ALGEBRAIC GROUPS AND TRIANGULAR QUASI-QUADRATIC SYSTEMS

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ABSTRACT. We introduce the notion of q'-compactness for connected algebraic groups. We characterize the conditions when the lattice of all closed connected subgroups of connected algebraic group turn out to be prevalent in determining the algebraic structure of the group is a strongly algebraically closed algebra. Also, we study the relationships between the families of groups and the triangular quasi-quadratic systems.

1. INTRODUCTION

The objective of this paper is to study the algebraically closed objects in some classes of algebras. An algebraically closed algebraic structure is formulated and recent studies in algebraic geometry by A. Myasnikov, E. Daniyarova, O. Kharalampovich, V. Remeslennikov and others (see [3], [10]-[14], [16], [23], [25], and [31]) are further studied and discussed. We observe that R. Schmid [30] has first considered the subgroup lattices of a group and A. Shevlyikov has proposed some new problems in universal geometry which are related by Boolean equations in [32]. In this aspect, A. Shevlyikov and A. Molkhasi in [24] further proved that if a complete Boolean lattice is q'-compact, then it is a strong algebraically closed lattice.

It is noted that G. Gratzer [19] observed that all complemented distributive lattices are closely related to Boolean algebras and A. Sherlyikov [32] also observed that two Boolean algebras B_1, B_2 are geometrically equivalent if and only if any system equations over B_1, B_2 are isomorphic.

In this paper, we first define the concept of q'-compactness for a connected algebraic group G so that we obtain some necessary conditions

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for those lattices of its closed connected subgroup G to be a strong closed algebraic closed algebras. Finally, we also study the the triangular quasi- quadratic systems on free orderable groups. Furthermore, we also consider the kernel of Frobeniu groups of even order, the equally Noetherian groups to torsion-free hyperbolic groups and Engel groups [36].

2. STRONGLY ALGEBRAICALLY CLOSED CONNECTED ALGEBRAIC GROUPS

In this section we mention some basic notions of systems of equations and algebraic sets. We refer the reader to [2] for details.

Throughout this section, $\mathcal{L}$ is an algebraic language and A is an algebra of type $\mathcal{L}$. If we attach the elements of A as constants to $\mathcal{L}$, then we simply denote the new language by $\mathcal{L}(A)$. Now, we call an algebra B of type $\mathcal{L}(A)$ an $A-$algebra of the map $a \hookrightarrow a^B$ is an embedding of A into B. Note that a^B denotes the interpretation of the constant symbol a in B. We assume that $X = \{x_1, \ldots, x_k\}$ is a finite set of variables. We denote the term algebra in the language $\mathcal{L}$ and variables from X by $T_{\mathcal{L}}(X)$, and similarly the term algebra in the extended language $\mathcal{L}(A)$ will denoted by $T_{\mathcal{L}(A)}(X)$. An equation is a pair (p, q) of the elements of the term algebra $T_{\mathcal{L}}(X)$. In many cases, we assume that such an equation is the same as the atomic formula $p(x_1, \ldots, x_n) \approx q(x_1, \ldots, x_n)$ or briefly $p \approx q$. Hence, the set $At_{\mathcal{L}}(X)$ of atomic formula in the language $\mathcal{L}$. Any subset $S \subseteq At_{\mathcal{L}}(X)$ is called a *system of equations* in the language $\mathcal{L}$. Suppose B is an A-algebra. An element $(b_1, \ldots, b_n) \in B^n$ will be denoted by $\bar{b}$. Let S be a system of equations with coefficients in A. Then the set

$$V_B(S) = \{\bar{b} \in B^n \mid \forall(p \approx q) \in S,\, p^B(\bar{b}) = q^B(\bar{b})\}$$

is called an *algebraic set*. A structure A is algebraically closed iff every finite set of equations with parameters in A has a solution in some extension of A.

We recall from [19] that a lattice L is relatively complemented if every interval $[x, y]$ of lattice is complemented. A complement in $[x, y]$ of $a \in [x, y]$ is called a relative complement of a. We say that the algebra A is finitary equational Noetherian, if every finitary system of equations in the language $\mathcal{L}(A)$ is reducible over A to a finite system.

