FIRST p-STEKLOV EIGENVALUE UNDER GEODESIC CURVATURE FLOW

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ABSTRACT. We study the first nonzero p-Steklov eigenvalue on a two-dimensional compact Riemannian manifold with a smooth boundary along the geodesic curvature flow. We prove that the first nonzero p-Steklov eigenvalue is nondecreasing if the initial metric has positive geodesic curvature on boundary ∂M and Gaussian curvature is identically equal to zero in M along the un-normalized geodesic curvature flow. An eigenvalue estimation is also obtained along the normalized geodesic curvature flow.

1. INTRODUCTION

Let (M^n, g) be a compact Riemannian manifold of dimension n with smooth boundary ∂M. For $u \in C^\infty(M)$, we consider the following p-Steklov eigenvalue problem

$$\Delta_p u = 0, \quad \text{in } M,$$
$$|\nabla u|^{p-2}\frac{\partial u}{\partial \nu} = \lambda |u|^{p-2}u, \quad \text{on } \partial M, \tag{1.1}$$

where $\Delta_p u = \nabla(|\nabla u|^{p-2}\nabla u)$, $p \in (1, \infty)$, is the p-Laplace operator and $\frac{\partial u}{\partial \nu}$ is the outer normal derivative of u. The above problem reduces to the classical Steklov eigenvalue problem when $p = 2$. For the p-Steklov eigenvalue problem [17, 18], there is a sequence of nonnegative eigenvalues

$$0 \leq \lambda_1(p) \leq \lambda_2(p) \leq \lambda_3(p) \leq \cdots .$$

The operator Δ_p is conformally covariant [6], i.e., functions which are p-harmonic with respect to g are also p-harmonic with respect to $\tilde{g}$ and vice versa, where $\tilde{g} = e^u g$ is a conformal metric. Variational formula for the first nonzero p-Steklov eigenvalue $\lambda_1(p)$ is given by

$$\lambda_1(p) = inf\Big\{\frac{\int_M |\nabla_g u(t)|^p dA_g}{\int_{\partial M} |u(t)|^p dS_g} : 0 \neq u \in C^\infty(M), \int_{\partial M} |u(t)|^{p-2}u(t)dS_g = 0\Big\}. \tag{1.2}$$

Definition 1.1. *A Riemannian metric on a two-dimensional manifold is called a flat metric if its Gaussian curvature is identically equal to zero.*

Definition 1.2. *A two-dimensional Riemannian manifold with flat metric is called a flat Riemannian surface.*

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Throughout the paper we consider (M, g_0) is a compact flat Riemannian surface with a smooth boundary ∂M.

In determining geometry and topology of a Riemannian manifold, the study of eigenvalue of geometric operators plays a crucial role. Perelman [13] proved that the first eigenvalue of $-4\Delta + R$, where R is the scalar curvature, is nondecreasing along the Ricci flow. After that eigenvalues of different geometric operators on a Riemannian manifold evolves by geometric flows were studied by many authors, for instance see [4, 5, 8, 14, 15, 16]. Studying geometric flows is also an active area of research in geometry. Osgood, Phillips and Sarnak [12] proved the existence of a conformal metric with Gaussian curvature identically equal to zero in M and constant geodesic curvature on ∂M. In [2, 3], Brendle studied geodesic curvature flow on a surface with boundary. To study more results related to prescribing geodesic curvature, one can see [1, 7, 19]. Recently in [9], Ho and Koo studied the first nonzero Steklov eigenvalue on a compact Riemannian surface with a smooth boundary along the geodesic curvature flow. In [10], the so called canonical deformation is introduced. The canonical deformation applies to any smooth simply connected (probably multi-sheet) planar domain regardless to the geodesic curvature of the boundary. Given such a domain Ω, let Ω_t $(t \in [0, \infty))$ be the canonical deformation of the domain and $\zeta_{\Omega_t}(s)$, the Steklov zeta-function of Ω_t. The main result of the paper is that $\zeta_{\Omega_t}(s)$ does not increase in t for any real s. The domain Ω_t converges to the round disk of the same perimeter as Ω when $t \to \infty$ in the C^∞ topology.

In section 2, we study the first nonzero p-Steklov eigenvalue along the un-normalized geodesic curvature flow and proved that the first nonzero p-Steklov eigenvalue is nondecreasing along the flow if the initial metric has positive geodesic curvature on ∂M and Gaussian curvature is identically equal to zero in M. In section 3, we derive an eigenvalue estimation of the first nonzero p-Steklov eigenvalue along the normalized geodesic curvature flow.

