

FIRST p -STEKLOV EIGENVALUE UNDER GEODESIC CURVATURE FLOW

APURBA SAHA, SHAHROUD AZAMI AND SHYAMAL KUMAR HUI

ABSTRACT. We study the first nonzero p -Steklov eigenvalue on a two-dimensional compact Riemannian manifold with a smooth boundary along the geodesic curvature flow. We prove that the first nonzero p -Steklov eigenvalue is nondecreasing if the initial metric has positive geodesic curvature on boundary ∂M and vanishing Gaussian curvature in M along the unnormalized geodesic curvature flow. An eigenvalue estimation is also obtained along the normalized geodesic curvature flow.

1. INTRODUCTION

Let (M^n, g) be a compact Riemannian manifold of dimension n with smooth boundary ∂M . We consider the following p -Steklov eigenvalue problem

$$\begin{aligned} \Delta_p u &= 0, \quad \text{in } M, \\ |\nabla u|^{p-2} \frac{\partial u}{\partial \nu} &= \lambda |u|^{p-2} u, \quad \text{on } \partial M, \end{aligned} \tag{1.1}$$

where $\Delta_p u = \Delta(|\nabla u|^{p-2} \nabla u)$ is the p -Laplace operator and $\frac{\partial u}{\partial \nu}$ is the outer normal derivative of u . The above problem reduces to the classical Steklov eigenvalue problem when $p = 2$. The p -Steklov eigenvalue problem admits a sequence of nonnegative eigenvalues

$$0 \leq \lambda_1 \leq \lambda_2 \leq \lambda_3 \leq \dots$$

The operator Δ_p is conformally covariant, i.e., functions which are p -harmonic with respect to g is also p -harmonic with respect to \tilde{g} and vice versa, where $\tilde{g} = e^u g$ is a conformal metric. Variational formula for the first nonzero p -Steklov eigenvalue λ_1 is given by

$$\lambda_1 = \inf \left\{ \frac{\int_M |\nabla_g u(t)|^p dA_g}{\int_{\partial M} |u(t)|^p dS_g} : 0 \neq u \in C^\infty(M), \int_{\partial M} |u(t)|^{p-2} u(t) dS_g = 0 \right\}. \tag{1.2}$$

Throughout the paper we consider (M, g_0) is a two-dimensional compact Riemannian manifold with a smooth boundary ∂M .

In determining geometry and topology of a Riemannian manifold, the study of eigenvalue of geometric operators plays a crucial role. Perelman in [9] proved that the first eigenvalue of $-4\Delta + R$, where R is the scalar curvature, is nondecreasing along the Ricci flow. After that eigenvalue of different geometric operators on a Riemannian manifold evolves by geometric flows were studied by many authors, for instance see [3, 4, 6, 10, 11, 12]. Studying geometric flows is also an active

2010 *Mathematics Subject Classification.* 53C44, 58C40.

Key words and phrases. p -Steklov eigenvalue, geodesic curvature, geodesic curvature flow.

area of research in geometry. In this article we consider a Riemannian surface with vanishing Gaussian curvature and constant geodesic curvature on its boundary. Osgood, Phillips and Sarnak in [8] proved the existence of a conformal metric with vanishing Gaussian curvature in M and constant geodesic curvature on ∂M . In [1, 2], Brendle studied geodesic curvature flow on a surface with boundary. To study more results related to prescribing geodesic curvature, one can see [5, 13]. Recently in [7], Ho and Koo studied the first nonzero Steklov eigenvalue on a compact Riemannian surface with a smooth boundary along the geodesic curvature flow.

In section 2, we study the first nonzero p -Steklov eigenvalue along the unnormalized geodesic curvature flow and proved that the first nonzero p -Steklov eigenvalue is nondecreasing along the flow if the initial metric has positive geodesic curvature on ∂M and vanishing Gaussian curvature in M . In the section 3, we derive an eigenvalue estimation of the first nonzero eigenvalue along the normalized geodesic curvature flow.

2. p -STEKLOV EIGENVALUE ALONG UNNORMALIZED GEODESIC CURVATURE FLOW

Let (M, g_0) be a two-dimensional compact Riemannian manifold with smooth boundary ∂M . We consider the unnormalized geodesic curvature flow [2] by

$$\begin{aligned} \frac{\partial}{\partial t} g(t) &= -2k_{g(t)}g(t) \quad \text{on } \partial M, \\ K_{g(t)} &= 0 \quad \text{in } M, \end{aligned} \quad (2.1)$$

where $k_{g(t)}$ is the geodesic curvature of ∂M and $K_{g(t)}$ is the Gaussian curvature of M .

