14.06.2023

Referee report
on the article "First p-Steklov eigenvalue under geodesic curvature flow"
by A. Saha, S. Azami, and S.K. Hui

The subject and results of the paper are very interesting as well as methods of proofs. Nevertheless, I have a number of questions, remarks and recommendations.

1. Abstract and many other points of the paper. The authors use the term "vanishing" as a synonym of "identically equal to zero". But the conventional sense of "vanishing" is "equal to zero at some points". I strongly recommend to check all enters of "vanishing" and to replace with "identically equal to zero" if needed.

The expression "vanishing Gauassian curvature on M" is used several times (for instance in Theorem 2.1). The following definitions are widely used in geometry:

(a) A Riemannian metric on a two-dimensional manifold is called *a flat metric* if its Gaussian curvature is identically equal to zero.

(b) A two-dimensional Riemannian manifold with flat metric is called *a flat Riemannian surface.*

I recommend to recall these definitions in Introduction and then to use the terms *flat metric* and *flat Riemannian surface* throughout the paper.

2. Page 1. The definition of the p-Laplacian is unclear for me. I guess there is a misprint: the formula $\Delta_p u = \Delta(|\nabla u|^{p-2}\nabla u)$ should be replaced with $\Delta_p u = \nabla(|\nabla u|^{p-2}\nabla u)$.

3. Page 1. The statement "The operator Δ_p is conformally covariant" is not quite obvious. A reference should be presented to a paper where this fact is proved.

4. Page 1. There are many careless sentences in the paper written in a bad presentation style. For example "The p-Steklov eigenvalue problem admits a sequence of nonnegative eigenvalues." A problem cannot "admit eigenvalues", but an operator can possess eigenvalues.

5. Page 1. The statement "The operator Δ_p has a sequence of nonnegative eigenvalues $\{0 \le \lambda_1 \le \lambda_2 \le \lambda_3 \le \dots\}$" is not quite obvious. A reference should be presented to a paper where this fact is proved. How about the multiplicity of eigenvalues? Is it true that $\lambda_k \to \infty$ as $k \to \infty$? I recommend to write $\lambda_k(p)$ instead of λ_k in order to emphasize the dependence on p, or may be the full notation $\lambda_k(M, g; p)$ should be used. What is known about the dependence of $\lambda_k(p)$ on p?

6. Page 1. The sentence "... functions which are p-harmonic with respect to g is also p-harmonic with respect to ..." should be replaced with "... functions which are p-harmonic with respect to g are also p-harmonic with respect to ...". This is an example of grammatically wrong sentence. There are many such examples in the paper.

8. Pages 1–2. A short survey of publications on the subject is presented in Introduction. I recommend to include the recent paper

[A. Jollivet and V. Sharafutdinov. An estimate for the Steklov zeta function of a planar domain derived from a first variation formula. J. Geom. Anal. (2022), 32:161.]

The so called *canonical deformation* is introduced in the paper. The canonical deformation applies to any smooth simply connected (probably multi-sheet) planar domain regardless to the geodesic curvature of the boundary. Given such a domain Ω, let Ω_t $(t \in [0,\infty))$ be the canonical deformation of the domain and $\zeta_{\Omega_t}(s)$, the Steklov zeta-function of Ω_t. The main result of the paper is that $\zeta_{\Omega_t}(s)$ does not increase in t for any real s. The domain Ω_t converges to the round disk of the same perimeter as Ω as $t \to \infty$ in the C^∞ topology.

By the way, a simply connected flat Riemannian surface is isometric to a planar (probably multi-sheet) domain $\Omega \subset \mathbb{R}^2$ with the metric induced by the standard Euclidean metric of $\mathbb{R}^2$. Therefore Jollivet – Sharafutdinov's result is somewhat similar to Saha–Azami–Hui's results. But in my opinion, the canonical deformation is much more general and interesting example of a geometric flow than the geodesic curvature flow.

Does the *p-Steklov zeta-function* make sense for a simply connected flat Riemannian surface? The question is closely related to the next one: What is known about the asymptotics of $\lambda_k(p)$ as $k \to \infty$? Besides this, the behavior of the first positive p-Steklov eigenvalue $\lambda_1(p)$ under the canonical deformation is worth of studying.

9. Page 2. We read at the beginning of page 2: "In this article we consider a Riemannian surface with vanishing Gaussian curvature and constant geodesic curvature on its boundary." But the definition presented at the beginning of Section 2 does not assume that the geodesic curvature is constant. As far as I understand the geodesic curvature is not assumed to be constant in Theorems 2.1 and 3.1. The hypothesis $k_g = \mathrm{const}$ appears in Theorem 3.2 only.