2. p-Steklov eigenvalue along un-normalized geodesic curvature flow

Let (M, g_0) be a compact flat Riemannian surface with smooth boundary ∂M. The un-normalized geodesic curvature flow [9] is defined by

$$\begin{aligned} &\frac{\partial}{\partial t} g(t) = -2k_{g(t)} g(t) \quad on \quad \partial M, \\ &K_{g(t)} = 0 \quad \text{in} \quad M, \ g(0) = g_0, \end{aligned} \tag{2.1}$$

where $k_{g(t)}$ is the geodesic curvature of ∂M and $K_{g(t)}$ is the Gaussian curvature of M.

Following [9], clearly for a general metric $g(t) = e^{2u(t)} g_0$ conformal to g_0, the un-normalized geodesic curvature flow (2.1) reduces to

$$\frac{\partial}{\partial t} u(t) = -k_{g(t)} \quad \text{on} \quad \partial M. \tag{2.2}$$

Lemma 2.1. [9] *Along the un-normalized geodesic curvature flow, we have*

$$\min_{\partial M} k_{g(t)} \geq \min_{\partial M} k_{g_0}. \tag{2.3}$$

Lemma 2.2. *Let $g(t),\ t \in [0,T)$ be a solution of the un-normalized geodesic curvature flow (2.1) and $\lambda(t)$ be the corresponding first nonzero p-Steklov eigenvalue. Then for any $t_2 \geq t_1,\ t_1, t_2 \in [0,T)$, we have*

$$\lambda(t_2) \geq \lambda(t_1) + p \int_{t_1}^{t_2} \int_{\partial M} |\nabla_{g(t)} f(t)|^{p-2} \frac{\partial f(t)}{\partial t} \frac{\partial f(t)}{\partial \nu_{g(t)}} dS_{g(t)} dt, \tag{2.4}$$

where $f(t)$ is a smooth function on $M \times [0,T)$ satisfying

$$\Delta_{p,g(t)} f(t) = 0 \text{ in } M, \int_{\partial M} |f(t)|^{p-2} f(t) dS_{g(t)} = 0 \text{ and } \int_{\partial M} |f(t)|^p dS_{g(t)} = 1, \tag{2.5}$$

such that $f(t_2)$ is the corresponding eigenfunction of $\lambda(t_2)$.

Proof. At time $t = t_2$, $f(t_2)$ is the corresponding eigenfunction of the first p-Steklov eigenvalue $\lambda(t_2)$. Now, we consider a smooth function on ∂M by

$$h(t) = \left(\frac{e^{u(t_2)}}{e^{u(t)}} \right)^{\frac{1}{p-1}} f(t_2), \tag{2.6}$$

where $u(t)$ is the solution of (2.2). We normalized this function on ∂M by

$$f(t) = \frac{h(t)}{(\int_{\partial M} |h(t)|^p dS_{g(t)})^{\frac{1}{p}}}. \tag{2.7}$$

Extend this function to a p-harmonic function in M with respect to $g(t)$, which we shall continue to denote as $f(t)$ (see [11]). Now, we have

$$\begin{aligned}
\int_{\partial M} |f(t)|^{p-2} f(t) dS_{g(t)} &= \frac{1}{(\int_{\partial M} |h(t)|^p dS_{g(t)})^{1-\frac{1}{p}}} \int_{\partial M} |h(t)|^{p-2} h(t) dS_{g(t)} \\
&= \frac{1}{(\int_{\partial M} |h(t)|^p dS_{g(t)})^{1-\frac{1}{p}}} \int_{\partial M} \left(\frac{e^{u(t_2)}}{e^{u(t)}} \right) |f(t_2)|^{p-2} f(t_2) e^{u(t)} dS_{g_0} \\
&= \frac{1}{(\int_{\partial M} |h(t)|^p dS_{g(t)})^{1-\frac{1}{p}}} \int_{\partial M} |f(t_2)|^{p-2} f(t_2) dS_{g(t_2)} = 0,
\end{aligned}$$

and

$$\int_{\partial M} |f(t)|^p dS_{g(t)} = \frac{1}{(\int_{\partial M} |h(t)|^p dS_{g(t)})} \int_{\partial M} |h(t)|^p dS_{g(t)} = 1.$$

Set

$$G(g(t), f(t)) = \int_M |\nabla_{g(t)} f(t)|^p dA_{g(t)}, \tag{2.8}$$

which is a smooth function on t. Taking derivative with respect to t, we obtain

$$\begin{aligned}
\mathcal{G}(g(t), f(t)) := \frac{d}{dt} G(g(t), f(t)) &= \int_M \frac{\partial}{\partial t} |\nabla_{g(t)} f(t)|^p dA_{g(t)} \\
&= p \int_M |\nabla_{g(t)} f(t)|^{p-2} \langle \nabla_{g(t)} f(t), \nabla_{g(t)} f_t(t) \rangle dA_{g(t)}.
\end{aligned}$$

Now using the Stokes theorem, we have

$$\frac{d}{dt} G(g(t), f(t)) = p \int_{\partial M} |\nabla_{g(t)} f(t)|^{p-2} \frac{\partial f(t)}{\partial t} \frac{\partial f(t)}{\partial \nu_{g(t)}} dS_{g(t)}.$$

