1+\kappa} \varsigma_1 + \frac{1+\kappa-\zeta}{1+\kappa} \varsigma_2 \right) \right|^q d\zeta \right)^{\frac{1}{q}}$$

and

$$(22) \quad \int_0^1 |w(\zeta)| \left| \psi' \left( \frac{1+\kappa-\zeta}{1+\kappa} \varsigma_1 + \frac{\zeta}{1+\kappa} \varsigma_2 \right) \right| d\zeta \\ \leq \left( \int_0^1 |w(\zeta)| d\zeta \right)^{1-\frac{1}{q}} \left( \int_0^1 |w(\zeta)| \left| \psi' \left( \frac{1+\kappa-\zeta}{1+\kappa} \varsigma_1 + \frac{\zeta}{1+\kappa} \varsigma_2 \right) \right|^q d\zeta \right)^{\frac{1}{q}}.$$

By using the modified  $(h, m)$ -convexity of  $|\psi'|^q$ , we get

$$(23) \quad \int_0^1 |w(\zeta)| \left| \psi' \left( \frac{\zeta}{1+\kappa} \varsigma_1 + \frac{1+\kappa-\zeta}{1+\kappa} \varsigma_2 \right) \right|^q d\zeta \\ \leq \int_0^1 |w(\zeta)| \left[ h^s \left( \frac{\zeta}{1+\kappa} \right) |\psi'(\varsigma_1)|^q + m \left( 1 - h \left( \frac{\zeta}{1+\kappa} \right) \right)^s \left| \psi' \left( \frac{\varsigma_2}{m} \right) \right|^q \right] d\zeta \\ = |\psi'(\varsigma_1)|^q \int_0^1 w(\zeta) h^s \left( \frac{\zeta}{1+\kappa} \right) d\zeta \\ + m \left| \psi' \left( \frac{\varsigma_2}{m} \right) \right|^q \int_0^1 |w(\zeta)| \left( 1 - h \left( \frac{\zeta}{1+\kappa} \right) \right)^s d\zeta.$$

Similarly,

$$(24) \quad \int_0^1 |w(\zeta)| \left| \psi' \left( \frac{1+\kappa-\zeta}{1+\kappa} \varsigma_1 + \frac{\zeta}{1+\kappa} \varsigma_2 \right) \right|^q d\zeta \\ \leq |\psi'(\varsigma_1)|^q \int_0^1 |w(\zeta)| h^s \left( \frac{1+\kappa-\zeta}{1+\kappa} \right) d\zeta \\ + m \left| \psi' \left( \frac{\varsigma_2}{m} \right) \right|^q \int_0^1 |w(\zeta)| \left( 1 - h \left( \frac{1+\kappa-\zeta}{1+\kappa} \right) \right)^s d\zeta.$$

If we put (23) and (24) in (21) and in (22), it allows us to obtain the inequality (19). The proof is completed.  $\square$

**Remark 10.** Results known in the literature, which can be obtained from Theorem 4, by taking  $\kappa = 0$ , we can get the following special cases:

- (i.01) For the  $s$ -convex functions we can obtain the Theorem 12 of [33] by using  $I_1$  with  $w(\zeta) = (1 - \zeta)^\alpha - \zeta^\alpha$ .
- (i.02) Theorem 3.2 of [26] (see Remarks 5(i09)).
- (i.03) Theorem 5 of [2], by using  $I_1$  for  $w(t) = p(t)$ .
- (i.04) Theorem 9 of [23] (see Remarks 3(ii)) and  $|\psi|^q$  is  $s$ -convex.
- (i.05) Theorem 2.3 of [3] by using  $I_1$  with  $w(\zeta) = 1 - 2\zeta$ .
- (i.06) Theorem 7 of [37] by using  $I_1$  with  $w(\zeta) = (1-\zeta)^\alpha - \zeta^\alpha$  and  $|\psi|^q$  is  $s$ -convex.
- (i.10) Theorem 3.6 of [14] by using  $I_1$  with  $w(\zeta) = (1 - \zeta)^{\frac{\alpha}{k}} - \zeta^{\frac{\alpha}{k}}$  and  $|\psi|^q$  is  $(h - m)$ -convex.

