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REIDEMEISTER CLASSES, WREATH PRODUCTS AND SOLVABILITY

E.V. TROITSKY

Abstract. Reidemeister (or twisted conjugacy) classes are considered in restricted wreath products of the form $G \wr \mathbb{Z}^k$, where G is a finite group.

For an automorphism φ of finite order with finite Reidemeister number $R(\varphi)$, this number is identified with the number of equivalence classes of finite-dimensional unitary irreducible representations that are fixed by the dual homeomorphism $\widehat{\varphi}$ (i.e. the so-called conjecture TBFT_f is proved in this case).

We construct a counterexample from this class of groups to disprove the following conjecture: if a finitely generated residually finite group has an automorphism with $R(\varphi) < \infty$ then it is solvable-by-finite (so-called conjecture R).

Keywords: Reidemeister number, twisted conjugacy class, Burnside-Frobenius theorem, solvable group, unitary dual, finite-dimensional representation, wreath product.

Reidemeister, or *twisted conjugacy classes* of an automorphism φ of a group Γ are equivalence classes with respect to $x \sim yx\varphi(y^{-1})$. Their number $R(\varphi)$ (finite or infinite) is called *Reidemeister number*.

The following three directions form the mainstream of the current study of Reidemeister classes:

1) To prove or disprove the so-called TBFT (twisted Burnside-Frobenius theory) conjecture: the Reidemeister number $R(\varphi)$ (if finite) coincides with the number of equivalence classes of irreducible unitary representations of

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Γ fixed by the induced homeomorphism $\widehat{\varphi}$ of the unitary dual $\widehat{\Gamma}$. Its finite-dimensional version TBFT_f was also studied (here one considers only finite-dimensional fixed representations). The most important classes of groups for which TBFT_f is true, are polycyclic-by-finite groups [7] and residually finite groups of finite Prüfer rank [21]. On the other hand, in [10] we have detected an example of infinitely generated residually finite group which has neither TBFT nor TBFT_f.

2) As an opposite case, to determine classes of groups for which any automorphism has infinite Reidemeister number (this property is called R_∞) — the list of results here is very extended, we mention only some expository or recent papers: [5, 1, 8, 18, 11, 21, 17, 19].
3) To study rationality and other properties of Reidemeister zeta function constructed from $R(\varphi^n)$ (see e.g. [4, 9] for a recent progress).

We will deal with the first two aspects in this paper. Inspired by [16] we have formulated in [8] the following conjecture:

Conjecture R. *Let Γ be a finitely generated residually finite group. Either Γ is R_∞, or Γ is solvable-by-finite.*

It was discussed in several papers, in particular in [13]. The conjecture was supported by the following recent result [21]: any residually finite group of finite upper rank admitting an automorphism φ with finite Reidemeister number $R(\varphi)$ is solvable-by-finite.

The main results of the present paper are:

A) The above conjecture (R) fails to be true. More specifically, we construct a series of counterexamples of the form $\Gamma = (G \oplus G) \wr \mathbb{Z}$, where G is a non-abelian finite simple group, and automorphisms $\varphi : \Gamma \to \Gamma$ such that $R(\varphi) < \infty$ but Γ is not solvable-by-finite (Theorem 10).

B) TBFT_f is true for automorphisms of finite order of $G \wr \mathbb{Z}^k$, where G is an arbitrary finite group (Theorem 14).

The second result generalizes to the non-commutative case a result from [11] and is much more expected than the first one.

Recall some facts to be used in the proofs.

Let $\mathbf{F}(\varphi) := \{g \in \Gamma\colon \varphi(g) = g\}$ be the subgroup of φ-fixed elements. We use the notation $\tau_g(x) = gxg^{-1}$ for an inner automorphism and for its restriction to a normal subgroup.

From the equality

$$yg\varphi(y^{-1})x = ygxx^{-1}\varphi(y^{-1})x = y(gx)(\tau_{x^{-1}} \circ \varphi)(y^{-1}),$$

it follows a very useful statement (see e.g. [7]):

Lemma 1. *Shifts of Reidemeister classes of φ are Reidemeister classes of $\tau_{x^{-1}} \circ \varphi$:*

$$\{g\}_\varphi x = \{gx\}_{\tau_{x^{-1}} \circ \varphi}.$$

Hence, $R(\tau_g \circ \varphi) = R(\varphi)$.