Let G is connected algebraic group. Then the set ΛG of an algebraic group G forms a lattice with the definition $A \wedge B = A \cap B$ and $A \vee B$ be the algebraic subgroup generated by A and B.

Theorem 2.1. *Suppose G is connected algebraic group and ΛG is the complete lattice of all closed connected subgroups of G. If G is IM-group and ΛG is q'-compact, then ΛG is a strongly algebraically closed algebra.*

Proof. By Theorem B of [28], ΛG is a relatively complemented lattice iff G is IM-group. Thus, ΛG is a relatively complemented lattice. On the other hand, we know that ΛG is algebraically closed lattice iff ΛG is a relatively complemented lattice [30]. Notice that ΛG is a complete lattice. Therefore, by [32], ΛG is a weak equational Noetherian Boolean algebra and a finite system T equivalent to S over ΛG. Using the result [30] of Schmid, every finite $S_0 \subseteq S$ is consistent and has a solution in ΛG and then S has a solution in some ultra-power $B = \Lambda G^I/\mathcal{U}$. Note that B is also a distributive lattice, and since it is an elementary extension of ΛG, and ΛG is q'-compact then $Rad_{\Lambda G}(S) = Rad_B(S)$. On the other hand, we have $Rad_{\Lambda G}(S) = Rad_{\Lambda G}(T)$. Since T is finite, we have $Rad_{\Lambda G}(T) = Rad_B(T)$. This shows that S and T are equivalent over B. Therefore, T has a solution in B and consequently in ΛG. Thus S has a solution in ΛG. Note that T is a finite system in the language $\mathcal{L}(\Lambda G)$. But by introducing a finite number of new variables and a finite number of new equations we can transform it to a finite system in $\mathcal{L}(\Lambda G)$. To do this, we perform the following actions:

1- If T contains the boolean constants 0 and 1, then there will be no change, since $0, 1 \in \Lambda G$.

2- If T contains therm x', then we introduce a new variable y and insert new equations $x \wedge y \approx 0$ and $x \vee y \approx 1$, instead.

3- If there appears a term of the form a', then again there will not be any changes. □

3. Systems of equations and the NTQ systems

In this section, we recall some definitions that we need to them. Suppose that G is a fixed group and $X = \{x_1, \ldots, x_k\}$ is a finite set of variables. The free product of G and the free group $F(X) = \langle x_1, \ldots, x_k \rangle$ freely generated by $\{x_, \ldots, x_k\}$ denoted by:

$$G[X] = G[x_1, \ldots, x_k] = G * F(X).$$

We sometimes say that $\{x_1, \ldots, x_k\}$ freely generates the free G-group $G[X]$. Recall that G-groups can be linked to algebras over a unitary commutative ring, more specially a field, with G playing the role of the coefficient a ring. We now call a group H is a G-group if it contains a designed copy of G which will for the most part identify with G. Obviously, the group G itself is a G-group. Let H be a G-group. Then we write $H^k = \{(a_1, \ldots, a_k) \mid a_i \in H\}$. We can easily see that H^k is an affine $k-$space over H. The elements of H^k can be regarded as points. A subset of G^n is closed if it is the intersection of an arbitrary number of finite unions of algebraic sets. Thus, we have defined a topology on G^n which is the well known Zariski Topology on G^n. Now, we call a group G equationally Noetherian if and only if for each $n \geq 1$, the Zariski Topology on G^n is Noetherian. With the above notion, we call a subgroup F of the group G Noetherian if for every k greater than 0 and every subset S of $G[x_1, \ldots, x_k]$, there exists a finite subset S_0 of S such that $V_H(S) = V_H(S_0)$, as usual. Now, we observe that $V_H(S) = \{v \in H^n \mid f(v) = 1, \text{ for all } f \in S\}$. When $G = H$, we simply call the group G equational Noetherian. Recall that term

$$v = (a_1, \ldots, a_k) \in H^k$$

is a root of a polynomial in the non-commuting variables $x_1, \ldots, x_k$, with coefficient in G if $f(v) = f(a_1, \ldots, a_k, g_1, \ldots, g_t) = 1$, where the constants $g_1, \ldots, g_t \in G$ [32].