10. Page 2. The definition of *the un-normalized geodesic curvature flow* is presented in the form:

"Let (M, g_0) be a two-dimensional compact Riemannian manifold with smooth boundary ∂M. We consider the un-normalized geodesic curvature flow [2] by

$$\begin{aligned} &\frac{\partial}{\partial t} g(t) = -2k_{g(t)} g(t) \quad \text{on} \quad \partial M, \\ &K_{g(t)} = 0 \quad \text{in} \quad M, \end{aligned} \tag{2.1}$$

where $k_{g(t)}$ is the geodesic curvature of ∂M and $K_{g(t)}$ is the Gaussian curvature of M."

In my opinion, the first sentence "Let (M, g_0) be a two-dimensional compact Riemannian manifold with smooth boundary ∂M." should be replaced with "Let (M, g_0) be a compact flat Riemannian surface with smooth boundary ∂M." Otherwise the second line of (2.1) is hard understandable.

The second sentence "We consider the un-normalized geodesic curvature flow by" is another example of the careless style. The expression "We consider ... by" is wrong. The sentence should be replaces either with "We consider the un-normalized geodesic curvature flow defined by" or with "The un-normalized geodesic curvature flow is defined by".

Besides the above style remarks, I have the following mathematical question. Unfortunately, I do not understand how the cited definition is related to another definition of the geodesic curvature flow which I have known before. For simplicity, let us consider the case of $M \subset \mathbb{R}^2$ furnished with the metric induced by the standard Euclidean metric of $\mathbb{R}^2$. Can (M, g_t) be interpreted as a family of planar domains smoothly depending on t? If such an interpretation exists, it should be mentioned.

11. Page 2. For a general metric $g(t) = e^{2u(t)}g_0$ conformal to g_0, the un-normalized geodesic curvature flow (2.1) reduces to

$$\frac{\partial}{\partial t}u(t) = -k_{g(t)} \quad \text{on} \quad \partial M. \tag{2.2}$$

This is hard to understand. What of two metrics, g_0 and $g(t)$ is assumed to be flat? Again no reference is presnted to a paper where this fact is proved.

12. Page 2. The statement of Lemma 2.1 is finished with "where $f(t_2)$ is the corresponding eigenfunction of $\lambda(t_2)$." But $f(t_2)$ does not participate in formulas (2.4)–(2.5). According to the formulation of Lemma 2.1, "$f(t_2)$ is an eigenfunction of $\lambda(t_2)$" is one of statements of the lemma. But it is clear from the proof that "$f(t_2)$ is an eigenfunction of $\lambda(t_2)$" is actually one of hypotheses of the lemma. Thus, Lemma 2.1 is formulated in a bad form.

Besides that, the assumption $t_1, t_2 \in [0, T)$ should be included to hypotheses of Lemma 2.1.

13. Page 3. According to the definition of Δ_p presented on page 1, there is no restriction on the value of p, i.e. Δ_p is defined for all $p \in \mathbb{R}$ (or may be for all $p \in \mathbb{C}$). But formulas (2.6)–(2.7) assume that $p \neq 0$ and $p \neq 1$. These inequalities do not participate in hypotheses of Lemma 2.1. This remark is valid for many other formulas in the paper.

14. Page 3. The authors widely use the following fact: the Dirichlet problem

$$\Delta_p u = 0 \quad \text{on} \quad M, \qquad u|_{\partial M} = f$$

is uniquely solvable for a compact Riemannian manifold with boundary. In my opinion, the fact is not obvious since Δ_p is a nonlinear operator. The fact should be explicitly mentioned with corresponding references. Besides this, some properties of the solution (for example, smooth dependence on f) are used in the paper. All such properties should also be explicitly mentioned.

15. Page 3. We read after formula (2.7): "... which we still denote it by $f(t)$." It is a grammatically wrong sentence.

16. The articles "a", "an", "the" are used in a wrong way in many sentences of the paper and are absent in many other sentences where they are needed. I strongly recommend to check articles usage throughout the paper.

I stopped my detailed reading at the end of Section 2 and looked quickly through Section 3. I hope the authors have understood the requirement level for the rigor and presentation style to make corresponding corrections in Section 3 too.

Most probably, I will recommend the paper for publishing in SEMR. But I would like to look at the revised paper before giving the recommendation.