Lemma 2 ([16])**.** *Suppose that Γ is a residually finite group and $\varphi : \Gamma \to \Gamma$ is an automorphism of finite order with $R(\varphi) < \infty$. Then $|\mathbf{F}(\varphi)| < \infty$.*

Lemma 3 (Prop. 3.4 in [6])**.** *Suppose that Γ is a finitely generated residually finite group and $\varphi : \Gamma \to \Gamma$ is an automorphism with $R(\varphi) < \infty$. Then $|\mathbf{F}(\varphi)| < \infty$.*

Lemma 4 ([3, 12], see also [7, 15]). *Suppose, $\varphi : \Gamma \to \Gamma$ is an automorphism of a discrete group, H is a normal φ-invariant subgroup of Γ, so φ induces automorphisms $\varphi' : H \to H$ and $\widetilde{\varphi} : \Gamma/H \to \Gamma/H$. Then*

- *the projection $\Gamma \to \Gamma/H$ maps Reidemeister classes of φ onto Reidemeister classes of $\widetilde{\varphi}$, in particular $R(\widetilde{\varphi}) \le R(\varphi)$;*
- *if $|\mathbf{F}(\widetilde{\varphi})| = n$, then $R(\varphi') \le R(\varphi) \cdot n$;*
- *if $\mathbf{F}(\widetilde{\varphi}) = \{e\}$, then each Reidemeister class of φ' is an intersection of the appropriate Reidemeister class of φ and H;*
- *if $\mathbf{F}(\widetilde{\varphi}) = \{e\}$, then $R(\varphi) = \sum_{j=1}^{R} R(\tau_{g_j} \circ \varphi')$, where $g_1, \dots g_R$ are some elements of Γ such that $p(g_1), \dots, p(g_R)$ are representatives of all Reidemeister classes of $\widetilde{\varphi}$, $p : \Gamma \to \Gamma/H$ is the natural projection and $R = R(\widetilde{\varphi})$.*

Recall also the following folklore observation.

Lemma 5. *For an automorphism $f : F \to F$ of a finite group, $R(f) > 1$ if and only if $\mathbf{F}(f) \neq \{e\}$.*

Proof. Indeed, consider the Reidemeister class $\{e\}_f$ as an orbit of the twisted action of G on itself: $g : x \mapsto gxf(g^{-1})$. Then by the orbit-stabilizer theorem, $R(f) = 1$, i.e. $\{e\}_f = G$, if and only if the stabilizer of e under the twisted action is trivial. But $xef(x^{-1}) = e$ if and only if $x \in \mathbf{F}(f)$. □

We say that Reidemeister classes of $\varphi : \Gamma \to \Gamma$ are separated by an epimorphism $f : \Gamma \to F$ onto a finite group F if f induces a bijection of Reidemeister classes.

Lemma 6 (see [7, 6]). *Let $\varphi : \Gamma \to \Gamma$ have $R(\varphi) < \infty$. Then $TBFT_f$ is true for φ if and only if Reidemeister classes of φ are separated by an epimorphism $f : \Gamma \to F$ onto a finite group F.*

Now we pass to a construction of the desired counterexample.

We have by definition, $F \wr \mathbb{Z}^k = \Sigma \rtimes_\alpha \mathbb{Z}^k$, where Σ denotes $\oplus_{x \in \mathbb{Z}^k} F_x$, and $\alpha(x)(g_y) = g_{x+y}$. Here g_x denotes g as an element of $F \cong F_x$.

The following statement was discussed in [14, 11] in the abelian case.

Lemma 7. *An authomorphism $\varphi : F \wr \mathbb{Z}^k \to F \wr \mathbb{Z}^k$, where $|F| < \infty$, has $R(\varphi) < \infty$ if and only if $R(\overline{\varphi}) < \infty$ and $R(\tau_m \circ \varphi') < \infty$ for any $m \in \mathbb{Z}^k$, where $\varphi' : \oplus_m F_m \to \oplus_m F_m$ and $\overline{\varphi} : \mathbb{Z}^k \to \mathbb{Z}^k$ are induced by φ (in fact, it is sufficient to verify this for representatives of Reidemeister classes of $\overline{\varphi}$).*

Proof. Suppose, $R(\varphi) < \infty$. By Lemma 4, we have $R(\overline{\varphi}) < \infty$. Then by Lemma 3, we obtain $|\mathbf{F}(\overline{\varphi})| < \infty$ (in fact, $|\mathbf{F}(\overline{\varphi})| = 1$, because an automorphism of $\mathbb{Z}^k$ can not have finitely many fixed elements except of 0). So, by Lemma 4, $R(\varphi') < \infty$. Considering $\tau_z \circ \varphi$, which has $R(\tau_z \circ \varphi) = R(\varphi) < \infty$, instead of φ, we obtain in the same way that $R(\tau_z \circ \varphi') < \infty$.

