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INFINITELY MANY SOLUTIONS FOR A NEW CLASS OF FRACTIONAL SCHRÖDINGER-MAXWELL SYSTEMS

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Abstract. This work concerned with the fractional Schrödinger-Maxwell system involving the distributional Riesz fractional derivative operator. By using variational techniques in combination with the symmetric mountain pass theorem, under certain assumptions on g, we prove the existence of infinitely many solutions in the Bessel potential space without the Ambrosetti-Rabinowitz's 4-superlinearity condition.

Keywords: Fractional Schrödinger-Maxwell system, symmetric mountain pass theorem, Cerami condition, Bessel potential space.

1. Introduction

In this paper, we study the following fractional Schrödinger-Maxwell system

$$\begin{cases} -D^{\alpha}.(D^{\alpha}u) + V(x)u + \phi u = g(x,u) & \text{in } \mathbb{R}^3, \\ -D^{\alpha}.(D^{\alpha}\phi) = u^2 & \text{in } \mathbb{R}^3, \end{cases} \tag{1}$$

where $\alpha \in (0,1)$, $V : \mathbb{R}^3 \to (0,\infty)$ is a continuous functions, $g : \mathbb{R}^3 \times \mathbb{R} \to \mathbb{R}$ $V : \mathbb{R}^3 \to (0,\infty)$ is continuous, and $-D^{\alpha}.(D^{\alpha})$ is the distributional Riesz fractional derivative. This operator will shortly be defined, and we will show its consistency with the usual fractional Laplacian.
In recent years, this kind of systems with the fractional Laplacian operator was studied in some papers, as a result of the fact that solutions $(u(x), \phi(x))$ of (1) involving the fractional Laplacian operator correspond to standing waves solutions

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$(e^{-iEt}u(x), \phi(x))$ of the following system

$$(2) \qquad \begin{cases} i\frac{\partial \psi}{\partial t} = (-\Delta)^{\alpha}\psi + \tilde{V}(x)\psi + \phi\psi - \tilde{g}(x, |\psi|)\psi & \text{in } \mathbb{R}^3 \times \mathbb{R}, \\ (-\Delta)^{\alpha}\phi = |\psi|^2 & \text{in } \mathbb{R}^3, \end{cases}$$

where $\tilde{V}(x) = V(x) + E$ is the potential function, i is the imaginary unit and $\tilde{g}(x, |u|)u = \tilde{g}(x, u)$.

In [15, 16] Laskin introduced the first equation in (2) as a result of extending the Feynman path integral from the Brownian-like to the Lévy-like quantum mechanical paths, it is a fundamental equation in fractional quantum mechanics. It also appeared in several areas such as plasma physics, water waves, optimization, and condensed matter physics. For more details on the physical background, we refer the readers to [2, 4, 6] and their references.

Taking $\alpha = 1$, system (2) reduces to the following system

$$(3) \qquad \begin{cases} -\Delta u + V(x)u + \phi u = g(x, u) & \text{in } \mathbb{R}^3, \\ -\Delta\phi = u^2 & \text{in } \mathbb{R}^3. \end{cases}$$

which was first introduced by Benci and Fortunato in [3], to describe solitary waves for nonlinear Schrödinger equations interacting with an electrostatic field. In the fractional case, the classical analysis is not available and elliptic PDEs cannot be treated pointwisely. This is one reason why, in the past several years, fractional problems are extensively studied by many scholars. On the other hand, another important reason is their various applications in many fields, such as fractional quantum mechanics [15, 16], obstacle problems [17], and so on. Caffarelli and Silvestre in their celebrated work [7] introduce the reduction method to overcome these difficulties. Since then and with the aid of [11], there have been many interesting works that considered the existence, multiplicity, ground state, and infinitely many solutions to the fractional Schrödinger-Poisson system (2) through variational tools and critical point theory, see for instance [9, 12, 13, 14, 23, 25].