For a general metric $g(t) = e^{2u(t)}g_0$ conformal to g_0 , the unnormalized geodesic curvature flow (2.1) reduces to

$$\frac{\partial}{\partial t} u(t) = -k_{g(t)} \quad \text{on } \partial M. \quad (2.2)$$

Lemma 2.1. [7] *Along the unnormalized geodesic curvature flow, we have*

$$\min_{\partial M} k_{g(t)} \geq \min_{\partial M} k_{g_0}. \quad (2.3)$$

Lemma 2.2. *Let $g(t)$, $t \in [0, T]$ be a solution of the unnormalized geodesic curvature flow (2.1) and $\lambda(t)$ be the corresponding first nonzero p -Steklov eigenvalue. Then for any $t_2 \geq t_1$, there exists a smooth function $f(t)$ on $M \times [0, T]$ satisfying*

$$\Delta_{p, g(t)} f(t) = 0 \quad \text{in } M, \quad \int_{\partial M} |f(t)|^{p-2} f(t) dS_{g(t)} = 0 \quad \text{and} \quad \int_{\partial M} |f(t)|^p dS_{g(t)} = 1, \quad (2.4)$$

such that

$$\lambda(t_2) \geq \lambda(t_1) + p \int_{t_1}^{t_2} \int_{\partial M} |\nabla_{g(t)} f(t)|^{p-2} \frac{\partial f(t)}{\partial t} \frac{\partial f(t)}{\partial \nu_{g(t)}} dS_{g(t)} dt, \quad (2.5)$$

where $f(t_2)$ is the corresponding eigenfunction of $\lambda(t_2)$.

Proof. It is given that at a time $t = t_2$, $f(t_2)$ is the corresponding eigenfunction of the first p -Steklov eigenvalue $\lambda(t_2)$. Now we consider a smooth function on ∂M by

$$h(t) = \left(\frac{e^{u(t_2)}}{e^{u(t)}} \right)^{\frac{1}{p-1}} f(t_2), \quad (2.6)$$

where $u(t)$ is the solution of (2.2). We normalized this function on ∂M by

$$f(t) = \frac{h(t)}{\left(\int_{\partial M} |h(t)|^p dS_{g(t)} \right)^{\frac{1}{p}}}. \quad (2.7)$$

Extend this function to a p -harmonic function in M with respect to $g(t)$, which we still denote it by $f(t)$. Now we get

$$\begin{aligned} \int_{\partial M} |f(t)|^{p-2} f(t) dS_{g(t)} &= \frac{1}{\left(\int_{\partial M} |h(t)|^p dS_{g(t)} \right)^{1-\frac{1}{p}}} \int_{\partial M} |h(t)|^{p-2} h(t) dS_{g(t)} \\ &= \frac{1}{\left(\int_{\partial M} |h(t)|^p dS_{g(t)} \right)^{1-\frac{1}{p}}} \int_{\partial M} \left(\frac{e^{u(t_2)}}{e^{u(t)}} \right) |f(t_2)|^{p-2} f(t_2) e^{u(t)} dS_{g_0} \\ &= \frac{1}{\left(\int_{\partial M} |h(t)|^p dS_{g(t)} \right)^{1-\frac{1}{p}}} \int_{\partial M} |f(t_2)|^{p-2} f(t_2) dS_{g(t_2)} = 0, \end{aligned}$$

and

$$\int_{\partial M} |f(t)|^p dS_{g(t)} = \frac{1}{\left(\int_{\partial M} |h(t)|^p dS_{g(t)} \right)} \int_{\partial M} |h(t)|^p dS_{g(t)} = 1.$$

Set

$$G(g(t), f(t)) = \int_M |\nabla_{g(t)} f(t)|^p dA_{g(t)}, \quad (2.8)$$

which is a smooth function on t . Taking derivative with respect to t , we get

$$\begin{aligned} \mathcal{G}(g(t), f(t)) &:= \frac{d}{dt} G(g(t), f(t)) = \int_M \frac{\partial}{\partial t} |\nabla_{g(t)} f(t)|^p dA_{g(t)} \\ &= p \int_M |\nabla_{g(t)} f(t)|^{p-2} \langle \nabla_{g(t)} f(t), \nabla_{g(t)} f_t(t) \rangle dA_{g(t)}. \end{aligned}$$

Now using the Stokes theorem we have

$$\frac{d}{dt} G(g(t), f(t)) = p \int_{\partial M} |\nabla_{g(t)} f(t)|^{p-2} \frac{\partial f(t)}{\partial t} \frac{\partial f(t)}{\partial \nu_{g(t)}} dS_{g(t)}.$$

Using the definition of $\mathcal{G}(g(t), f(t))$, we get

$$G(g(t_2), f(t_2)) - G(g(t_1), f(t_1)) = \int_{t_1}^{t_2} \mathcal{G}(g(t), f(t)) dt. \quad (2.9)$$

Since $f(t_2)$ is the corresponding eigenfunction of the p -Steklov eigenvalue $\lambda(t_2)$, we have

$$G(g(t_2), f(t_2)) = \lambda(t_2) \int_{\partial M} |f(t_2)|^p dS_{g(t_2)} = \lambda(t_2). \quad (2.10)$$