Conversely, having $|\mathbf{F}(\overline{\varphi})| = 1$, one can apply Lemma 4 (the formula from the last item) and use the equality $R(\tau_{\sigma s} \varphi') = R(\tau_s \varphi')$, where $\sigma \in \oplus_m F_m$, $s \in \mathbb{Z}^k$, see Lemma 1. □

Now we consider a particular case of F being a simple non-abelian group with $f \in F$, $f^2 \neq e$, and $k = 1$. For example, one can take $F = A_5$ and $f = (123)$.

Lemma 8. *Any group $\Gamma = F \wr \mathbb{Z}$ with such F admits an automorphism φ with $R(\varphi) < \infty$, i.e. Γ does not have the R_∞ property.*

Proof. We need a description of automorphisms of a semidirect product $H \rtimes K$, where H is characteristic, as matrices $\begin{pmatrix} a & b \\ 0 & d \end{pmatrix}$, where $a \in \mathrm{Aut}(H)$, $d \in \mathrm{Aut}(K)$, $b : K \to H$ satisfy

(i) $b(kk') = b(k)\tau_{d(k)}(b(k'))$ for any $k, k' \in K$,

(ii) $a(\tau_k(h)) = \tau_{b(k)d(k)}(a(h))$ for any $h \in H$, $k \in K$,

(see [2, Theorem 1]). If $b = 0$ (i.e. $b(k) = e_H$ for any $k \in K$), the first equality is satisfied for any d and one needs to verify only $a(\tau_k(h)) = \tau_{d(k)}(a(h))$.

Hence, since Σ is characteristic in Γ (as its torsion subgroup), taking $b = 0$ we see, that an automorphism φ can be defined by $\varphi' : \Sigma \to \Sigma$ and $\overline{\varphi} : \mathbb{Z} \to \mathbb{Z}$ restricted to satisfy

$$\varphi'(\alpha(m)(h)) = \alpha(\overline{\varphi}(m))(\varphi'(h)), \qquad h \in \Sigma, \quad m \in \mathbb{Z}^k. \tag{1}$$

For $\overline{\varphi}$ we have the only possibility $\overline{\varphi} = -\mathrm{Id} : \mathbb{Z} \to \mathbb{Z}$.

We define φ' as $\varphi'_0 : F_0 \to F_0$ by $\tau_f : F \to F$ (with the above mentioned f, $f^2 \neq e$) and

$$\varphi'_i : F_i \oplus F_{-i} \to F_i \oplus F_{-i} \qquad \text{by} \qquad \begin{pmatrix} 0 & \tau_f \\ \tau_f & 0 \end{pmatrix},$$

i.e. $\varphi' = \tau_f : F_i \to F_{-i}$. Then $R(\varphi') = \prod_{i \geq 0} R(\varphi'_i)$. We claim that $R(\varphi') = 1$.

By Lemma 5 to prove that $R(\varphi') = 1$ we need to verify that φ'_0 and φ'_i have no non-trivial fixed elements. Let us find them. Observe that $C(\tau_f) = C(\tau_{f^2}) = \{e\}$. Indeed, since F is simple, the group of fixed elements is either $\{e\}$ or F, but in the second case $f \neq e$ belongs to the center of F, which is trivial, because F is non-abelian. The same for f^2.

Thus, for $\varphi'_0 = \tau_f$ we have no non-trivial fixed elements. For φ'_i:

$$\begin{pmatrix} 0 & \tau_f \\ \tau_f & 0 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x \\ y \end{pmatrix}, \qquad \left\{ \begin{array}{lcl} x & = & \tau_f(y) \\ y & = & \tau_f(x) \end{array} \right., \qquad x = \tau_{f^2}(x), \qquad \left\{ \begin{array}{lcl} x & = & e \\ y & = & \tau_f(x) = e \end{array} \right. .$$

Thus, $R(\varphi') = 1$.

Let us verify (1). Since both sides are homomorphisms in h, one can assume that $h = h_i \in F_i$. Then

$$\varphi'(\alpha(m)(h_i)) = \varphi'(h_{m+i}) = (\tau_f(h))_{-m-i},$$
$$\alpha(\overline{\varphi}(m))(\varphi'(h_i)) = \alpha(-m)(\tau_f(h))_{-i} = (\tau_f(h))_{-m-i}$$

and (1) is proved.