More recently, an increasing number of authors have focused their attention on problems involving the distributional Riesz fractional derivative in their works, see e.g [5, 17, 20, 21]. Shieh and Spector in their pioneered work [20] gave a new definition of fractional derivative, as they were the first to study a new kind of fractional PDEs related to the distributional fractional gradient operator D^{α} of order $\alpha \in (0, 1)$, which is called here the α-gradient, they showed that the latter operator satisfies three basic physical requirements as proved in [22] on fractional gradient analysis, as well as it is an intrinsic object of interest for the study of fractional PDEs. Then, they introduced an appropriate functional space to study fractional problems in which the α-gradient is present.

Motivated by all the works just described, this aims paper is to prove the existence of infinitely many solutions for (1) in the Bessel potential space. in particular, we do not use the Ambrosetti-Rabinowitz condition.

The starting point of this work is the α-gradient for $1 < p < \infty$, if $u \in L^p(\mathbb{R}^N)$, such that $I_{1-\alpha} * u$ is well defined, D^{α} can be characterized as

$$(D^{\alpha}u)_j = \frac{\partial^{\alpha} u}{\partial x_j^{\alpha}} = \frac{\partial}{\partial x_j} I_{1-\alpha} * u \quad 0 < \alpha < 1, \quad j = 1, ..., N,$$

where $\dfrac{\partial}{\partial x_j}$ is defined in the distributional sense, for every $\varphi \in C_0^\infty(\mathbb{R}^N)$,

$$\langle \frac{\partial^\alpha u}{\partial x_i^\alpha}, \varphi\rangle = (-1)\langle (I_{1-\alpha} * u), \frac{\partial \varphi}{\partial x_j}\rangle = -\int_{\mathbb{R}^N} (I_{1-\alpha} * u)\frac{\partial \varphi}{\partial x_j} dx,$$

where I_α denotes the Riesz potential of order α, $0 < \alpha < 1$:

$$I_\alpha u\,(x) := \gamma\,(N,\alpha) \int\limits_{\mathbb{R}^N} \frac{u\,(y)}{|x-y|^{N-\alpha}} dy, \quad \text{with} \quad \gamma(N,\alpha) := \frac{\Gamma(\frac{N-\alpha}{2})}{\pi^{\frac{N}{2}} 2^\alpha \Gamma(\frac{\alpha}{2})}.$$

Moreover, the α-gradient (D^α) and the α-divergence ($D^\alpha.$) can be written for sufficiently regular functions u and vector φ [10, 17, 21], respectively by

$$D^\alpha u(x) := \gamma(N,\alpha) \int_{\mathbb{R}^N} [u(x) - u(y)] \frac{x-y}{|x-y|} \frac{1}{|x-y|^{N+\alpha}} dy,$$

$$D^\alpha.\varphi(x) := \gamma(N,\alpha) \int_{\mathbb{R}^N} [\varphi(x) - \varphi(y)].\frac{x-y}{|x-y|} \frac{1}{|x-y|^{N+\alpha}} dy.$$

As it was proved in [10, 20], for $u \in C_0^\infty(\mathbb{R}^N)$, the composition of α-divergence and α-gradient it coincides with the well known fractional Laplacian as follows:

$$\begin{aligned} (-\Delta)^\alpha u &= -\sum_{j=1}^N \frac{\partial^\alpha}{\partial x_j^\alpha} \frac{\partial^\alpha}{\partial x_j^\alpha} u \\ &= -D^\alpha.(D^\alpha u), \end{aligned} \tag{4}$$

where the fractional Laplacian may be given (see [11]), for $\alpha \in (0,1)$ and some constant $\gamma(N,\alpha)$ by

$$(-\Delta)^\alpha u(x) = \gamma(N,\alpha) P.V. \int_{\mathbb{R}^N} \frac{u(x)-u(y)}{|y|^{N+2\alpha}} dy.$$

Furthermore, for $u, w \in C_0^\infty(\mathbb{R}^N)$ equation (4) is to be understood in the following sense

$$\int\limits_{\mathbb{R}^N} D^\alpha u.D^\alpha w dx = \int\limits_{\mathbb{R}^N} (-\Delta)^\alpha u.w dx = \int\limits_{\mathbb{R}^N} (-\Delta)^{\frac{\alpha}{2}} u.(-\Delta)^{\frac{\alpha}{2}} w dx,$$

which is particularily useful for the variational formulation of PDEs involving fractional operator. We refer to [10, 17, 20, 21, 22] for more details about this new fractional operator.

