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REIDEMEISTER CLASSES, WREATH PRODUCTS AND SOLVABILITY

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ABSTRACT. Reidemeister (or twisted conjugacy) classes are considered in restricted wreath products of the form $G \wr \mathbb{Z}^k$, where G is a finite group.

For an automorphism φ of finite order with finite Reidemeister number $R(\varphi)$, this number is identified with the number of equivalence classes of finite-dimensional unitary irreducible representations that are fixed by the dual homeomorphism $\widehat{\varphi}$ (i.e. the so-called conjecture TBFT_f is proved in this case).

We construct a counterexample from this class of groups to disprove the following conjecture: if a finitely generated residually finite group has an automorphism with $R(\varphi) < \infty$ then it is solvable-by-finite (so-called conjecture R).

Keywords: Reidemeister number, twisted conjugacy class, Burnside-Frobenius theorem, solvable group, unitary dual, finite-dimensional representation, wreath product.

Reidemeister, or twisted conjugacy classes of an automorphism φ of a group Γ are equivalence classes with respect to $x \sim yx\varphi(y^{-1})$. Their number $R(\varphi)$ (finite or infinite) is called *Reidemeister number*.

The following three directions form the mainstream of the current study of Reidemeister classes:

- 1) To prove or disprove the so-called TBFT (twisted Burnside-Frobenius theory) conjecture: the Reidemeister number $R(\varphi)$ (if finite) coincides with the number of equivalence classes of irreducible unitary representations of

Γ fixed by the induced homeomorphism $\widehat{\varphi}$ of the unitary dual $\widehat{\Gamma}$. Its finite-dimensional version TBFT_f was also studied (here one considers only finite-dimensional fixed representations). The most important classes of groups for which TBFT_f is true, are polycyclic-by-finite groups [7] and residually finite groups of finite Prüfer rank [21]. On the other hand, in [10] we have detected an example of infinitely generated residually finite group which has neither TBFT nor TBFT_f .

- 2) As an opposite case, to determine classes of groups for which any automorphism has infinite Reidemeister number (this property is called R_∞) — the list of results here is very extended, we mention only some expository or recent papers: [5, 1, 8, 18, 11, 21, 17, 19].
- 3) To study rationality and other properties of Reidemeister zeta function constructed from $R(\varphi^n)$ (see e.g. [4, 9] for a recent progress).

We will deal with the first two aspects in this paper. Inspired by [16] we have formulated in [8] the following conjecture:

Conjecture R. *Let Γ be a finitely generated residually finite group. Either Γ is R_∞ , or Γ is solvable-by-finite.*

It was discussed in several papers, in particular in [13]. The conjecture was supported by the following recent result [21]: any residually finite group of finite upper rank admitting an automorphism φ with finite Reidemeister number $R(\varphi)$ is solvable-by-finite.

The main results of the present paper are:

A) The above conjecture (R) fails to be true. More specifically, we construct a series of counterexamples of the form $\Gamma = (G \oplus G) \wr \mathbb{Z}$, where G is a non-abelian finite simple group, and automorphisms $\varphi : \Gamma \rightarrow \Gamma$ such that $R(\varphi) < \infty$ but Γ is not solvable-by-finite (Theorem 10).

B) TBFT_f is true for automorphisms of finite order of $G \wr \mathbb{Z}^k$, where G is an arbitrary finite group (Theorem 14).

The second result generalizes to the non-commutative case a result from [11] and is much more expected than the first one.

Recall some facts to be used in the proofs.

Let $\mathbf{F}(\varphi) := \{g \in \Gamma : \varphi(g) = g\}$ be the subgroup of φ -fixed elements. We use the notation $\tau_g(x) = gxg^{-1}$ for an inner automorphism and for its restriction to a normal subgroup.

From the equality

$$yg\varphi(y^{-1})x = ygx x^{-1}\varphi(y^{-1})x = y(gx)(\tau_{x^{-1}} \circ \varphi)(y^{-1}),$$

it follows a very useful statement (see e.g. [7]):

Lemma 1. *Shifts of Reidemeister classes of φ are Reidemeister classes of $\tau_{x^{-1}} \circ \varphi$:*

$$\{g\}_\varphi x = \{gx\}_{\tau_{x^{-1}} \circ \varphi}.$$

Hence, $R(\tau_g \circ \varphi) = R(\varphi)$.