We continue to study the systems of equations over free product of groups which were considered by M. Casals-Ruiz and I.V. Kazachkov in [8]. For an arbitrary element $s \in G[x]$ the for all equation $S = 1$ can be treated as an equation over G, where G is a fixed group generated by a set A and $X = \{X_1, X_2, \ldots, X_n\}$. In general, for a subset $S \in G[X]$, the formal equation $S = 1$ can be regarded as a system of equations over G with coefficients in A.

A finite system of equations $S = 1$ is called *triangular quasi-quadratic* over group G if it can be partitioned into the following subsystems

$$\begin{gathered} S_1(X_1, X_2, \ldots, X_n, A) = 1, \\ S_2(X_2, \ldots, X_n, A) = 1, \\ \vdots \\ S_n(X_n, A) = 1, \end{gathered}$$

where for each $1 \leq i \leq n$ one of the following holds:

(1) S_i is a quadratic equation in variables X_i;
(2) $S_i = \{[y, z] = 1, [y, u] = 1, [z, u] = 1 \mid y, z \in X_i\}$, where u is a group word in $X_{i+1} \cup \ldots \cup X_n \cup A$;
(3) $S_i = \{[y, z] = 1 \mid y, z \in X_i\}$;

(4) S_i is the empty equation.

With the above notations, we define $G_i = G_{R(S_i, \ldots, S_n)}$ and we set $G_{n+1} = G$. Now, we call the triangular quasi-Quadratic system $S = 1$, the non-degenerated or simply NTQ if the following two conditions hold:

(A) Each system $S_i = 1$, where $X_{i+1}, \ldots, X_n$ are viewed as the corresponding constants from G_{i+1} (under the canonical maps $X_j \longrightarrow G_{i+1}$, $j = i+1, \ldots, n$) has a solution in G_{i+1};

(B) The element in G_{i+1} represented by the word u from case (3) above, is not a proper power in G_{i+1}.

It is noted that the Frobenius groups and the 2-Frobenius groups were first studied by G.Y. Chen in [9]. Throughout this section, we let G be a finite group acting transitively on a set X , with more than one elements. It is known that for an element $g \in G$ and two distinct elements x and y if $gx = x$ and $gy = y$, then $g = 1$ (see [9]). Now, we state the following main theorem in our paper.

Theorem 3.1. *Let G be a the kernel of a Frobenius group of even order. If there exists an abelian group such that the its subgroups lattice is isomorphic to the lattice of normal subgroups of G, then G is an equationally Noetherian group and every closed set in affine k-space H^k is a union of finitely many images of algebraic sets of the NTQ systems, which $H = G[x_1, \ldots, x_k]$ is the free product of G and the free group $F(X) = \langle x_1, \ldots, x_n \rangle$ freely generated by $\{x_1, \ldots, x_k\}$.*

Proof. Let G be the kernel of a Frobenius group of even order. By ([22], Lemma 2.1), G is a nilpotent group. Suppose that there exists an abelian group C such that $N(G)$ the lattice of normal subgroups of G is isomorphic to $N(C)$, the subgroup lattice of C, and

$$\theta : N(G) \rightarrow N(C)$$

be an isomorphism and G is non-abelian group. We observe that the group C is decomposed as a direct sum of its p-components such that $C = \bigoplus_p C_p$ and we will have $G = \prod_p \theta^{-1}(C_p)$. Since G is non-abelian, so there exists a prime q such that $\theta^{-1}(C_q)$ is a non-abelian because G is a torsian group (see [30]). Consequently, we may assume that C is a G-group. By apply Theorem 9.1.11, in [30], we can easily see that the group C is a locally cyclic group and the lattice of groups of C forms a chain (see [23]). Since G is not abelian, then the nilpotency class t of G is greater than 2 for $t \in \mathbb{N}$ and we can find a k such that