The following assumptions on g and V will be needed throughout the paper:

(H_1) : $g \in C(\mathbb{R}^3 \times \mathbb{R}; \mathbb{R})$ for every $x \in \mathbb{R}^3$ and $u \in \mathbb{R}$, there exists constant $C_1 > 0$, and $p \in]2; 2_\alpha^*[$ such that

$$|g(x,u)| \le C_1(|u| + |u|^{p-1}),$$

where $2_\alpha^* = \frac{6}{3-2\alpha}$ the fractional critical Sobolev exponent .

(H_2) : $g(x,-u) = -g(x,u)$, $x \in \mathbb{R}^3$,$u \in \mathbb{R}$.

(H_3) : For every $x \in \mathbb{R}^3$ and $u \in \mathbb{R}$, there exists $\mu > 4$ and $\lambda > 0$ such that

$$\mu G(x,u) \le u g(x,u) + \lambda |u|^2,$$

where $G(x,u) = \int\limits_0^u g(x,s)ds.$

(H_4) : We have for all $(x,u) \in \mathbb{R}^3 \times \mathbb{R}$

$$\lim_{|u|\to\infty} \frac{G(x,u)}{|u|^4} = +\infty,$$

uniformly in $x \in \mathbb{R}^3$ and $G(x,u) \geq 0$, $G(x,0) \equiv 0$.

(V) : $V \in C(\mathbb{R}^3,\mathbb{R}), V_0 := \inf_{x\in\mathbb{R}^3} V(x) > 0$, where V_0 is a constant, for every $M > 0$ meas $\left\{x \in \mathbb{R}^3 : V(x) \leq M\right\} < \infty$, where meas represents the Lesbesgue measure. Our main result is as follows:

Theorem 1. *Assume that system* (1) *satisfies* (H_1)-(H_4) *and* (V), *then* (1) *has infinitely many non-trivial solutions.*

The rest of the paper is organized as follows, in section 2, we present some preliminary results that will be used in this paper. In section 3, we give the proof of Theorem 1. In section 4, we give a discussion about our results.

Remark 2. *(i) In this paper, we do not impose the Ambrosetti-Rabinowitz's 4-superlinearity condition:*

$$\exists\mu > 4 \text{ such that } 0 < \mu G(x,u) \leq ug(x,u) \text{ for all } x \in \mathbb{R}^3, \tag{5}$$

which was first introduced by Ambrosetti and Rabinowitz in [1]*, this condition is important to ensure that the corresponding functional I has the mountain pass geometry, and to guarantee that the* (PS)*, or* (C) *sequence of* I *is bounded.*
(ii) Hypothesis (H_3)*, is weaker than the (AR)-condition.*

2. Preliminaries and variational settings

In this section, we state some preliminary results that will be needed later.
For $\alpha \in (0,1)$, the fractional Sobolev space $W^{\alpha,2}(\mathbb{R}^N)$ is defined as follows

$$W^{\alpha,2}(\mathbb{R}^N) := \left\{ u \in L^2(\mathbb{R}^N) : \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{|u(x)-u(y)|^2}{|x-y|^{N+2\alpha}} dxdy < +\infty \right\}.$$

Is simply denoted by $H^\alpha(\mathbb{R}^N)$. For $\alpha \in (0,1)$ and $u \in C_0^\infty(\mathbb{R}^N)$, we can thus define the vector space of fractional differentiable functions $S^{\alpha,2}(\mathbb{R}^N)$ as the closure of $C_0^\infty(\mathbb{R}^N)$ naturally endowed with the norm

$$\|u\|^2_{S^{\alpha,2}(\mathbb{R}^N)} = \|u\|^2_{L^2(\mathbb{R}^N)} + \|D^\alpha u\|^2_{L^2(\mathbb{R}^N)}. \tag{6}$$