**Remark 11.** With  $\kappa = 1$  and using the new variables  $\zeta \rightarrow t+1$ , we obtain Theorem 7 from [32] for convex functions,  $m = s = 1$  and  $h(t) = t$ , with  $w(t) = \frac{(tb+(1-t)a}{4}$  for  $I_1$  and  $w(t) = -\frac{(ta+(1-t)b}{4}$  for  $I_2$ .

**Remark 12.** Under the assumptions of above remark, also we have the Theorem 12 of [33].

**Theorem 5.** Let  $\psi : [\varsigma_1, \varsigma_2] \rightarrow \mathbb{R}$  is a  $L^1$  function. If  $|\psi'|$  on the interval  $[\varsigma_1, \varsigma_2]$  satisfies the condition of the Lagrange Theorem, then the following inequality holds:

$$(25) \quad \left| \mathbb{D} + \frac{1 + \kappa}{\varsigma_2 - \varsigma_1} \left( J_{\left(\frac{\varsigma_1 + \kappa \varsigma_2}{1 + \kappa}\right)^+}^w \psi(\varsigma_2) + J_{\left(\frac{\kappa \varsigma_1 + \varsigma_2}{1 + \kappa}\right)^-}^w \psi(\varsigma_1) \right) \right| \leq \frac{(\varsigma_2 - \varsigma_1) |\psi''(\zeta)|}{1 + \kappa} \int_0^1 |w(\zeta)| \left| 1 - \frac{2\zeta}{1 + \kappa} \right| d\zeta,$$

where  $\mathbb{D}$  was defined earlier,

$$\zeta \in \begin{cases} (c, d), & \text{for } \zeta \leq \frac{1+\kappa}{2}; \\ (d, c), & \text{for } \zeta > \frac{1+\kappa}{2}, \end{cases} \quad c = \frac{\zeta}{1 + \kappa} \varsigma_1 + \frac{1 + \kappa - \zeta}{1 + \kappa} \varsigma_2, \quad d = \frac{1 + \kappa - \zeta}{1 + \kappa} \varsigma_1 + \frac{\zeta}{1 + \kappa} \varsigma_2.$$

*Proof.* From Lemma 1 and modulus properties, we can write

$$(26) \quad \left| \mathbb{D} + \frac{1 + \kappa}{\varsigma_2 - \varsigma_1} \left( J_{\left(\frac{\varsigma_1 + \kappa \varsigma_2}{1 + \kappa}\right)^+}^w \psi(\varsigma_2) + J_{\left(\frac{\kappa \varsigma_1 + \varsigma_2}{1 + \kappa}\right)^-}^w \psi(\varsigma_1) \right) \right| \leq \int_0^1 |w(\zeta)| \left| \psi' \left( \frac{\zeta}{1 + \kappa} \varsigma_1 + \frac{1 + \kappa - \zeta}{1 + \kappa} \varsigma_2 \right) - \psi' \left( \frac{1 + \kappa - \zeta}{1 + \kappa} \varsigma_1 + \frac{\zeta}{1 + \kappa} \varsigma_2 \right) \right| d\zeta.$$

By using the Lagrange Theorem condition, for the right side of the inequality (26), we get

$$(27) \quad \left| \psi' \left( \frac{\zeta}{1 + \kappa} \varsigma_1 + \frac{1 + \kappa - \zeta}{1 + \kappa} \varsigma_2 \right) - \psi' \left( \frac{1 + \kappa - \zeta}{1 + \kappa} \varsigma_1 + \frac{\zeta}{1 + \kappa} \varsigma_2 \right) \right| \leq |\psi''(\zeta)| \left| \frac{\zeta}{1 + \kappa} \varsigma_1 + \frac{1 + \kappa - \zeta}{1 + \kappa} \varsigma_2 - \frac{1 + \kappa - \zeta}{1 + \kappa} \varsigma_1 - \frac{\zeta}{1 + \kappa} \varsigma_2 \right| = |\psi''(\zeta)| \left| \left( 1 - \frac{2\zeta}{1 + \kappa} \right) (\varsigma_2 - \varsigma_1) \right|.$$