Using the definition of $\mathcal{G}(g(t), f(t))$, we get

$$G(g(t_2), f(t_2)) - G(g(t_1), f(t_1)) = \int_{t_1}^{t_2} \mathcal{G}(g(t), f(t))dt. \tag{2.9}$$

Since $f(t_2)$ is the corresponding eigenfunction of the p-Steklov eigenvalue $\lambda(t_2)$, we deduce

$$G(g(t_2), f(t_2)) = \lambda(t_2) \int_{\partial M} |f(t_2)|^p dS_{g(t_2)} = \lambda(t_2). \tag{2.10}$$

Again from the variational formula for the first p-Stekolv eigenvalue, we infer

$$G(g(t_1), f(t_1)) \geq \lambda(t_1) \int_{\partial M} |f(t_1)|^p dS_{g(t_1)} = \lambda(t_1). \tag{2.11}$$

Finally using (2.10) and (2.11) in (2.9), we have (2.4). □

Theorem 2.1. *Under the un-normalized geodesic curvature flow on a compact Riemannian manifold M with smooth boundary ∂M, the first p-Steklov eigenvalue is nondecreasing if the initial metric g_0 has positive geodesic curvature on ∂M and the Gaussian curvature is identically equal to zero in M.*

Proof. Since $f(t_2)$ is the corresponding eigenfunction of the p-Steklov eigenvalue $\lambda(t_2)$, we have

$$\begin{aligned} &\int_{\partial M} |\nabla_{g(t_2)} f(t_2)|^{p-2} \frac{\partial f(t_2)}{\partial t} \frac{\partial f(t_2)}{\partial \nu_{g(t_2)}} dS_{g(t_2)} \\ &\qquad = \lambda(t_2) \int_{\partial M} |f(t_2)|^{p-2} f(t_2) \frac{\partial f(t_2)}{\partial t} dS_{g(t_2)}. \end{aligned} \tag{2.12}$$

Differentiating $\int_{\partial M} |f(t)|^p dS_{g(t)} = 1$, we get

$$\begin{aligned} p \int_{\partial M} |f(t)|^{p-2} f(t) \frac{\partial f(t)}{\partial t} dS_{g(t)} &= - \int_{\partial M} |f(t)|^p \frac{\partial}{\partial t} (e^{u(t)} dS_{g(0)}) \\ &= - \int_{\partial M} |f(t)|^p \frac{\partial u(t)}{\partial t} dS_{g(t)} \\ &= \int_{\partial M} |f(t)|^p k_{g(t)} dS_{g(t)} \\ &\geq (\min_{\partial M} k_{g(0)}) \int_{\partial M} |f(t)|^p dS_g(t) = \min_{\partial M} k_{g(0)}. \end{aligned} \tag{2.13}$$

Thus,

$$\int_{\partial M} |\nabla_{g(t_2)} f(t_2)|^{p-2} \frac{\partial f(t_2)}{\partial t} \frac{\partial f(t_2)}{\partial \nu_{g(t_2)}} dS_{g(t_2)} \geq \frac{\lambda(t_2)}{p} (\min_{\partial M} k_{g(0)}). \tag{2.14}$$

It is clear by assumption that $\min_{\partial M} k_{g(0)} > 0$, hence for t sufficiently close to t_2, we deduce

$$\int_{\partial M} |\nabla_{g(t)} f(t)|^{p-2} \frac{\partial f(t)}{\partial t} \frac{\partial f(t)}{\partial \nu_{g(t)}} dS_{g(t)} \geq 0. \tag{2.15}$$

Hence using Lemma 2.2, we can conclude that $\lambda(t_2) \geq \lambda(t_1)$ for any $t_1 (< t_2)$ sufficiently close to t_2. Since t_2 is arbitrary, hence the proof is complete. □

3. p-Steklov eigenvalue along normalized geodesic curvature flow

With the initial metric g_0, in this section we consider the following normalized geodesic curvature flow [9] defined by

$$\begin{aligned} \frac{\partial}{\partial t} g(t) &= -2(k_{g(t)} - \bar{k}_{g(t)})g(t) \quad \text{on} \quad \partial M, \\ K_{g(t)} &= 0 \quad \text{in} \quad M, \qquad g(0) = g_0, \end{aligned} \tag{3.1}$$

where $k_{g(t)}$ and $K_{g(t)}$ are defined as in (2.1). Here $\bar{k}_{g(t)}$ is the average of geodesic curvature on ∂M given by

$$\bar{k}_{g(t)} = \frac{\int_{\partial M} k_{g(t)} dS_{g(t)}}{\int_{\partial M} dS_{g(t)}}. \tag{3.2}$$