To estimate $R(\varphi)$ using Lemma 7, it is sufficient to estimate

$$R(\tau_\gamma \circ \varphi') = R(\tau_\sigma \tau_m \circ \varphi') = R(\tau_m \circ \varphi') = R(\alpha(m) \circ \varphi'),$$

where $\gamma = \sigma \cdot m \in \Gamma$, $\sigma \in \Sigma$, $m \in \mathbb{Z}$, and the middle equality is obtained similarly to the end of the proof of Lemma 7 above.

Then $\alpha(m) \circ \varphi'$ maps with the help of τ_f

$$F_i \to F_{-i+m}, \qquad F_{-i+m} \to F_{-m+i+m} = F_i,$$
$$F_{m/2} \to F_{m/2} \quad \text{(if } m \text{ is even)}.$$

Thus, for an even m, the invariant subgroup structure and the restrictions of automorphism are isomorphic. The same for an odd m, but in this case there will be no orbit of length 1 in i. In both cases $R(\tau_\gamma \circ \varphi') = R(\alpha(m) \circ \varphi') = 1$. So $R(\varphi) < \infty$, moreover $R(\varphi) = 2$ to be precise (using Lemma 4). □

Lemma 9. *Suppose that $\Gamma = F \wr \mathbb{Z}$, where F is a finite non-abelian simple group as in the previous lemma. Then Γ is not a solvable-by-finite group.*

Proof. If Γ is a (finitely generated) solvable-by-finite group, then it has a characteristic solvable subgroup Γ' of finite index, $\Gamma/\Gamma' = F'$, $|F'| < \infty$. Then $\Sigma' := \Sigma \cap \Gamma'$ is a solvable subgroup of finite index in Σ. Hence, for some summand, the projection $\Sigma' \to F$ is non-trivial. Thus, it is an epimorphism of a solvable group onto a non-abelian simple group. A contradiction. □

Thus we have proved the following statement.

Theorem 10. *There exists a residually finite finitely generated group Γ and its automorphism φ such that $R(\varphi) < \infty$ but Γ is not a solvable-by-finite group.*

In the same time the following statement can be easily proved.

Proposition 11. *For the specific automorphisms φ constructed above, the TBFT$_f$ is fulfilled.*

Proof. From $R(\tau_\gamma \circ \varphi') = R(\alpha(m) \circ \varphi') = 1$ it follows that Reidemeister classes of φ have "cylindrical" form, i.e. $\{\gamma\}_\varphi = p^{-1}(\{p(\gamma)\}_{\overline{\varphi}})$, where $p : \Gamma \to \mathbb{Z}$ is the natural projection. Then $\rho_i \circ p$, $i = 0, 1$, are the desired $\widehat{\varphi}$-fixed classes of representations of Γ, if $\rho_0 = 1$ and $\rho_1 = -1$ in $S^1 \subset \mathbb{C}$ are $\widehat{\overline{\varphi}}$-fixed representations of $\mathbb{Z}$ (see [20, 11] for an extended discussion of a similar situation). □

Remark 12. An important observation related this example series is that the subgroup Σ does *not* satisfy neither TBFT nor TBFT$_f$ as it was proved in [10] (of course with failure for *another* automorphism then φ').

Now we will develop Proposition 11.

Theorem 13. *Suppose that φ is an automorphism of finite order of the restricted wreath product $G \wr \mathbb{Z}^k = \oplus_{m \in \mathbb{Z}^k} G_m \rtimes_\alpha \mathbb{Z}^k$, where G is a finite group. If $R(\varphi') < \infty$, then $R(\varphi') = 1$.*

Proof. Suppose, $R(\varphi') > 1$. Then there exists an element $\sigma \in \Sigma$ such that $\sigma \notin \{e\}_{\varphi'}$. Hence $\sigma \notin \{e\}_{\varphi'_\sigma}$, where φ'_σ is the restriction of φ' onto the φ'-invariant subgroup Σ_σ generated by σ. This follows from the evident observation

$$\{e\}_{\varphi'_\sigma} = \{g(\varphi'_\sigma)^{-1}(g) \colon g \in \Sigma_\sigma\} = \{g(\varphi')^{-1}(g) \colon g \in \Sigma_\sigma\} \subseteq \subseteq \{g(\varphi')^{-1}(g) \colon g \in \Sigma\} = \{e\}_{\varphi'}.$$