By taking into account (26) and (27), we have (25). □

**Corollary 4.** If we choose  $w(\zeta) = 1 - 2\zeta$  and  $\kappa = 0$  in Theorem 5, then we get

$$(28) \quad \left| \frac{\psi(\varsigma_2) + \psi(\varsigma_1)}{2} - \frac{1}{\varsigma_2 - \varsigma_1} \int_{\varsigma_1}^{\varsigma_2} \psi(\zeta) d\zeta \right| \leq \frac{\varsigma_2 - \varsigma_1}{12} \|\psi''\|_\infty,$$

where  $\|\psi''\|_\infty = \max_{\zeta \in [\varsigma_1, \varsigma_2]} (|\psi''(\zeta)|)$ .

*Proof.* From (25), we get

$$\begin{aligned} \mathbb{D} &= w(0) (\psi(\varsigma_1) + \psi(\varsigma_2)) - w(1) \left( \psi\left(\frac{\kappa\varsigma_1 + \varsigma_2}{1 + \kappa}\right) + \psi\left(\frac{\varsigma_1 + \kappa\varsigma_2}{1 + \kappa}\right) \right) \\ &= (w(0) - w(1)) (\psi(\varsigma_2) + \psi(\varsigma_1)) = 2 (\psi(\varsigma_2) + \psi(\varsigma_1)), \end{aligned}$$

since  $w'(\zeta) = -2$ ,

$$\begin{aligned} &\left| \mathbb{D} + \frac{2}{\varsigma_2 - \varsigma_1} \left( J_{\varsigma_1^+}^w \psi(\varsigma_2) + J_{\varsigma_2^-}^w \psi(\varsigma_1) \right) \right| \\ &= \left| 2 [\psi(\varsigma_2) + \psi(\varsigma_1)] - \frac{4}{\varsigma_2 - \varsigma_1} \int_{\varsigma_1}^{\varsigma_2} \psi(\zeta) d\zeta \right| \end{aligned}$$

and

$$\int_0^1 |w(\zeta)| \left| 1 - \frac{2\zeta}{1 + \kappa} \right| d\zeta = \int_0^1 |1 - 2\zeta| |1 - 2\zeta| d\zeta = \int_0^1 (1 - 2\zeta)^2 d\zeta = \frac{1}{3}.$$

Thus, we have

$$\left| 2 [\psi(\varsigma_2) + \psi(\varsigma_1)] - \frac{4}{\varsigma_2 - \varsigma_1} \int_{\varsigma_1}^{\varsigma_2} \psi(\zeta) d\zeta \right| \leq \frac{(\varsigma_2 - \varsigma_1) \|\psi''\|_\infty}{3}.$$

By dividing both parts of the last inequality by 4, we obtain (28).  $\square$

**Remark 13.** For  $\varsigma_2 - \varsigma_1 > 1$ , it is easy to verify that the upper estimate (28) was better than the estimate obtained in [27, Proposition 2] and confirm in a number of subsequent studies, for example, see [4, Corollary 3.1] and [5, Corollary 2], for more details.