It is proved in [3], the above initial value problem (3.1) has a solution on a small time interval. Also it is clear form [9], under the conformal change $g(t) = e^{2u(t)}g_0$, the normalized geodesic curvature flow (3.1) reduces to

$$\frac{\partial}{\partial t} u(t) = -(k_{g(t)} - \bar{k}_{g(t)}) \quad \text{on} \quad \partial M. \tag{3.3}$$

Along the normalized geodesic curvature flow

$$\frac{d}{dt}\left(\int_{\partial M} dS_{g(t)}\right) = -\int_{\partial M} (k_{g(t)} - \bar{k}_{g(t)}) dS_{g(t)} = 0, \tag{3.4}$$

which implies that

$$\int_{\partial M} dS_{g(t)} = \int_{\partial M} dS_{g_0} \quad \text{for all} \quad t \geq 0. \tag{3.5}$$

Lemma 3.1. *Let $g(t),\ t \in [0, T)$ be a solution of the normalized geodesic curvature flow (3.1) and $\lambda(t)$ be the corresponding first nonzero p-Steklov eigenvalue. Then for any $t_2 \geq t_1,\ t_1, t_2 \in [0, T)$, we have*

$$\lambda(t_2) \geq \lambda(t_1) + p \int_{t_1}^{t_2} \int_{\partial M} |\nabla_{g(t)} f(t)|^{p-2} \frac{\partial f(t)}{\partial t} \frac{\partial f(t)}{\partial \nu_{g(t)}} dS_{g(t)} dt, \tag{3.6}$$

where $f(t)$ is a smooth function on $M \times [0, T)$ satisfying

$$\Delta_{p,g(t)} f(t) = 0 \text{ in } M, \ \int_{\partial M} |f(t)|^{p-2} f(t) dS_{g(t)} = 0 \text{ and } \int_{\partial M} |f(t)|^p dS_{g(t)} = 1, \tag{3.7}$$

such that $f(t_2)$ is the corresponding eigenfunction of $\lambda(t_2)$.

Proof. The proof is similar as Lemma 2.2. □

Theorem 3.1. *Under the normalized geodesic curvature flow on a compact Riemannian manifold M with smooth boundary ∂M, the first nonzero p-Steklov eigenvalue is nondecreasing if for the initial metric g_0, $(\min_{\partial M} k_{g(t)} - \bar{k}_{g(t)}) \geq 0$ on ∂M and Gaussian curvature is identically equal to zero in M.*

Proof. Since $f(t_2)$ is the corresponding eigenfunction of the p-Steklov eigenvalue $\lambda(t_2)$, we have

$$\begin{aligned}\int_{\partial M} |\nabla_{g(t_2)} f(t_2)|^{p-2} \frac{\partial f(t_2)}{\partial t} \frac{\partial f(t_2)}{\partial \nu_{g(t_2)}} dS_{g(t_2)} &= \lambda(t_2) \int_{\partial M} |f(t_2)|^{p-2} f(t_2) \frac{\partial f(t_2)}{\partial t} dS_{g(t_2)} \\ &= -\frac{\lambda(t_2)}{p} \int_{\partial M} |f(t_2)|^p \frac{\partial u(t_2)}{\partial t} dS_{g(t_2)} \\ &= \frac{\lambda(t_2)}{p} \int_{\partial M} |f(t_2)|^p (k_{g(t_2)} - \bar{k}_{g(t_2)}) dS_g(t_2) \\ &\geq \frac{\lambda(t_2)}{p} \left(\min_{\partial M} k_{g(t_2)} - \bar{k}_{g(t_2)} \right). \end{aligned} \tag{3.8}$$

Rest of the proof is same as the method applied in Theorem 2.1. □

Proposition 3.1. *Along the normalized geodesic curvature flow (3.1), the first nonzero p-Steklov eigenvalue $\lambda(t)$ satisfies*

$$\frac{d}{dt} \log \lambda(t) \geq \left(\min_{\partial M} k_{g(t)} - \bar{k}_{g(t)} \right) \quad \textit{for all} \quad t, \tag{3.9}$$

where on the left side, the derivative is in the sense of the lim inf *of backward difference quotients.*

Proof. Using (3.7) and the fact that $f(t_2)$ is the corresponding eigenfunction of the first nonzero p-Steklov eigenvalue $\lambda(t_2)$, we have

$$\begin{aligned}\int_{\partial M} |\nabla_{g(t_2)} f(t_2)|^{p-2} \frac{\partial f(t_2)}{\partial t} \frac{\partial f(t_2)}{\partial \nu_{g(t_2)}} dS_{g(t_2)} &= \lambda(t_2) \int_{\partial M} |f(t_2)|^{p-2} f(t_2) \frac{\partial f(t_2)}{\partial t} dS_{g(t_2)} \\ &= -\frac{\lambda(t_2)}{p} \int_{\partial M} |f(t_2)|^p \frac{\partial u(t_2)}{\partial t} dS_{g(t_2)} \\ &= \frac{\lambda(t_2)}{p} \int_{\partial M} |f(t_2)|^p (k_{g(t_2)} - \bar{k}_{g(t_2)}) dS_g(t_2) \\ &\geq \frac{\lambda(t_2)}{p} \left(\min_{\partial M} k_{g(t_2)} - \bar{k}_{g(t_2)} \right). \end{aligned} \tag{3.10}$$