Lemma 2 ([16]). *Suppose that Γ is a residually finite group and $\varphi : \Gamma \rightarrow \Gamma$ is an automorphism of finite order with $R(\varphi) < \infty$. Then $|\mathbf{F}(\varphi)| < \infty$.*

Lemma 3 (Prop. 3.4 in [6]). *Suppose that Γ is a finitely generated residually finite group and $\varphi : \Gamma \rightarrow \Gamma$ is an automorphism with $R(\varphi) < \infty$. Then $|\mathbf{F}(\varphi)| < \infty$.*

Lemma 4 ([3, 12], see also [7, 15]). *Suppose, $\varphi : \Gamma \rightarrow \Gamma$ is an automorphism of a discrete group, H is a normal φ -invariant subgroup of Γ , so φ induces automorphisms $\varphi' : H \rightarrow H$ and $\tilde{\varphi} : \Gamma/H \rightarrow \Gamma/H$. Then*

- *the projection $\Gamma \rightarrow \Gamma/H$ maps Reidemeister classes of φ onto Reidemeister classes of $\tilde{\varphi}$, in particular $R(\tilde{\varphi}) \leq R(\varphi)$;*
- *if $|\mathbf{F}(\tilde{\varphi})| = n$, then $R(\varphi') \leq R(\varphi) \cdot n$;*
- *if $\mathbf{F}(\tilde{\varphi}) = \{e\}$, then each Reidemeister class of φ' is an intersection of the appropriate Reidemeister class of φ and H ;*
- *if $\mathbf{F}(\tilde{\varphi}) = \{e\}$, then $R(\varphi) = \sum_{j=1}^R R(\tau_{g_j} \circ \varphi')$, where g_1, \dots, g_R are some elements of Γ such that $p(g_1), \dots, p(g_R)$ are representatives of all Reidemeister classes of $\tilde{\varphi}$, $p : \Gamma \rightarrow \Gamma/H$ is the natural projection and $R = R(\tilde{\varphi})$.*

Recall also the following folklore observation.

Lemma 5. *For an automorphism $f : F \rightarrow F$ of a finite group, $R(f) > 1$ if and only if $\mathbf{F}(f) \neq \{e\}$.*

Proof. Indeed, consider the Reidemeister class $\{e\}_f$ as an orbit of the twisted action of G on itself: $g : x \mapsto gfx(g^{-1})$. Then by the orbit-stabilizer theorem, $R(f) = 1$, i.e. $\{e\}_f = G$, if and only if the stabilizer of e under the twisted action is trivial. But $xf(x^{-1}) = e$ if and only if $x \in \mathbf{F}(f)$. \square

We say that Reidemeister classes of $\varphi : \Gamma \rightarrow \Gamma$ are separated by an epimorphism $f : \Gamma \rightarrow F$ onto a finite group F if f induces a bijection of Reidemeister classes.

Lemma 6 (see [7, 6]). *Let $\varphi : \Gamma \rightarrow \Gamma$ have $R(\varphi) < \infty$. Then $TBFT_f$ is true for φ if and only if Reidemeister classes of φ are separated by an epimorphism $f : \Gamma \rightarrow F$ onto a finite group F .*

Now we pass to a construction of the desired counterexample.

We have by definition, $F \wr \mathbb{Z}^k = \Sigma \rtimes_{\alpha} \mathbb{Z}^k$, where Σ denotes $\bigoplus_{x \in \mathbb{Z}^k} F_x$, and $\alpha(x)(g_y) = g_{x+y}$. Here g_x denotes g as an element of $F \cong F_x$.

The following statement was discussed in [14, 11] in the abelian case.

Lemma 7. *An automorphism $\varphi : F \wr \mathbb{Z}^k \rightarrow F \wr \mathbb{Z}^k$, where $|F| < \infty$, has $R(\varphi) < \infty$ if and only if $R(\tilde{\varphi}) < \infty$ and $R(\tau_m \circ \varphi') < \infty$ for any $m \in \mathbb{Z}^k$, where $\varphi' : \bigoplus_m F_m \rightarrow \bigoplus_m F_m$ and $\tilde{\varphi} : \mathbb{Z}^k \rightarrow \mathbb{Z}^k$ are induced by φ (in fact, it is sufficient to verify this for representatives of Reidemeister classes of $\tilde{\varphi}$).*