According to the Theorem 1.7 in [20], it is exactly the Bessel potential space defined by

$$L^{\alpha,2}(\mathbb{R}^N) := G_\alpha(L^2(\mathbb{R}^N)) = \{G_\alpha * f : f \in L^2(\mathbb{R}^N)\},$$

for $\alpha \in \mathbb{R}_+$, where the bessel potential G_α is defined by (see [19, 20])

$$G_\alpha(x) := \frac{1}{(4\pi)^{\frac{\alpha}{2}}\Gamma(\frac{\alpha}{2})} \int_0^{+\infty} e^{\frac{-\pi|x|^2}{t}} e^{\frac{-t}{4\pi}} t^{\frac{\alpha-N}{2}-1} dt.$$

The norm of this Bessel space is $\|u\|_{L^{\alpha,2}(\mathbb{R}^N)} = \|f\|_{L^2(\mathbb{R}^N)}$ if $G_\alpha * f$.
We summarize the main properties of this Bessel space (see [20]).

Theorem 3. (1) *If* $\alpha \in (0,1)$, *then* $H^\alpha(\mathbb{R}^N) = W^{\alpha,2}(\mathbb{R}^N) = L^{\alpha,2}(\mathbb{R}^N) = S^{\alpha,2}(\mathbb{R}^N)$ *with the norm given by* (6).

(2) *If* $\alpha \geq 0$ *and* $2 \leq q \leq 2^*_\alpha$, *then* $L^{\alpha,2}(\mathbb{R}^N)$ *is continuously embedded in* $L^q(\mathbb{R}^N)$, *and the embedding is locally compact if* $2 \leq q < 2^*_\alpha$,

Remark 4. *According to the Theorem 3, The Bessel potential space* $L^{\alpha,2}(\mathbb{R}^N)$ *is topologically undistinguishable from the well known fractional Sobolev space* $H^\alpha(\mathbb{R}^N)$.

The homogeneous Sobolev space $D^{\alpha,2}(\mathbb{R}^N)$ for $\alpha \in (0,1)$, is defined by

$$D^{\alpha,2}(\mathbb{R}^N) = \left\{ u \in L^{2^*_\alpha}(\mathbb{R}^N) : D^\alpha u \in L^2(\mathbb{R}^N \right\},$$

which is the completion of $C_0^\infty(\mathbb{R}^N)$ under the norm

$$\|u\|_{D^{\alpha,2}(\mathbb{R}^N)} = (\int_{\mathbb{R}^N} |D^\alpha u|^2 dx)^{\frac{1}{2}}.$$

endowed with the inner product

$$\langle u, w\rangle_{D^{\alpha,2}(\mathbb{R}^N)} = \int_{\mathbb{R}^N} (D^\alpha u.D^\alpha w)dx.$$

Now, we introduce our working space

$$E = \left\{ u \in L^{\alpha,2}\left(\mathbb{R}^N\right) : \int_{\mathbb{R}^N} |D^\alpha u|^2 + V(x)u^2)dx < \infty \right\},$$

which is a Hilbert space equipped with the norm and the inner product respectively,

$$\|u\|_E = \int_{\mathbb{R}^N} (|D^\alpha u|^2 + V(x)|u|^2)^{\frac{1}{2}} dx.$$

$$\langle u, w\rangle_E = \int_{\mathbb{R}^N} (D^\alpha u.D^\alpha w + V(x)uw)dx.$$

Remark 5. *The two definitions above of the fractional Spaces* $D^{\alpha,2}(\mathbb{R}^N)$ *and* E *coincides with any other standard definitions of them found in the literature.*

Assumption (V) implies that $\|u\|_E = \|u\|_{L^{\alpha,2}(\mathbb{R}^N)}$, and we have the following embedding properties of E.

Lemma 1. [23] *E is compactly embedded in* $L^q\left(\mathbb{R}^N\right)$ *for* $q \in [2, 2^*_\alpha)$, *and continuously embedded in* $L^q\left(\mathbb{R}^N\right)$ *for* $q \in [2, 2^*_\alpha]$.

The following Sobolev embedding theorem are necessary.

Lemma 2. *(*[11]*) For any* $\alpha \in \left(0, \frac{3}{2}\right)$, $D^{\alpha,2}(\mathbb{R}^N)$ *is continuously embedded in* $L^{2^*_\alpha}(\mathbb{R}^N)$, *i.e there exists* $C_\alpha > 0$ *such that :*

$$\left(\int_{\mathbb{R}^N} |u|^{2^*_\alpha} dx\right)^{\frac{2}{2^*_\alpha}} \leq C_\alpha \int_{\mathbb{R}^N} |D^\alpha u|^2 dx,\ u \in D^{\alpha,2}\left(\mathbb{R}^N\right).$$

The existence of a solution to a linear fractional PDEs with variable coefficients is established by the following theorem.