Hence for any $\epsilon > 0$, we have that

$$\int_{\partial M} |\nabla_{g(t)} f(t)|^{p-2} \frac{\partial f(t)}{\partial t} \frac{\partial f(t)}{\partial \nu_{g(t)}} dS_{g(t)} \geq \frac{\lambda(t_2)}{p} \left(\min_{\partial M} k_{g(t)} - \bar{k}_{g(t)} - \epsilon \right) \tag{3.11}$$

for t sufficiently closed to t_2. Thus the Lemma 3.1 gives

$$\lambda(t_2) - \lambda(t_1) \geq \lambda(t_2) \int_{t_1}^{t_2} \left(\min_{\partial M} k_{g(t)} - \bar{k}_{g(t)} - \epsilon \right) dt. \tag{3.12}$$

for t_1 sufficiently closed to t_2 and $t_2 > t_1$. Now dividing the equation (3.12) by $t_2 - t_1$ and taking $t_1 \to t_2$, we obtain

$$\liminf_{t_1 \to t_2} \frac{\lambda(t_2) - \lambda(t_1)}{t_2 - t_1} \geq \lambda(t_2) \left(\min_{\partial M} k_{g(t_2)} - \bar{k}_{g(t_2)} - \epsilon \right). \tag{3.13}$$

Using the same argument used (in (2.21), [8]), we can say that

$$\liminf_{t_1 \to t_2} \frac{\log \lambda(t_2) - \log \lambda(t_1)}{t_2 - t_1} \geq \frac{1}{\lambda(t_2)} \liminf_{t_1 \to t_2} \frac{\lambda(t_2) - \lambda(t_1)}{t_2 - t_1}. \tag{3.14}$$

Now (3.13) and (3.14) yields

$$\liminf_{t_1\to t_2}\frac{\log\lambda(t_2)-\log\lambda(t_1)}{t_2-t_1}\geq \min_{\partial M}\, k_{g(t_2)}-\bar{k}_{g(t_2)}-\epsilon. \tag{3.15}$$

Since ϵ is arbitrary, we have our result. □

Lemma 3.2. *Let $g(t),\ t\in[0,T)$ be a solution of the normalized geodesic curvature flow (3.1) and $\lambda(t)$ be the corresponding first nonzero p-Steklov eigenvalue. Then for any $t_2\geq t_1,\ t_1,t_2\in[0,T)$, we have*

$$\lambda(t_2)\leq\lambda(t_1)+p\int_{t_1}^{t_2}\int_{\partial M}|\nabla_{g(t)}f(t)|^{p-2}\frac{\partial f(t)}{\partial t}\frac{\partial f(t)}{\partial\nu_{g(t)}}dS_{g(t)}dt, \tag{3.16}$$

where $f(t)$ is a smooth function on $M\times[0,T)$ satisfying

$$\Delta_{p,g(t)}f(t)=0\ \textit{in}\ M,\ \int_{\partial M}|f(t)|^{p-2}f(t)dS_{g(t)}=0\ \textit{and}\ \int_{\partial M}|f(t)|^p dS_{g(t)}=1, \tag{3.17}$$

such that $f(t_1)$ is the corresponding eigenfunction of $\lambda(t_1)$.

Proof. We define a function on the boundary ∂M of M by

$$h(t)=\left(\frac{e^{u(t_1)}}{e^{u(t)}}\right)^{\frac{1}{p-1}}f(t_1), \tag{3.18}$$

where $u(t)$ is the solution of (3.3). We normalized the function on ∂M by

$$f(t)=\frac{h(t)}{(\int_{\partial M}|h(t)|^p dS_{g(t)})^{\frac{1}{p}}}. \tag{3.19}$$

Extend this function to a p-harmonic function in M with respect to $g(t)$, which we shall continue to denote as$f(t)$. Now we have

$$\int_{\partial M}|f(t)|^{p-2}f(t)dS_{g(t)}=\frac{1}{(\int_{\partial M}|h(t)|^p dS_{g(t)})^{1-\frac{1}{p}}}\int_{\partial M}|f(t_1)|^{p-2}f(t_1)dS_{g(t_1)}=0,$$

and

$$\int_{\partial M}|f(t)|^p dS_{g(t)}=\frac{1}{(\int_{\partial M}|h(t)|^p dS_{g(t)})}\int_{\partial M}|h(t)|^p dS_{g(t)}=1.$$