### 3. SOME APPLICATION TO SPECIAL MEANS

Let us present some applications in which the obtained results can be used. Known mean values of the real numbers  $\alpha$  and  $\beta$ ; see Pearce and J. Pečarić [25]:

- (i) Arithmetic mean:  $A(\alpha, \beta) = \frac{\alpha + \beta}{2}$ ;
- (ii) Quadratic mean:  $Q(\alpha, \beta) = \sqrt{\alpha^2 + \beta^2}$ ;
- (iii) Geometric mean:  $G(\alpha, \beta) = \sqrt{\alpha\beta}$ , where  $\alpha\beta \geq 0$ ;
- (iv) Harmonic mean:  $H(\alpha, \beta) = \frac{2\alpha\beta}{\alpha + \beta}$ , where  $\alpha + \beta \neq 0$ ;
- (v) Logarithmic mean:  $L(\alpha, \beta) = \frac{\beta - \alpha}{\ln \beta - \ln \alpha}$ , where  $\alpha, \beta > 0$  and  $\alpha \neq \beta$ .

Now, by using some results, we give special means of positive real numbers to some applications.

**Example 1.** In Corollary 1, if we take  $\psi(x) = \sqrt{1 + x^2}$  defined on the interval  $(0, \varsigma_2)$  as a function, then we get

$$\begin{aligned} &\left| A \left\{ A[1, Q(\varsigma_2, 1)], Q\left(\frac{\varsigma_2}{2}, 1\right) \right\} - A \left[ Q(\varsigma_2, 1), \ln(2A(\varsigma_2, Q(\varsigma_2, 1)))^{\frac{1}{\varsigma_2}} \right] \right| \\ &\leq \frac{\varsigma_2^2}{16} Q^{-1}(\varsigma_2, 1) \end{aligned}$$

and from (28), we have

$$\left| A(1, Q(\varsigma_2, 1)) - A \left[ Q(\varsigma_2, 1), \ln(2A(\varsigma_2, Q(\varsigma_2, 1)))^{\frac{1}{\varsigma_2}} \right] \right| \leq \frac{\varsigma_2}{12}.$$

**Example 2.** In Corollary 1, if we take  $\psi(x) = \frac{1}{x^2}$  defined on the interval  $(\varsigma_1, \varsigma_2)$ , where  $\varsigma_1, \varsigma_2 \in \mathbb{R}^+$  as a function, then we get

$$|A [H^{-1}(\varsigma_1^2, \varsigma_2^2), A^{-2}(\varsigma_1, \varsigma_2)] - G^{-2}(\varsigma_1^2, \varsigma_2^2)| \leq \frac{\varsigma_2 - \varsigma_1}{3} H^{-1}\left(\varsigma_1^{\frac{3}{2}}, \varsigma_2^{\frac{3}{2}}\right)$$

and from (28), we have

$$|H^{-1}(\varsigma_1^2, \varsigma_2^2) - G^{-2}(\varsigma_1^2, \varsigma_2^2)| \leq \frac{\varsigma_2 - \varsigma_1}{2\varsigma_1^4}.$$

**Example 3.** In Corollary 1, if we take  $\psi(x) = \frac{1}{x}$  defined on the interval  $(\varsigma_1, \varsigma_2)$ , where  $\varsigma_1, \varsigma_2 \in \mathbb{R}^+$  as a function, then we get

$$|A [H^{-1}(\varsigma_1, \varsigma_2), A^{-1}(\varsigma_1, \varsigma_2)] - L^{-1}(\varsigma_1, \varsigma_2)| \leq \frac{\varsigma_2 - \varsigma_1}{16} H^{-1}(\varsigma_1^2, \varsigma_2^2)$$

and from (28), we have

$$|H^{-1}(\varsigma_1, \varsigma_2) - L^{-1}(\varsigma_1, \varsigma_2)| \leq \frac{\varsigma_2 - \varsigma_1}{6\varsigma_1^3}.$$

#### 4. CONCLUSIONS

In this article, we considered new integral inequalities of the Hadamard type for the weighted integrals. Throughout the article, we have shown that many known results are our particular cases. The generality of the results lies in the fact that they are valid for various classes of convex functions defined in some closed subset of a non-negative numerical semi-axis, for example,  $s$ -convex in the second sense,  $h$ -convex,  $m$ -convex, and for a number of other composite classes of convex functions. These results can easily be extended to the case of  $(h, m)$ -convex functions of the first type.

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