Again from the variational formula for the first p -Steklov eigenvalue we have that

$$G(g(t_1), f(t_1)) \geq \lambda(t_1) \int_{\partial M} |f(t_1)|^p dS_{g(t_1)} = \lambda(t_1). \quad (2.11)$$

Finally using (2.10) and (2.11) in (2.9), we have (2.5). \square

Theorem 2.1. *Under the unnormalized geodesic curvature flow on a compact Riemannian manifold M with smooth boundary ∂M , the first p -Steklov eigenvalue is nondecreasing if the initial metric g_0 has positive geodesic curvature on ∂M and vanishing Gaussian curvature on M .*

Proof. Since $f(t_2)$ is the corresponding eigenfunction of the p -Steklov eigenvalue $\lambda(t_2)$, we have

$$\begin{aligned} \int_{\partial M} |\nabla_{g(t_2)} f(t_2)|^{p-2} \frac{\partial f(t_2)}{\partial t} \frac{\partial f(t_2)}{\partial \nu_{g(t_2)}} dS_{g(t_2)} \\ = \lambda(t_2) \int_{\partial M} |f(t_2)|^{p-2} f(t_2) \frac{\partial f(t_2)}{\partial t} dS_{g(t_2)}. \end{aligned} \quad (2.12)$$

Differentiating $\int_{\partial M} |f(t)|^p dS_{g(t)} = 1$, we get

$$\begin{aligned} p \int_{\partial M} |f(t)|^{p-2} f(t) \frac{\partial f(t)}{\partial t} dS_{g(t)} &= - \int_{\partial M} |f(t)|^p \frac{\partial}{\partial t} (e^{u(t)} dS_{g(0)}) \\ &= - \int_{\partial M} |f(t)|^p \frac{\partial u(t)}{\partial t} dS_{g(t)} \\ &= \int_{\partial M} |f(t)|^p k_{g(t)} dS_{g(t)} \\ &\geq (\min_{\partial M} k_{g(0)}) \int_{\partial M} |f(t)|^p dS_{g(t)} = \min_{\partial M} k_{g(0)}. \end{aligned} \quad (2.13)$$

Thus,

$$\int_{\partial M} |\nabla_{g(t_2)} f(t_2)|^{p-2} \frac{\partial f(t_2)}{\partial t} \frac{\partial f(t_2)}{\partial \nu_{g(t_2)}} dS_{g(t_2)} \geq \frac{\lambda(t_2)}{p} (\min_{\partial M} k_{g(0)}). \quad (2.14)$$

It is clear by assumption that $\min_{\partial M} k_{g(0)} > 0$, hence for t sufficiently closed to t_2 , we have

$$\int_{\partial M} |\nabla_{g(t)} f(t)|^{p-2} \frac{\partial f(t)}{\partial t} \frac{\partial f(t)}{\partial \nu_{g(t)}} dS_{g(t)} \geq 0. \quad (2.15)$$

Hence using Lemma 2.2 we can conclude that $\lambda(t_2) \geq \lambda(t_1)$ for any $t_2 > t_1$ sufficiently closed to t_2 . Since t_2 is arbitrary, hence the proof is complete. \square

3. p -STEKLOV EIGENVALUE ALONG NORMALIZED GEODESIC CURVATURE FLOW

With the initial metric g_0 , in this section we consider the following normalized geodesic curvature flow [2] given by

$$\begin{aligned} \frac{\partial}{\partial t} g(t) &= -2(k_{g(t)} - \bar{k}_{g(t)})g(t) \text{ on } \partial M, \\ K_{g(t)} &= 0 \text{ in } M, \end{aligned} \quad (3.1)$$

where $k_{g(t)}$ is the geodesic curvature of ∂M and $K_g(t)$ is the Gaussian curvature of M . Here $\bar{k}_{g(t)}$ is the average of geodesic curvature on ∂M defined by

$$\bar{k}_{g(t)} = \frac{\int_{\partial M} k_{g(t)} dS_{g(t)}}{\int_{\partial M} dS_{g(t)}}. \quad (3.2)$$

Under the conformal change $g(t) = e^{2u(t)}g_0$, normalized geodesic curvature flow (3.1) reduces to

$$\frac{\partial}{\partial t} u(t) = -(k_{g(t)} - \bar{k}_{g(t)}) \text{ on } \partial M. \quad (3.3)$$

Along the normalized geodesic curvature flow

$$\frac{d}{dt} \left(\int_{\partial M} dS_{g(t)} \right) = - \int_{\partial M} (k_{g(t)} - \bar{k}_{g(t)}) dS_{g(t)} = 0, \quad (3.4)$$

which implies that

$$\int_{\partial M} dS_{g(t)} = \int_{\partial M} dS_{g_0} \text{ for all } t \geq 0. \quad (3.5)$$