Set

$$G(g(t),f(t))=\int_M|\nabla_{g(t)}f(t)|^p dA_{g(t)}, \tag{3.20}$$

which is a smooth function on t. Taking derivative with respect to t, we get

$$\begin{aligned}\mathcal{G}(g(t),f(t)):=\frac{d}{dt}G(g(t),f(t))&=\int_M\frac{\partial}{\partial t}|\nabla_{g(t)}f(t)|^p dA_{g(t)}\\&=p\int_M|\nabla_{g(t)}f(t)|^{p-2}\langle\nabla_{g(t)}f(t),\nabla_{g(t)}f_t(t)\rangle dA_{g(t)}.\end{aligned}$$

So by using the Stoke's theorem, we obtain

$$\frac{d}{dt}G(g(t),f(t))=p\int_{\partial M}|\nabla_{g(t)}f(t)|^{p-2}\frac{\partial f(t)}{\partial t}\frac{\partial f(t)}{\partial\nu_{g(t)}}dS_{g(t)}.$$

Using the definition of $\mathcal{G}(g(t), f(t))$, we deduce

$$G(g(t_2), f(t_2)) - G(g(t_1), f(t_1)) = \int_{t_1}^{t_2} \mathcal{G}(g(t), f(t))dt. \tag{3.21}$$

Since $f(t_1)$ is the corresponding eigenfunction of the p-Steklov eigenvalue $\lambda(t_1)$, we conclude

$$G(g(t_1), f(t_1)) = \lambda(t_1) \int_{\partial M} |f(t_1)|^p dS_{g(t_1)} = \lambda(t_1). \tag{3.22}$$

Again form the variational formula for the first p-Stekolv eigenvalue, we have

$$G(g(t_2), f(t_2)) \geq \lambda(t_2) \int_{\partial M} |f(t_2)|^p dS_{g(t_2)} = \lambda(t_2). \tag{3.23}$$

Finally using (3.22) and (3.23) in (3.21), we arrive at (3.16). □

Proposition 3.2. *Under the normalized geodesic curvature flow the first nonzero p-Steklov eigenvalue $\lambda(t)$ satisfies*

$$\frac{d}{dt} \log \lambda(t) \leq \left(\max_{\partial M} k_{g(t)} - \bar{k}_{g(t)} \right) \quad \textit{for all} \quad t, \tag{3.24}$$

where on the left hand side, the derivative is in the sense of the lim sup *of backward difference quotients.*

Proof. By using (3.17) and since $f(t_1)$ is the corresponding eigenfunction of the first nonzero p-Steklov eigenvalue $\lambda(t_1)$, we have

$$\begin{aligned}
\int_{\partial M} |\nabla_{g(t_1)} f(t_1)|^{p-2} \frac{\partial f(t_1)}{\partial t} \frac{\partial f(t_1)}{\partial \nu_{g(t_1)}} dS_{g(t_1)} &= \lambda(t_1) \int_{\partial M} |f(t_1)|^{p-2} f(t_1) \frac{\partial f(t_1)}{\partial t} dS_{g(t_1)} \\
&= -\frac{\lambda(t_1)}{p} \int_{\partial M} |f(t_1)|^p \frac{\partial u(t_1)}{\partial t} dS_{g(t_1)} \\
&= \frac{\lambda(t_1)}{p} \int_{\partial M} |f(t_1)|^p (k_{g(t_1)} - \bar{k}_{g(t_1)}) dS_{g(t_1)} \\
&\leq \frac{\lambda(t_1)}{p} \left(\max_{\partial M} k_{g(t_1)} - \bar{k}_{g(t_1)} \right).
\end{aligned} \tag{3.25}$$

Thus, for any $\epsilon > 0$ we get

$$\int_{\partial M} |\nabla_{g(t)}|^{p-2} \frac{\partial f(t)}{\partial t} \frac{\partial f(t)}{\partial \nu_{g(t)}} dS_{g(t)} \leq \frac{\lambda(t_1)}{p} \left(\max_{\partial M} k_{g(t)} - \bar{k}_{g(t)} + \epsilon \right), \tag{3.26}$$

for t sufficiently closed to t_1 and $t_2 > t_1$. Hence by using (3.16), we find

$$\lambda(t_2) - \lambda(t_1) \leq \lambda(t_1) \int_{t_1}^{t_2} \left(\max_{\partial M} k_{g(t)} - \bar{k}_{g(t)} + \epsilon \right), \tag{3.27}$$

for t_1 sufficiently closed to t_2. Dividing both sides by $t_2 - t_1$ and taking $t_2 \to t_1$, it follows

$$\limsup_{t_2 \to t_1} \frac{\lambda(t_2) - \lambda(t_1)}{t_2 - t_1} \leq \lambda(t_1) \left(\max_{\partial M} k_{g(t_1)} - \bar{k}_{g(t_1)} + \epsilon \right). \tag{3.28}$$