$G_{t-1} = M_k$, where $N(G) = \{1 < M_1 < \ldots M_k < \ldots\}$ (see [30]). Hence, we can immediately see that $G[X] = G[x_1, ..., x_k]$ is a $G-$equationally Noetherian group. Now, we consider group G/G_{t-2}. We set $H = G/G_{t-2}$ and $Z(H) = G_{t-1}/G_{t-2}$. It is clear that $H/Z(H)$ is an abelian group. If we note that the form subgroups of $H/Z(H)$ and using Section 3 of [15] we can say that $H/Z(H)$ is union of an ascending chain of cyclic subgroups. We know that if $x, y \in H$, then we can find an element u of H such that $xZ(H), yZ(H) \in \langle uZ(H)\rangle$ and $x = u^k v$, $y = u^\ell z$ for some integers k and ℓ and some elements $v, z \in Z(H)$. Then $xy = yx$ and H is an abelian that it is a contradiction. So, G is an abelian. On the other hand, every abelian group is equationally Noetherian (see Theorem 1 of [2]). Also, we know if G is an equationally Noetherian group, then free G-group $G[x_1, \ldots, x_k]$ is G-equationally Noetherian (see [2]). So, $G[X] = G[x_1, \ldots, x_k]$ is G-equationally Noetherian group. By ([2], Corollary 11), we have that if H is G-group and it is a equationally Noetherian group, then every closed set in $H^k = \{(a_1, \ldots, a_k) \mid a_i \in H\}$, affine k-space over H, is a finite union of algebra sets. Here, we set $H = G[X]$. By [8], every algebraic set over G can be decomposed as a union of finitely many images of algebraic sets of the NTQ systems. Finally, we conclude that every closed set in affine k-space H^k is union of finitely many images of algebraic sets of the NTQ systems. Thus, we have proved that if G is the kernel of a Frobenius group of even order and the lattice of normal subgroups of G is isomorphic to the subgroup lattice of an abelian group, then every close set in an affine k-space over H^k is union of finitely many images of algebraic sets of the NTQ systems. □

Corollary 3.2. *Let G be the kernel of a Frobenius group of even order and N be a normal subgroup of G which is a finite union of algebraic sets over G. If there exists an abelian group such that the its subgroups lattice is isomorphic to the lattice of normal subrgoups of G, then G/N is also an equationally Noetherian.*

Proof. By Theorem 4.1, G is an abelian group. In [2], it is proved that abelian groups are equationally Noetherian. By ([3], Theorem 1), we see that of G is equationally Noetherian and V is a normal subgroup of G which is a finite number of alphabets over G, then G/N is also equationally Noetherian. This completes the proof. □

We recall the definition of a CSA-group with an involution. Note that CSA-groups are in particular commutative transitive, i.e. groups in which the relation $[x, y] = 1$ is an equivalence relation on the set of nontrivial elements, or equivalently in which the centralizer of every nontrivial element is an abelian. For more details about CSA-groups, we refer the reader [17] and [26]. A *CSA-group* is a group G in which every maximal abelian subgroup $\mathbf{A}$ is malnormal i.e. satisfies $\mathbf{A}\cap\mathbf{A}^g = 1$ for all $g \in G\backslash\mathbf{A}$.

A presentation$(\mathtt{A}, \mathfrak{R})$ is called finite if both sets $\mathtt{A}$ and $\mathfrak{R}$ are finite. A group is *finitely presented* if it has a finite presentation. Let λ be a number in $[0, 1]$. Let $(\mathtt{A}, \mathfrak{R})$ be a group presentation, and $\widehat{\mathfrak{R}}$ be the symmetrization of $\mathfrak{R}$. Then the set $\mathfrak{R}$, or the presentation $(\mathtt{A}, \mathfrak{R})$, is said to satisfy the condition $C'(\lambda)$ if $|X| \leq \lambda|\mathtt{R}|$ for any $\mathtt{R} \in \widehat{\mathfrak{R}}$ and for any subword X of $\mathtt{R}$ which is a piece relative to $\widehat{\mathfrak{R}}$.