Lemma 3.1. *Let $g(t)$, $t \in [0, T)$ be a solution of the normalized geodesic curvature flow (3.1) and $\lambda(t)$ be the corresponding first nonzero p -Steklov eigenvalue. Then for any $t_2 \geq t_1$, there exists a smooth function $f(t)$ on $M \times [0, T)$ satisfying*

$$\Delta_{p,g(t)} f(t) = 0 \text{ in } M, \int_{\partial M} |f(t)|^{p-2} f(t) dS_{g(t)} = 0 \text{ and } \int_{\partial M} |f(t)|^p dS_{g(t)} = 1, \quad (3.6)$$

such that

$$\lambda(t_2) \geq \lambda(t_1) + p \int_{t_1}^{t_2} \int_{\partial M} |\nabla_{g(t)} f(t)|^{p-2} \frac{\partial f(t)}{\partial t} \frac{\partial f(t)}{\partial \nu_{g(t)}} dS_{g(t)} dt, \quad (3.7)$$

where $f(t_2)$ is the corresponding eigenfunction of $\lambda(t_2)$.

Proof. The proof is similar as Lemma 2.2. \square

Theorem 3.1. *Under the normalized geodesic curvature flow on a compact Riemannian manifold M with smooth boundary ∂M , the first nonzero p -Steklov eigenvalue is nondecreasing if for the initial metric g_0 , $(\min_{\partial M} k_{g_0} - \bar{k}_{g_0}) \geq 0$ on ∂M and Gaussian curvature of M is zero.*

Proof. Since $f(t_2)$ is the corresponding eigenfunction of the p -Steklov eigenvalue $\lambda(t_2)$, we have

$$\begin{aligned} \int_{\partial M} |\nabla_{g(t_2)} f(t_2)|^{p-2} \frac{\partial f(t_2)}{\partial t} \frac{\partial f(t_2)}{\partial \nu_{g(t_2)}} dS_{g(t_2)} &= \lambda(t_2) \int_{\partial M} |f(t_2)|^{p-2} f(t_2) \frac{\partial f(t_2)}{\partial t} dS_{g(t_2)} \\ &= -\frac{\lambda(t_2)}{p} \int_{\partial M} |f(t_2)|^p \frac{\partial u(t_2)}{\partial t} dS_{g(t_2)} \\ &= \frac{\lambda(t_2)}{p} \int_{\partial M} |f(t_2)|^p (k_{g(t_2)} - \bar{k}_{g(t_2)}) dS_{g(t_2)} \\ &\geq \frac{\lambda(t_2)}{p} \left(\min_{\partial M} k_{g(t_2)} - \bar{k}_{g(t_2)} \right). \end{aligned} \quad (3.8)$$

Rest of the proof is same as the method applied in Theorem 2.1. \square

Proposition 3.1. *Along the normalized geodesic curvature flow (3.1), the first nonzero p -Steklov eigenvalue $\lambda(t)$ satisfies*

$$\frac{d}{dt} \log \lambda(t) \geq \left(\min_{\partial M} k_{g(t)} - \bar{k}_{g(t)} \right) \text{ for all } t, \quad (3.9)$$

where on the left side the derivative is in the sense of the \liminf of backward difference quotients.

Proof. Using (3.6) and the fact that $f(t_2)$ is the corresponding eigenfunction of the first nonzero p -Steklov eigenvalue $\lambda(t_2)$, we have

$$\begin{aligned} \int_{\partial M} |\nabla_{g(t_2)} f(t_2)|^{p-2} \frac{\partial f(t_2)}{\partial t} \frac{\partial f(t_2)}{\partial \nu_{g(t_2)}} dS_{g(t_2)} &= \lambda(t_2) \int_{\partial M} |f(t_2)|^{p-2} f(t_2) \frac{\partial f(t_2)}{\partial t} dS_{g(t_2)} \\ &= -\frac{\lambda(t_2)}{p} \int_{\partial M} |f(t_2)|^p \frac{\partial u(t_2)}{\partial t} dS_{g(t_2)} \\ &= \frac{\lambda(t_2)}{p} \int_{\partial M} |f(t_2)|^p (k_{g(t_2)} - \bar{k}_{g(t_2)}) dS_{g(t_2)} \\ &\geq \frac{\lambda(t_2)}{p} \left(\min_{\partial M} k_{g(t_2)} - \bar{k}_{g(t_2)} \right). \end{aligned} \quad (3.10)$$