By similar argument used (in (2.21), [8]), we get

$$\limsup_{t_2 \to t_1} \frac{\log \lambda(t_2) - \log \lambda(t_1)}{t_2 - t_1} \leq \max_{\partial M} k_{g(t_1)} - \bar{k}_{g(t_1)} + \epsilon. \tag{3.29}$$

Since $\epsilon > 0$ is arbitrary, we have (3.24). □

Theorem 3.2. *Assume that for a initial metric g_0, Gaussian curvature is identically equal to zero in M and ∂M has negative geodesic curvature. Also g_c is the metric conformal to g_0 with respect to which the Gaussian curvature identically equal to zero in M and constant geodesic curvature on ∂M such that the lengths of ∂M of g_c and g_0 are the same. If $\lambda(g_c)$ and $\lambda(g_0)$ are the first nonzero p-Steklov eigenvalue of g_c and g_0 respectively, then*

$$\left(1-\frac{\min\limits_{\partial M} k_{g_0}}{\max\limits_{\partial M} k_{g_0}}\right) \leq \log\frac{\lambda(g_c)}{\lambda(g_0)} \leq -\left(1-\frac{\min\limits_{\partial M} k_{g_0}}{\max\limits_{\partial M} k_{g_0}}\right). \tag{3.30}$$

Proof. It was proved in [3] that $g \to g_\infty$ as $t \to \infty$ along the normalized geodesic curvature flow (3.1) such that g_∞ is conformal to g_0 and has constant geodesic curvature on ∂M and Gaussian curvature is identically equal to zero in M. Now from (3.5), we have

$$\int_{\partial M} dS_{g_\infty} = \int_{\partial M} dS_{g_0}. \tag{3.31}$$

By assumption it is given that

$$\int_{\partial M} dS_{g_c} = \int_{\partial M} dS_{g_0}. \tag{3.32}$$

From (3.31) and (3.32), we get

$$\int_{\partial M} dS_{g_\infty} = \int_{\partial M} dS_{g_c}. \tag{3.33}$$

Now from Gauss-Bonnet theorem, it follows that

$$k_{g_\infty}\int_{\partial M} dS_{g_\infty} = \int_M K_{g_\infty} dA_{g_\infty} + \int_{\partial M} k_{g_\infty} dS_{g_\infty} = 2\pi\chi(M) \tag{3.34}$$

and

$$k_{g_c}\int_{\partial M} dS_{g_c} = \int_M K_{g_c} dA_{g_c} + \int_{\partial M} k_{g_c} dS_{g_c} = 2\pi\chi(M), \tag{3.35}$$

where $\chi(M)$ is the Euler characteristic on M. It is given that for the initial metric g_0, M has Gaussian curvature which is identically equal to zero and ∂M has negative geodesic curvature, so it is clear that the Euler characteristic function is negative. So using (3.33), we find

$$k_{g_\infty} = k_{g_c} < 0. \tag{3.36}$$

If $g(t) = e^{2u(t)} g_0$ then we obtain

$$-\Delta_{g_0} u + k_{g_0} = k_g e^{2u} \quad \text{in} \quad M, \tag{3.37}$$

$$\frac{\partial u}{\partial \nu_{g_0}} + k_{g_0} = k_g e^{u} \quad \text{on} \quad \partial M, \tag{3.38}$$

where $\frac{\partial}{\partial \nu_{g_0}}$ is the normal derivative with respect to g_0.

From the Gauss-Bonnet theorem, (3.1), (2.3), and (3.5), we have

$$\bar{k}_{g(t)} = \frac{\int_M K_{g(t)} dA_{g(t)} + \int_{\partial M} k_{g(t) dS_{g(t)}}}{\int_{\partial M} dS_{g(t)}} = \frac{2\pi\chi(M)}{\int_{\partial M} dS_{g(t)}} \quad \text{for} \quad t \geq 0. \tag{3.39}$$

Hence g_c and g_∞ are conformal to g_0. With respect to all of them Gaussian curvature is identically equal to zero, if $g_c = e^{2v} g_0$ then we infer

$$\begin{cases} \Delta_{g_0} u = 0 \ \text{ in } \ M, \\ \frac{\partial u}{\partial \nu_{g_0}} + k_{g_0} = k_\infty e^u \ \text{ on } \ \partial M, \end{cases} \quad \textit{and} \quad \begin{cases} \Delta_{g_0} v = 0 \ \text{ in } \ M, \\ \frac{\partial v}{\partial \nu_{g_0}} + k_{g_0} = k_{g_c} e^v \ \text{ on } \ \partial M. \end{cases}$$

Since $k_\infty = k_{g_0}$, we obtain

$$\begin{aligned} &\Delta_{g_0}(u-v) = 0 \ \text{ in } \ M, \\ &\frac{\partial(u-v)}{\partial \nu_{g_0}} = k_{g_c}(e^u - e^v) \ \text{ on } \ \partial M. \end{aligned}$$