Proof. Suppose, $R(\varphi) < \infty$. By Lemma 4, we have $R(\tilde{\varphi}) < \infty$. Then by Lemma 3, we obtain $|\mathbf{F}(\tilde{\varphi})| < \infty$ (in fact, $|\mathbf{F}(\tilde{\varphi})| = 1$, because an automorphism of \mathbb{Z}^k can not have finitely many fixed elements except of 0). So, by Lemma 4, $R(\varphi') < \infty$. Considering $\tau_z \circ \varphi$, which has $R(\tau_z \circ \varphi) = R(\varphi) < \infty$, instead of φ , we obtain in the same way that $R(\tau_z \circ \varphi') < \infty$.

Conversely, having $|\mathbf{F}(\tilde{\varphi})| = 1$, one can apply Lemma 4 (the formula from the last item) and use the equality $R(\tau_{\sigma s} \varphi') = R(\tau_s \varphi')$, where $\sigma \in \bigoplus_m F_m$, $s \in \mathbb{Z}^k$, see Lemma 1. \square

Now we consider a particular case of $F = G \oplus G$ and $k = 1$. In the abelian case the next statement was proved in [14].

Lemma 8. *Any such group $\Gamma = (G \oplus G) \wr \mathbb{Z}$ admits an automorphism φ with $R(\varphi) < \infty$, i.e. Γ does not have the R_{∞} property.*

Proof. We need a description of automorphisms of a semidirect product $H \rtimes K$, where H is characteristic, as matrices $\begin{pmatrix} a & b \\ 0 & d \end{pmatrix}$, where $a \in \text{Aut}(H)$, $d \in \text{Aut}(K)$, $b : K \rightarrow H$ satisfy

- (i) $b(kk') = b(k)\tau_{d(k)}(b(k'))$ for any $k, k' \in K$,
- (ii) $a(\tau_k(h)) = \tau_{b(k)d(k)}(a(h))$ for any $h \in H, k \in K$,

(see [2, Theorem 1]). If $b = 0$ (i.e. $b(k) = e_H$ for any $k \in K$), the first equality is satisfied for any d and one needs to verify only $a(\tau_k(h)) = \tau_{d(k)}(a(h))$.

Hence, since Σ is characteristic in Γ (as its torsion subgroup), taking $b = 0$ we see, that an automorphism φ can be defined by $\varphi' : \Sigma \rightarrow \Sigma$ and $\bar{\varphi} : \mathbb{Z} \rightarrow \mathbb{Z}$ restricted to satisfy

$$(1) \quad \varphi'(\alpha(m)(h)) = \alpha(\bar{\varphi}(g))(\varphi'(h)), \quad h \in \Sigma, \quad m \in \mathbb{Z}^k.$$

We define φ' as $\varphi'_0 : (G \oplus G)_0 \rightarrow (G \oplus G)_0$ by the matrix $M = \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}$ and

$$\varphi'_i : (G \oplus G)_i \oplus (G \oplus G)_{-i} \rightarrow (G \oplus G)_i \oplus (G \oplus G)_{-i} \quad \text{by} \quad \begin{pmatrix} 0 & M \\ M & 0 \end{pmatrix},$$

i.e. $\varphi' = M : (G \oplus G)_i \rightarrow (G \oplus G)_{-i}$. Then $R(\varphi') = \prod_{i \geq 0} R(\varphi'_i)$. We claim that $R(\varphi') = 1$.

By Lemma 5 to prove that $R(\varphi') = 1$ we need to verify that φ'_0 and φ'_i have no non-trivial fixed elements. Let us find them. For φ'_0 :

$$\begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x \\ y \end{pmatrix}, \quad \begin{cases} y = x \\ xy = y \end{cases}, \quad x = y = e.$$

For φ'_i :

$$\begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \\ u \\ v \end{pmatrix} = \begin{pmatrix} x \\ y \\ u \\ v \end{pmatrix}, \quad \begin{cases} v = x \\ uv = y \\ y = u \\ xy = v \end{cases}, \quad x = y = u = v = e.$$

Thus, $R(\varphi') = 1$.