Hence for any $\epsilon > 0$, we have that

$$\int_{\partial M} |\nabla_{g(t)} f(t)|^{p-2} \frac{\partial f(t)}{\partial t} \frac{\partial f(t)}{\partial \nu_{g(t)}} dS_{g(t)} \geq \frac{\lambda(t_2)}{p} \left(\min_{\partial M} k_{g(t)} - \bar{k}_{g(t)} - \epsilon \right) \quad (3.11)$$

for t sufficiently closed to t_2 . Thus the Lemma 3.1 gives

$$\lambda(t_2) - \lambda(t_1) \geq \lambda(t_2) \int_{t_1}^{t_2} \left(\min_{\partial M} k_{g(t)} - \bar{k}_{g(t)} - \epsilon \right) dt. \quad (3.12)$$

for t_1 sufficiently closed to t_2 and $t_2 > t_1$. Now dividing the equation (3.12) by $t_2 - t_1$ and taking $t_1 \rightarrow t_2$, we obtain

$$\liminf_{t_1 \rightarrow t_2} \frac{\lambda(t_2) - \lambda(t_1)}{t_2 - t_1} \geq \lambda(t_2) \left(\min_{\partial M} k_{g(t_2)} - \bar{k}_{g(t_2)} - \epsilon \right). \quad (3.13)$$

Using the same argument used (in (2.21), [6]), we can say that

$$\liminf_{t_1 \rightarrow t_2} \frac{\log \lambda(t_2) - \log \lambda(t_1)}{t_2 - t_1} \geq \frac{1}{\lambda(t_2)} \liminf_{t_1 \rightarrow t_2} \frac{\lambda(t_2) - \lambda(t_1)}{t_2 - t_1}. \quad (3.14)$$

Now (3.13) and (3.14) yields

$$\liminf_{t_1 \rightarrow t_2} \frac{\log \lambda(t_2) - \log \lambda(t_1)}{t_2 - t_1} \geq \min_{\partial M} k_{g(t_2)} - \bar{k}_{g(t_2)} - \epsilon. \quad (3.15)$$

Since ϵ is arbitrary, we have our result. \square

Lemma 3.2. *Let $g(t)$, $t \in [0, T)$ be a solution of the normalized geodesic curvature flow (3.1) and $\lambda(t)$ be the corresponding first nonzero p -Steklov eigenvalue. Then for any $t_2 \geq t_1$, there exists a smooth function $f(t)$ on $M \times [0, T)$ satisfying*

$$\Delta_{p, g(t)} f(t) = 0 \text{ in } M, \quad \int_{\partial M} |f(t)|^{p-2} f(t) dS_{g(t)} = 0 \text{ and } \int_{\partial M} |f(t)|^p dS_{g(t)} = 1, \quad (3.16)$$

such that

$$\lambda(t_2) \leq \lambda(t_1) + p \int_{t_1}^{t_2} \int_{\partial M} |\nabla_{g(t)} f(t)|^{p-2} \frac{\partial f(t)}{\partial t} \frac{\partial f(t)}{\partial \nu_{g(t)}} dS_{g(t)} dt, \quad (3.17)$$

where $f(t_1)$ is the corresponding eigenfunction of $\lambda(t_1)$.

Proof. We define a function on the boundary ∂M of M by

$$h(t) = \left(\frac{e^{u(t_1)}}{e^{u(t)}} \right)^{\frac{1}{p-1}} f(t_1), \quad (3.18)$$

where $u(t)$ is the solution of (3.3). We normalized the function on ∂M by

$$f(t) = \frac{h(t)}{\left(\int_{\partial M} |h(t)|^p dS_{g(t)}\right)^{\frac{1}{p}}}. \quad (3.19)$$

Extend this function to a p -harmonic function in M with respect to $g(t)$, which we still denote it by $f(t)$. Now we have

$$\int_{\partial M} |f(t)|^{p-2} f(t) dS_{g(t)} = \frac{1}{\left(\int_{\partial M} |h(t)|^p dS_{g(t)}\right)^{1-\frac{1}{p}}} \int_{\partial M} |f(t_1)|^{p-2} f(t_1) dS_{g(t_1)} = 0,$$

and

$$\int_{\partial M} |f(t)|^p dS_{g(t)} = \frac{1}{\left(\int_{\partial M} |h(t)|^p dS_{g(t)}\right)} \int_{\partial M} |h(t)|^p dS_{g(t)} = 1.$$

Set

$$G(g(t), f(t)) = \int_M |\nabla_{g(t)} f(t)|^p dA_{g(t)}, \quad (3.20)$$

which is a smooth function on t . Taking derivative with respect to t , we get

$$\begin{aligned} \mathcal{G}(g(t), f(t)) &:= \frac{d}{dt} G(g(t), f(t)) = \int_M \frac{\partial}{\partial t} |\nabla_{g(t)} f(t)|^p dA_{g(t)} \\ &= p \int_M |\nabla_{g(t)} f(t)|^{p-2} \langle \nabla_{g(t)} f(t), \nabla_{g(t)} f_t(t) \rangle dA_{g(t)}. \end{aligned}$$

So by using the Stoke's theorem we have

$$\frac{d}{dt} G(g(t), f(t)) = p \int_{\partial M} |\nabla_{g(t)} f(t)|^{p-2} \frac{\partial f(t)}{\partial t} \frac{\partial f(t)}{\partial \nu_{g(t)}} dS_{g(t)}.$$