We say that a group presentation $(\mathtt{A}, \mathfrak{R})$ is *singularly aspherical* if it is diagrammatically aspherical, concise, and no element of $\mathfrak{R}$ can be decomposed as a concatenation of several copies.

Theorem 3.3. *Suppose G is a finitely presented group with an involution which has a presentation satisfying $C'(\frac{1}{6})$. If G is a group with singularly spherical presentation. Then every closed set in affine k-space H^k is a union of finitely many images of algebraic sets of the NTQ systems, which $H = G[x_1, \ldots, x_k]$ is the free product of G and the free group $F(X) = \langle x_1, \ldots, x_n\rangle$ freely generated by $\{x_1, \ldots, x_k\}$.*

Proof. By ([20], Theorem 33), a finitely presented group which has a presentation satisfying $C'(\frac{1}{6})$ is hyperbolic. On the other hand, we know every group with singularly spherical presentation is torsion-free ([35], Lemma 64). Thus, G torsion-free hyperbolic group. By ([26], Proposition 12), G is a CAS-group. So by ([26], Remark 7), G is an abelian group. Therefore, the assertion holds by Theorem 4.1. □

We recall the definition of an H-limit group. Suppose H and G are groups. Then $(f_i \mid f_i : G \longrightarrow H,\ i \in \mathbb{N})$ is a sequence of morphisms from G to H. We say that $f_{i\in\mathbb{N}}$ is convergent or stable if for every $g \in G$ one of the following sets is finite:

$$\{i \in \mathbb{N} \mid f_i(g) = 1\},\ \{i \in \mathbb{N} \mid f_i(g) \neq 1\}.$$

We set $\underrightarrow{\ker}(f_i) = \{g \in G \mid |\{i \in \mathbb{N} \mid f_i(g) \neq 1\}| \text{ is finite}\}$.

A group K is said to be *H-limit* if there exist a group G and a converging sequence $(f_i \mid f_i : G \longrightarrow H,\ i \in \mathbb{N})$ such that $k = G/\underrightarrow{\ker}(f_i)$, where $(f_i \mid f_i : G \longrightarrow H\ i \in \mathbb{N})$ is a sequence of morphisms from group G to group H.

Recall that limit groups are heavily connected to residual properties. Let G be a group and $\mathcal{K}$ a class of groups. We say that G is residually-$\mathcal{K}$ (or that $\mathcal{K}$ separates G) if for every $g \in G$ there exist $\mathbf{K} \in \mathcal{K}$ and a morphism $f : G \longrightarrow \mathbf{K}$ such that $f(g) \neq 1$.

Corollary 3.4. *Let G be a CSA-group with an involution. Then any finitely generated G-limit group is an equationally Noetherian group.*

Proof. Since, G is a CSA-group with an involution, then G is an equationally Noetherian group. Now, we prove that any finitely generated G-limit group is equationally Noetherian. We have G is finitely generated G-pseudo-limit group. Then any sequence of epimorphisms of finitely generated residually-G groups is also a sequence of epimorphisms of finitely generated G-pseudo-limit groups; thus terminates after finitely many steps G is an equationally Noetherian. So, G is an equationally Noetherian group ([27], Corollary 2.10). □

4. The NTQ systems for Engel groups

In this section we recall Engel groups of Lie algebras. We refer the reader to [6], [21], and [34] for details. Engel groups and left and right Engel elements in groups have been extensively studied since the 1950s and even justify a subsection in the AMS classification scheme. An element x of a group G is said a (left) *Engel element* if for any $g \in G$ there exists $n = n(x, g) \geq 1$ such that $[g_n, x] = 1$. Suppose first that $[g_0, x] = g$ and the commutator $[g_n, x] = 1$ is defined recursively by the rule

$$[g_n, x] = 1 = [[g_{n-1}, x], x].$$

If n can be chosen independent of g, then x is a (left) n-Engel element. A group G is called an *n-Engel group* if all elements of G are n-Engel. Recently groups with n-Engel word-values were considered [4], [33].