To estimate $R(\varphi)$ using Lemma 7, it is sufficient to estimate

$$R(\tau_\gamma \circ \varphi') = R(\tau_\sigma \tau_m \circ \varphi') = R(\tau_m \circ \varphi') = R(\alpha(m) \circ \varphi'),$$

where $\gamma = \sigma \cdot m \in \Gamma$, $\sigma \in \Sigma$, $m \in \mathbb{Z}$, and the middle equality is obtained similarly to the end of the proof of Lemma 7 above.

Then $\alpha(m) \circ \varphi'$ maps with the help of M

$$(G \oplus G)_i \rightarrow (G \oplus G)_{-i+m}, \quad (G \oplus G)_{-i+m} \rightarrow (G \oplus G)_{-m+i+m} = (G \oplus G)_i, \\ (G \oplus G)_{m/2} \rightarrow (G \oplus G)_{m/2} \quad (\text{if } m \text{ is even}).$$

Thus, for an even m , the invariant subgroup structure and the restrictions of automorphism are isomorphic. The same for an odd m , but in this case there will be no orbit of length 1 in i . In both cases $R(\tau_\gamma \circ \varphi') = R(\alpha(m) \circ \varphi') = 1$. So $R(\varphi) < \infty$, moreover $R(\varphi) = 2$ to be precise (using Lemma 4). \square

Lemma 9. *Suppose in the definition of Γ that G is a non-abelian simple group, hence, perfect, $[G, G] = G$. Then Γ is not a solvable-by-finite group.*

Proof. If Γ is a (finitely generated) solvable-by-finite group, then it has a characteristic solvable subgroup Γ' of finite index, $\Gamma/\Gamma' = F$, $|F| < \infty$. Then $\Sigma' := \Sigma \cap \Gamma'$ is a solvable subgroup of finite index in Σ . Hence, for some summand, the projection $\Sigma' \rightarrow G$ is non-trivial. Thus, it is an epimorphism of a solvable group onto a non-abelian simple group. A contradiction. \square

Thus we have proved the following statement.

Theorem 10. *There exists a residually finite finitely generated group Γ and its automorphism φ such that $R(\varphi) < \infty$ but Γ is not a solvable-by-finite group.*

In the same time the following statement can be easily proved.

Proposition 11. *For the specific automorphisms φ constructed above, the TBFT_f is fulfilled.*

Proof. From $R(\tau_\gamma \circ \varphi') = R(\alpha(m) \circ \varphi') = 1$ it follows that Reidemeister classes of φ have ‘‘cylindrical’’ form, i.e. $\{\gamma\}_\varphi = p^{-1}(\{p(\gamma)\}_{\widehat{\varphi}})$, where $p : \Gamma \rightarrow \mathbb{Z}$ is the natural projection. Then $\rho_i \circ p$, $i = 0, 1$, are the desired $\widehat{\varphi}$ -fixed classes of representations of Γ , if $\rho_0 = 1$ and $\rho_1 = -1$ in $S^1 \subset \mathbb{C}$ are $\widehat{\varphi}$ -fixed representations of \mathbb{Z} (see [20, 11] for an extended discussion of a similar situation). \square

Remark 12. An important observation related this example series is that the subgroup Σ does *not* satisfy neither TBFT nor TBFT_f as it was proved in [10] (of course with failure for *another* automorphism then φ').

Now we will develop Proposition 11.

Theorem 13. *Suppose that φ is an automorphism of finite order of the restricted wreath product $G \wr \mathbb{Z}^k = \bigoplus_{m \in \mathbb{Z}^k} G_m \rtimes_\alpha \mathbb{Z}^k$, where G is a finite group. If $R(\varphi') < \infty$, then $R(\varphi') = 1$.*

Proof. Suppose, $R(\varphi') > 1$. Then there exists an element $\sigma \in \Sigma$ such that $\sigma \notin \{e\}_{\varphi'}$. Hence $\sigma \notin \{e\}_{\varphi'_\sigma}$, where φ'_σ is the restriction of φ' onto the φ' -invariant subgroup Σ_σ generated by σ . This follows from the evident observation

$$\begin{aligned} \{e\}_{\varphi'_\sigma} &= \{g(\varphi'_\sigma)^{-1}(g) : g \in \Sigma_\sigma\} = \{g(\varphi')^{-1}(g) : g \in \Sigma_\sigma\} \subseteq \\ &\subseteq \{g(\varphi')^{-1}(g) : g \in \Sigma\} = \{e\}_{\varphi'}. \end{aligned}$$