Using the definition of $\mathcal{G}(g(t), f(t))$, we get

$$G(g(t_2), f(t_2)) - G(g(t_1), f(t_1)) = \int_{t_1}^{t_2} \mathcal{G}(g(t), f(t)) dt. \quad (3.21)$$

Since $f(t_1)$ is the corresponding eigenfunction of the p -Steklov eigenvalue $\lambda(t_1)$, we have

$$G(g(t_1), f(t_1)) = \lambda(t_1) \int_{\partial M} |f(t_1)|^p dS_{g(t_1)} = \lambda(t_1). \quad (3.22)$$

Again from the variational formula for the first p -Steklov eigenvalue we have that

$$G(g(t_2), f(t_2)) \geq \lambda(t_2) \int_{\partial M} |f(t_2)|^p dS_{g(t_2)} = \lambda(t_2). \quad (3.23)$$

Finally using (3.22) and (3.23) in (3.21), we have (3.17). \square

Proposition 3.2. *Under the normalized geodesic curvature flow the first nonzero p -Steklov eigenvalue $\lambda(t)$ satisfies*

$$\frac{d}{dt} \log \lambda(t) \leq \left(\max_{\partial M} k_{g(t)} - \bar{k}_{g(t)} \right) \quad \text{for all } t, \quad (3.24)$$

where on the left hand side, the derivative is in the sense of the lim sup of backward difference quotients.

Proof. By using (3.16) and since $f(t_1)$ is the corresponding eigenfunction of the first nonzero p -Steklov eigenvalue $\lambda(t_1)$, we have

$$\begin{aligned} \int_{\partial M} |\nabla_{g(t_1)} f(t_1)|^{p-2} \frac{\partial f(t_1)}{\partial t} \frac{\partial f(t_1)}{\partial \nu_{g(t_1)}} dS_{g(t_1)} &= \lambda(t_1) \int_{\partial M} |f(t_1)|^{p-2} f(t_1) \frac{\partial f(t_1)}{\partial t} dS_{g(t_1)} \\ &= -\frac{\lambda(t_1)}{p} \int_{\partial M} |f(t_1)|^p \frac{\partial u(t_1)}{\partial t} dS_{g(t_1)} \\ &= \frac{\lambda(t_1)}{p} \int_{\partial M} |f(t_1)|^p (k_{g(t_1)} - \bar{k}_{g(t_1)}) dS_{g(t_1)} \\ &\leq \frac{\lambda(t_1)}{p} \left(\max_{\partial M} k_{g(t_1)} - \bar{k}_{g(t_1)} \right). \end{aligned} \quad (3.25)$$

Thus, for any $\epsilon > 0$ we get

$$\int_{\partial M} |\nabla_{g(t)} f(t)|^{p-2} \frac{\partial f(t)}{\partial t} \frac{\partial f(t)}{\partial \nu_{g(t)}} dS_{g(t)} \leq \frac{\lambda(t_1)}{p} \left(\max_{\partial M} k_{g(t)} - \bar{k}_{g(t)} + \epsilon \right), \quad (3.26)$$

for t sufficiently closed to t_1 and $t_2 > t_1$. Hence by using (3.17), we get

$$\lambda(t_2) - \lambda(t_1) \leq \lambda(t_1) \int_{t_1}^{t_2} \left(\max_{\partial M} k_{g(t)} - \bar{k}_{g(t)} + \epsilon \right), \quad (3.27)$$

for t_1 sufficiently closed to t_2 . Dividing both sides by $t_2 - t_1$ and taking $t_2 \rightarrow t_1$, we have

$$\limsup_{t_2 \rightarrow t_1} \frac{\lambda(t_2) - \lambda(t_1)}{t_2 - t_1} \leq \lambda(t_1) \left(\max_{\partial M} k_{g(t_1)} - \bar{k}_{g(t_1)} + \epsilon \right). \quad (3.28)$$

By similar argument used (in (2.21), [6]), we get

$$\limsup_{t_2 \rightarrow t_1} \frac{\log \lambda(t_2) - \log \lambda(t_1)}{t_2 - t_1} \leq \max_{\partial M} k_{g(t_1)} - \bar{k}_{g(t_1)} + \epsilon. \quad (3.29)$$

Since $\epsilon > 0$ is arbitrary, we have (3.24). \square

Theorem 3.2. *Assume that for a initial metric g_0 , M has vanishing Gaussian curvature and ∂M has negative geodesic curvature and g_c is the metric conformal to g_0 with vanishing Gaussian curvature in M and constant geodesic curvature on ∂M such that the lengths of ∂M of g_c and g_0 are the same. If $\lambda(g_c)$ and $\lambda(g_0)$ are the first nonzero p -Steklov eigenvalue of g_c and g_0 respectively, then*

$$\left(1 - \frac{\min_{\partial M} k_{g_0}}{\max_{\partial M} k_{g_0}} \right) \leq \log \frac{\lambda(g_c)}{\lambda(g_0)} \leq - \left(1 - \frac{\min_{\partial M} k_{g_0}}{\max_{\partial M} k_{g_0}} \right). \quad (3.30)$$