Recall that a partially ordered set $(L, \leq)$ is a lattice if $\sup\{a, b\}$ and $\inf\{a, b\}$ exist for all $a, b \in L$. The group G is called *orderable* if there exists a full order relation $\leq$ on the set G such that every pair of elements has a least upper bound and greatest lower bound and for any $x, y, a, b \in G$:

$$x \leq y \Longrightarrow axb \leq ayb.$$

By $\gamma_i(G)$, we mean the i-th term of the lower central series of a group G. Now, we will require the following Theorem, due to Burns and Medvedev [5].

Theorem 4.1. *Let $n \geq 1$. Then exist constant $c = c(n)$ and $e = e(n)$ depending only on n such that, if G is a finite n-Engel group, then the exponent of $\gamma_c(G)$ divides e.*

We are now ready to prove the following theorem.

Theorem 4.2. *Let m, n be positive integers and G be orderable or free group. For all $x \in G$, if x^m is n-Engel and there exists an abelian group such that the lattice of normal subgroups of G is isomorphism to the subgroup lattice of it, then G is equationally Noetherian group and every closed set in H^k is a union of finitely many images of algebraic sets of NTQ systems, which $H = G[x_1, \ldots, x_k]$.*

Proof. In the free group G with $\aleph$ generators, let $[G, G] = G_1$, and let $G_{t+1} = [G, G_t]$, where $[H, K]$ stands for the subgroup generated by all commutator $x^{-1}y^{-1}xy$, $x \in H$, $y \in K$. Then, by a theorem of Magnus and Witt we will have $\bigcap G_t = 0$, the group identity. On the other hand, we know that if a group G has a well-ordered central series ending with the identity such that all quotient-groups $G_t \diagup G_{t+1}$ are torsion-free, then G is orderable. Using ([34], Theorem 1.2), we observe that G is nilpotent. Because assume that K is f.g torsion-free nilpotent. Then it is a residually finite p-group which p is a prime number. As a result for any finitely generated subgroup K of G we have $\gamma_c(H) = 1$ and here $\gamma_c(G) = 1$ and G is nilpotent of class at most $c - 1$ which $c = c(n)$ in Theorem 3.2. Notice that, we proved that G is nilpotent. By applying that ([1], Theorem 2), G is abelian group. On the other hand, every abelian group is equationally Noetherian and by attention if G is an equationally Noetherian group, then free G-group $H = G[x_1, \ldots, x_k]$ is G-equationally Noetherian. Also, if H is a G-equationally Noetherian group, then every closed set in H^k is a finite union of algebraic sets, see [2]. As a result, it is a union of finitely many images of algebraic sets of NTQ systems (see [8]). □

Example 4.3. Let m, n be positive integers and G be orderable or free group and N be a subgroup of G which is a finite union of algebraic sets over G. For all $x \in G$, if x^m is n-Engel and there exists an abelian group such that the lattice of normal subgroups of G is isomorphism to the subgroup lattice of it, then G/N is also equationally Noetherian. In Theorem 6.2, we proved that G is abelian group. Because in [2], it is proved that abelian groups are equationally Noetherian and consequently, G/N is equationally Noetherian (see [7]).

Example 4.4. Let m, n be positive integers and G be orderable or free group. For all $x \in G$, if x^m is n-Engel and there exists an abelian group such that the lattice of normal subgroups of G is isomorphism to the subgroup lattice of it, then any finitely generated G-limit group is equationally Noetherian group. Because we proved that G is abelian group and as a result, G is equationally Noetherian group. Now, by applying that Corollary 2.10 of [27], completes the proof.

Conclusion

Another description of the solution sets of a system of equations was first given by M. Casals-Ruiz and I.V. Kazachkov [8] in terms of the NTQ system (ω residually free towers). We notice that the NTQ system has now become an essentially tool in the study if the fully residually free groups (limit groups). We conclude that exist the relationships between the families of groups and the triangular quasi-quadratic systems and every closed set in H^k union of finitely many images of algebraic sets of NTQ systems, which H is a G-equationally Noetherian group.

Declarations

Conflicts of interest: The author declare that there is no conflict of interests regarding the publication of this paper.

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