In particular, $R(\varphi'_\sigma) > 1$. Denote by s the order of φ . Then by the definition, Σ_σ is a finite group with generators $\sigma, \varphi'(\sigma), \dots, (\varphi')^s(\sigma)$ (because Σ is locally finite). Hence, φ'_σ has a nontrivial fixed element σ_0 , $\varphi'_\sigma(\sigma_0) = \sigma_0$ and $\sigma_0 \neq 0$ (see Lemma 5). For an element $m \in \mathbb{Z}^k$, consider the orbit (using the property (1))

$$\alpha(m)\sigma_0, \quad \varphi'(\alpha(m)\sigma_0) = \alpha(\widehat{\varphi}(m))\sigma_0, \dots \quad (\varphi')^t(\alpha(m)\sigma_0) = \alpha(\widehat{\varphi}^t(m))\sigma_0.$$

Here $\widehat{\varphi}^{t+1}(m) = m$ and $t+1 \leq s$ is the smallest number with this property. Then $(\varphi')^{t+1}(\alpha(m)\sigma_0) = \alpha(m)\sigma_0$. For $\omega = \sum_m g_m \in \Sigma$, denote the support in \mathbb{Z}^k of ω by $\text{supp}_{\mathbb{Z}^k}(\omega) = \{m \in \mathbb{Z}^k : g_m \neq e\}$. Passing from m to $n_1 m$, $n_1 \in \mathbb{Z}$, $m \in \mathbb{Z}^k$, if necessary, we can assume that the supports $\text{supp}_{\mathbb{Z}^k}(\alpha(\widehat{\varphi}^j(n_1 m))\sigma_0)$, $j = 0, \dots, t$, do not intersect. Indeed, since $t+1$ is the smallest with the property $\widehat{\varphi}^{t+1}(m) = m$, the elements $\widehat{\varphi}^j(m) \in \mathbb{Z}^k$ are distinct, $j = 0, \dots, t$, and we can take a large n_1 such that

$$\begin{aligned} \|\widehat{\varphi}^j(n_1 m) - \widehat{\varphi}^l(n_1 m)\| &= \|n_1 \widehat{\varphi}^j(m) - n_1 \widehat{\varphi}^l(m)\| = \\ &= |n_1| \cdot \|\widehat{\varphi}^j(m) - \widehat{\varphi}^l(m)\| > 2 \text{diam}(\text{supp}_{\mathbb{Z}^k}(\sigma_0)). \end{aligned}$$

Then $\sum_{j=1}^t \alpha(\bar{\varphi}^j(n_1 m))\sigma_0$ is a fixed element of φ' , which is distinct from 0 and σ_0 . Increasing n “in sufficiently large steps” we obtain infinitely many distinct fixed elements in the same way. Then by Lemma 2, $R(\varphi') = \infty$. \square

Theorem 14. *Suppose that φ is an automorphism of finite order of the restricted wreath product $G \wr \mathbb{Z}^k = \bigoplus_{m \in \mathbb{Z}^k} G_m \rtimes_{\alpha} \mathbb{Z}^k$, where G is a finite group. Then φ has the TBFT_f property.*

Proof. By Lemma 7, $R(\varphi) < \infty$ implies $R(\varphi') < \infty$. Then Theorem 13 implies that $R(\varphi') = 1$. Considering $\tau_z \circ \varphi$ instead of φ from the very beginning, we see that $R(\tau_z \circ \varphi') = 1$, for any $z \in \mathbb{Z}^k$. Thus, by Lemma 4, Reidemeister classes $\{g\}_{\varphi}$ of φ are pull-backs of Reidemeister classes $\{z\}_{\bar{\varphi}}$ of $\bar{\varphi}$ under the natural projection $\pi : G \wr \mathbb{Z}^k \rightarrow \mathbb{Z}^k$, i.e. $\{g\}_{\varphi} = \pi^{-1}(\{\pi(g)\}_{\bar{\varphi}})$. So, if classes of $\bar{\varphi}$ are separated by an epimorphism $f : \mathbb{Z}^k \rightarrow A$ onto a finite abelian group A , then classes of φ are separated by $f \circ \pi$.

Thus by Lemma 6, the statement follows from TBFT_f for abelian groups [3] (see also [7]). \square

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