Thus

$$(u-v)\frac{\partial(u-v)}{\partial \nu_{g_0}} = k_{g_c}(e^u - e^v)(u-v) \ \text{ on } \ \partial M. \tag{3.40}$$

Integrating of above equation over ∂M with respect to g_0, we infer

$$0 \leq \int_M |\nabla_{g_0}(u-v)|^2 dA_{g_0} = \int_{\partial M} (u-v)\frac{\partial(u-v)}{\partial \nu_{g_0}} dS_{g_0} = k_{g_c} \int_{\partial M} (e^u - e^v)(u-v) dS_{g_0}. \tag{3.41}$$

On the other hand $k_{g_c} < 0$ and $(e^u - e^v)(u-v) \geq 0$, then the left hand side of (3.41) is non positive. Therefore $\int_{\partial M}(e^u - e^v)(u-v)dS_{g_0} = 0$ which yields $u = v$ on ∂M and since $u - v$ is harmonic in M, we get $u = v$ in M. It implies that $g_c = g_\infty$.

Again from Lemma 2.9 of [9], we have

$$k_{g(t)} \leq \bar{k}_{g_0} + \left(\max_{\partial M} k_{g_0} - \min_{\partial M} k_{g_0}\right) + \left(\max_{\partial M} k_{g_0}\right) \int_0^t \left(\max_{\partial M} k_{g(\tau)} - \bar{k}_{g(\tau)}\right) d\tau. \tag{3.42}$$

It follows from (3.39) and (3.42) that

$$\left(\max_{\partial M} k_{g_t} - \bar{k}_{g_t}\right) - \left(\max_{\partial M} k_{g_0} - \min_{\partial M} k_{g_0}\right) \leq \left(\max_{\partial M} k_{g_0}\right) \int_0^t \left(\max_{\partial M} k_{g(\tau)} - \bar{k}_{g(\tau)}\right) d\tau. \tag{3.43}$$

If $t \to \infty$, then

$$-\left(1 - \frac{\min_{\partial M} k_{g_0}}{\max_{\partial M} k_{g_0}}\right) \geq \int_0^\infty \left(\max_{\partial M} k_{g(\tau)} - \bar{k}_{g(\tau)}\right) d\tau. \tag{3.44}$$

Integrating (3.24) with respect to t on interval $[0, \infty)$ and using (3.44) and $g_c = g_\infty$, we conclude

$$\log \frac{\lambda(g_c)}{\lambda(g_0)} = \log \frac{\lambda(g_\infty)}{\lambda(g_0)} \leq \int_0^\infty \left(\max_{\partial M} k_{g(\tau)} - \bar{k}_{g(\tau)}\right) d\tau \leq -\left(1 - \frac{\min_{\partial M} k_{g_0}}{\max_{\partial M} k_{g_0}}\right). \tag{3.45}$$

From Lemma 2.10 of [9], we obtain

$$k_{g(t)} \geq \bar{k}_{g_0} - \left(\max_{\partial M} k_{g_0} - \min_{\partial M} k_{g_0}\right) + \left(\max_{\partial M} k_{g_0}\right) \int_0^t \left(\min_{\partial M} k_{g(\tau)} - \bar{k}_{g(\tau)}\right) d\tau. \tag{3.46}$$

Then we get

$$\left(\bar{k}_{g(t)} - \min_{\partial M} k_{g(t)}\right) - \left(\max_{\partial M} k_{g_0} - \min_{\partial M} k_{g_0}\right) \leq -\left(\max_{\partial M} k_{g_0}\right) \int_0^t \left(\min_{\partial M} k_{g(\tau)} - \bar{k}_{g(\tau)}\right) d\tau. \tag{3.47}$$

As $t \to \infty$, we conclude

$$\left(1 - \frac{\min_{\partial M} k_{g_0}}{\max_{\partial M} k_{g_0}}\right) \leq \int_0^\infty \left(\min_{\partial M} k_{g(\tau)} - \bar{k}_{g(\tau)}\right) d\tau. \tag{3.48}$$

Integrating (3.24) and using (3.48) and $g_c = g_\infty$, we infer

$$\log \frac{\lambda(g_c)}{\lambda(g_0)} = \log \frac{\lambda(g_\infty)}{\lambda(g_0)} \geq \int_0^\infty \left(\min_{\partial M} k_{g(\tau)} - \bar{k}_{g(\tau)}\right) d\tau \geq \left(1 - \frac{\min_{\partial M} k_{g_0}}{\max_{\partial M} k_{g_0}}\right). \tag{3.49}$$

This completes the proof of theorem. □

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  We declare that we do not have any commercial or associative interest that represents a conflict of interest in connection with the work submitted.
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