In particular, $R(\varphi'_\sigma) > 1$. Denote by s the order of φ. Then by the definition, Σ_σ is a finite group with generators $\sigma, \varphi'(\sigma), \dots, (\varphi')^s(\sigma)$ (because Σ is locally finite). Hence, φ'_σ has a nontrivial fixed element σ_0, $\varphi'_\sigma(\sigma_0) = \sigma_0$ and $\sigma_0 \neq 0$ (see Lemma 5). For an element $m \in \mathbb{Z}^k$, consider the orbit (using the property (1))

$$\alpha(m)\sigma_0, \quad \varphi'(\alpha(m)\sigma_0) = \alpha(\overline{\varphi}(m))\sigma_0, \dots \quad (\varphi')^t(\alpha(m)\sigma_0) = \alpha(\overline{\varphi}^t(m))\sigma_0.$$

Here $\overline{\varphi}^{t+1}(m) = m$ and $t + 1 \le s$ is the smallest number with this property. Then $(\varphi')^{t+1}(\alpha(m)\sigma_0) = \alpha(m)\sigma_0$. For $\omega = \sum_m g_m \in \Sigma$, denote the support in $\mathbb{Z}^k$ of ω by $\operatorname{supp}_{\mathbb{Z}^k}(\omega) = \{m \in \mathbb{Z}^k \colon g_m \neq e\}$. Passing from m to $n_1 m$, $n_1 \in \mathbb{Z}$, $m \in \mathbb{Z}^k$, if necessary, we can assume that the supports $\operatorname{supp}_{\mathbb{Z}^k}(\alpha(\overline{\varphi}^j(n_1 m))\sigma_0)$, $j = 0, \dots, t$, do not intersect. Indeed, since $t+1$ is the smallest with the property $\overline{\varphi}^{t+1}(m) = m$, the elements $\overline{\varphi}^j(m)) \in \mathbb{Z}^k$ are distinct, $j = 0, \dots, t$, and we can take a large n_1 such that

$$\begin{aligned} \|\overline{\varphi}^j(n_1 m)) - \overline{\varphi}^l(n_1 m))\| &= \|n_1\overline{\varphi}^j(m)) - n_1\overline{\varphi}^l(m))\| = \\ &= |n_1| \cdot \|\overline{\varphi}^j(m)) - \overline{\varphi}^l(m))\| > 2\,\mathrm{diam}(\mathrm{supp}_{\mathbb{Z}^k}(\sigma_0)). \end{aligned}$$

Then $\sum_{j=1}^{t} \alpha(\overline{\varphi}^j(n_1 m))\sigma_0$ is a fixed element of φ', which is distinct from 0 and σ_0. Increasing n "in sufficiently large steps" we obtain infinitely many distinct fixed elements in the same way. Then by Lemma 2, $R(\varphi') = \infty$. □

Theorem 14. *Suppose that φ is an automorphism of finite order of the restricted wreath product $G \wr \mathbb{Z}^k = \oplus_{m\in\mathbb{Z}^k} G_m \rtimes_\alpha \mathbb{Z}^k$, where G is a finite group. Then φ has the TBFT$_f$ property.*

Proof. By Lemma 7, $R(\varphi) < \infty$ implies $R(\varphi') < \infty$. Then Theorem 13 implies that $R(\varphi') = 1$. Considering $\tau_z \circ \varphi$ instead of φ from the very beginning, we see that $R(\tau_z \circ \varphi') = 1$, for any $z \in \mathbb{Z}^k$. Thus, by Lemma 4, Reidemeister classes $\{g\}_\varphi$ of φ are pull-backs of Reidemeister classes $\{z\}_{\overline{\varphi}}$ of $\overline{\varphi}$ under the natural projection $\pi : G \wr \mathbb{Z}^k \to \mathbb{Z}^k$, i.e. $\{g\}_\varphi = \pi^{-1}(\{\pi(g)\}_{\overline{\varphi}})$. So, if classes of $\overline{\varphi}$ are separated by an epimorphism $f : \mathbb{Z}^k \to A$ onto a finite abelian group A, then classes of φ are separated by $f \circ \pi$.

Thus by Lemma 6, the statement follows from TBFT$_f$ for abelian groups [3] (see also [7]). □

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Evgenij Vadimovich Troitsky
Moscow Center for Fundamental and Applied Mathematics,
MSU Department;
Dept. of Mech. and Math.,
Lomonosov Moscow State University,
119991, Moscow, Russia
Email address: troitsky@mech.math.msu.su