Proof. It was proved in [2] that $g \rightarrow g_\infty$ as $t \rightarrow \infty$ along the normalized geodesic curvature flow (3.1) such that g_∞ is conformal to g_0 and has constant geodesic curvature on ∂M and vanishing Gaussian curvature in M . Now from (3.5) we have

$$\int_{\partial M} dS_{g_\infty} = \int_{\partial M} dS_{g_0}. \quad (3.31)$$

By assumption it is given that

$$\int_{\partial M} dS_{g_c} = \int_{\partial M} dS_{g_0}. \quad (3.32)$$

From (3.31) and (3.32), we get

$$\int_{\partial M} dS_{g_\infty} = \int_{\partial M} dS_{g_c}. \quad (3.33)$$

Now from Gauss-Bonnet theorem, it follows that

$$k_{g_\infty} \int_{\partial M} dS_{g_\infty} = \int_M K_{g_\infty} dA_{g_\infty} + \int_{\partial M} k_{g_\infty} dS_{g_\infty} = 2\pi\chi(M) \quad (3.34)$$

and

$$k_{g_c} \int_{\partial M} dS_{g_c} = \int_M K_{g_c} dA_{g_c} + \int_{\partial M} k_{g_c} dS_{g_c} = 2\pi\chi(M), \quad (3.35)$$

where $\chi(M)$ is the Euler characteristic on M . It is given that for the initial metric g_0 , M has vanishing Gaussian curvature and ∂M has negative geodesic curvature, so it is clear that the Euler characteristic function is negative. So using (3.33), we get

$$k_{g_\infty} = k_{g_c} < 0. \quad (3.36)$$

If $g(t) = e^{2u(t)}g_0$ then we have

$$-\Delta_{g_0} u + k_{g_0} = k_g e^{2u} \quad \text{in } M, \quad (3.37)$$

$$\frac{\partial u}{\partial \nu_{g_0}} + k_{g_0} = k_g e^u \quad \text{on } \partial M, \quad (3.38)$$

where $\frac{\partial}{\partial \nu_{g_0}}$ is the normal derivative with respect to g_0 .

From the Gauss-Bonnet theorem, (3.1), (2.3) and (3.5), we have

$$\bar{k}_{g(t)} = \frac{\int_M K_{g(t)} dA_{g(t)} + \int_{\partial M} k_{g(t)} dS_{g(t)}}{\int_{\partial M} dS_{g(t)}} = \frac{2\pi\chi(M)}{\int_{\partial M} dS_{g(t)}} \quad \text{for } t \geq 0. \quad (3.39)$$

Hence g_c and g_∞ are conformal to g_0 and all of them have vanishing Gaussian curvature, if $g_c = e^{2v}g_0$ we have

$$\begin{cases} \Delta_{g_0} u = 0 & \text{in } M, \\ \frac{\partial u}{\partial \nu_{g_0}} + k_{g_0} = k_\infty e^u & \text{on } \partial M, \end{cases} \quad \text{and} \quad \begin{cases} \Delta_{g_0} v = 0 & \text{in } M, \\ \frac{\partial v}{\partial \nu_{g_0}} + k_{g_0} = k_{g_c} e^v & \text{on } \partial M. \end{cases}$$

Since $k_\infty = k_{g_0}$, we obtain

$$\begin{aligned} \Delta_{g_0}(u - v) &= 0 \quad \text{in } M, \\ \frac{\partial(u - v)}{\partial \nu_{g_0}} &= k_{g_c}(e^u - e^v) \quad \text{on } \partial M. \end{aligned}$$

Thus

$$(u - v) \frac{\partial(u - v)}{\partial \nu_{g_0}} = k_{g_c}(e^u - e^v)(u - v) \quad \text{on } \partial M. \quad (3.40)$$

Integrating of above equation over ∂M with respect to g_0 , we infer

$$0 \leq \int_M |\nabla_{g_0}(u - v)|^2 dA_{g_0} = \int_{\partial M} (u - v) \frac{\partial(u - v)}{\partial \nu_{g_0}} dS_{g_0} = k_{g_c} \int_{\partial M} (e^u - e^v)(u - v) dS_{g_0}. \quad (3.41)$$

On the other hand $k_{g_c} < 0$ and $(e^u - e^v)(u - v) \geq 0$, then the left hand side of (3.41) is non positive. Therefore $\int_{\partial M} (e^u - e^v)(u - v) dS_{g_0} = 0$ which yields $u = v$ on ∂M and since $u - v$ is harmonic in M , we get $u = v$ in M . It implies that $g_c = g_\infty$.

From Lemma 2.9 of [7] we have

$$k_{g(t)} \leq \bar{k}_{g_0} + \left(\max_{\partial M} k_{g_0} - \min_{\partial M} k_{g_0} \right) + \left(\max_{\partial M} k_{g_0} \right) \int_0^t \left(\max_{\partial M} k_{g(\tau)} - \bar{k}_{g(\tau)} \right) d\tau. \quad (3.42)$$

It follows from (3.39) and (3.42) that

$$\left(\max_{\partial M} k_{g_t} - \bar{k}_{g_t} \right) - \left(\max_{\partial M} k_{g_0} - \min_{\partial M} k_{g_0} \right) \leq \left(\max_{\partial M} k_{g_0} \right) \int_0^t \left(\max_{\partial M} k_{g(\tau)} - \bar{k}_{g(\tau)} \right) d\tau. \quad (3.43)$$

If $t \rightarrow \infty$, then

$$- \left(1 - \frac{\min_{\partial M} k_{g_0}}{\max_{\partial M} k_{g_0}} \right) \geq \int_0^\infty \left(\max_{\partial M} k_{g(\tau)} - \bar{k}_{g(\tau)} \right) d\tau. \quad (3.44)$$

Integrating (3.24) with respect to t on interval $[0, \infty)$ and using (3.44) and $g_c = g_\infty$, we have

$$\log \frac{\lambda(g_c)}{\lambda(g_0)} = \log \frac{\lambda(g_\infty)}{\lambda(g_0)} \leq \int_0^\infty \left(\max_{\partial M} k_{g(\tau)} - \bar{k}_{g(\tau)} \right) d\tau \leq - \left(1 - \frac{\min_{\partial M} k_{g_0}}{\max_{\partial M} k_{g_0}} \right). \quad (3.45)$$

From Lemma 2.10 of [7] we have

$$k_{g(t)} \geq \bar{k}_{g_0} - \left(\max_{\partial M} k_{g_0} - \min_{\partial M} k_{g_0} \right) + \left(\max_{\partial M} k_{g_0} \right) \int_0^t \left(\min_{\partial M} k_{g(\tau)} - \bar{k}_{g(\tau)} \right) d\tau. \quad (3.46)$$

Then we get

$$\left(\bar{k}_{g(t)} - \min_{\partial M} k_{g(t)} \right) - \left(\max_{\partial M} k_{g_0} - \min_{\partial M} k_{g_0} \right) \leq - \left(\max_{\partial M} k_{g_0} \right) \int_0^t \left(\min_{\partial M} k_{g(\tau)} - \bar{k}_{g(\tau)} \right) d\tau. \quad (3.47)$$

As $t \rightarrow \infty$ we conclude

$$\left(1 - \frac{\min_{\partial M} k_{g_0}}{\max_{\partial M} k_{g_0}} \right) \leq \int_0^\infty \left(\min_{\partial M} k_{g(\tau)} - \bar{k}_{g(\tau)} \right) d\tau. \quad (3.48)$$

Integrating (3.24) and using (3.48) and $g_c = g_\infty$, we infer

$$\log \frac{\lambda(g_c)}{\lambda(g_0)} = \log \frac{\lambda(g_\infty)}{\lambda(g_0)} \geq \int_0^\infty \left(\min_{\partial M} k_{g(\tau)} - \bar{k}_{g(\tau)} \right) d\tau \geq \left(1 - \frac{\min_{\partial M} k_{g_0}}{\max_{\partial M} k_{g_0}} \right). \quad (3.49)$$

This completes the proof of theorem. \square

DECLARATIONS

- Funding
Not applicable
- Conflict of interest/Competing interests
We declare that we do not have any commercial or associative interest that represents a conflict of interest in connection with the work submitted.

- Ethics approval
All authors have agreed and consent to participate.
- Consent to participate
Not applicable
- Consent for publication
The authors agree to publication.
- Availability of data and materials
All data generated or analysed during this study are included in this published article.
- Code availability
Not applicable
- Authors' contributions
Not applicable

Acknowledgement: The first author (A. Saha) gratefully acknowledges to the CSIR (File No.: 09/025(0273)/2019-EMR-I), Government of India for the award of Senior Research Fellowship. This research work is also partially supported by DST FIST programme (No. SR/FST/MSII/2017/10(C)).

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Apurba Saha

Department of Mathematics, The University of Burdwan, Golapbag, Burdwan
713104, West Bengal, India

E-mail: apurbasahaju@gmail.com

Shahroud Azami

Department of Pure Mathematics, Faculty of Sciences, Imam Khomeini International University, Qazvin, Iran

E-mail: azami@sci.ikiu.ac.ir

Shyamal Kumar Hui

Department of Mathematics, The University of Burdwan, Golapbag, Burdwan 713104, West Bengal, India.

E-mail: skhui@math.buruniv.ac.in