Theorem 6. *([20]) Let $\Omega \subset \mathbb{R}^N$ is an arbitrary bounded open set. Assume that $h \in L^2(\Omega)$, such that $I_{1-\alpha} * u$ is well defined and $A : \mathbb{R}^N \longrightarrow \mathbb{R}^{N\times N}$ with coefficients bounded and measurable such that*

$$c_*|y|^2 \leq A(x)y.y \quad and \quad A(x)y.y \leq c^*|y|^2$$

For all $x, y \in \mathbb{R}^N$, and some $c_*, c^* > 0$. Then, there exists a unique $u \in L^{\alpha,2}(\mathbb{R}^N)$ such that

$$\int_{\mathbb{R}^N} A(x)D^\alpha u.D^\alpha w dx = \int_\Omega hw dx$$

for every $w \in L^{\alpha,2}(\mathbb{R}^N)$.

From now on, we restrict the work space in dimension $N = 3$.

2.1. **A reduced problem.** For any $u \in E$ and $w \in D^{\alpha,2}(\mathbb{R}^3)$ we have from Hölder inequality, Lemma 1 and Lemma 2

$$\begin{aligned} \int_{\mathbb{R}^3} u^2 w dx &\leq \|u\|^2_{L^{\frac{12}{3+2\beta}}} \|w\|_{L^{2^*_\alpha}} \\ &\leq C\|u\|^2_E \|w\|_{D^{\alpha,2}}. \end{aligned} \tag{7}$$

For any $u \in E$, the Lax-Milgram theorem implies that there exists a unique $\phi_u^\alpha \in D^{\alpha,2}\left(\mathbb{R}^3\right)$ such that

$$\int_{\mathbb{R}^3} D^\alpha \phi_u^\alpha . D^\alpha w dx = \int_{\mathbb{R}^3} u^2 w dx \quad \forall w \in D^{\alpha,2}\left(\mathbb{R}^3\right). \tag{8}$$

i.e. ϕ_u^α is a weak solution of $-D^\alpha.\left(D^\alpha \phi_u^\alpha\right) = u^2$. Moreover ϕ_u^α is given by

$$\phi_u^\alpha(x) = c_\alpha \int_{\mathbb{R}^3} \frac{u^2(y)}{|x-y|^{3-2\alpha}} dy, \tag{9}$$

which is the Riesz potential (see [19]), where

$$c_\alpha = \pi^{-\frac{3}{2}} 2^{-2\alpha} \frac{\Gamma\left(\frac{3-2\alpha}{2}\right)}{\Gamma(\alpha)}.$$

Taking $w = \phi_u^\alpha$ in (7) and (8), we derive

$$\|\phi_u^\alpha\|_{D^{\alpha,2}} \leq C\|u\|^2_E. \tag{10}$$

Substituting ϕ_u^α in (1), it leads to the equivalent form

$$-D^\alpha.(D^\alpha u) + V(x)u + \phi_u^\alpha u = g(x,u), \quad x \in \mathbb{R}^3, \tag{11}$$

whose corresponding functional $I : E \to \mathbb{R}$ is given as follows

$$I(u) = \frac{1}{2}\int_{\mathbb{R}^3} \left(|D^\alpha u|^2 + V(x)u^2\right) dx + \frac{1}{4}\int_{\mathbb{R}^3} \phi_u^\alpha u^2 dx - \int_{\mathbb{R}^3} G(x,u)\,dx. \tag{12}$$

Clearly, I is well defined in E and $I \in C^1(E,\mathbb{R})$. Moreover, its derivative is

$$< I'(u), w >= \int_{\mathbb{R}^3} (D^\alpha u.D^\alpha w + V(x)uw + \phi_u^\alpha uw - g(x,u)w)dx, \; w \in E. \tag{13}$$

Thus, we have the following result:

Theorem 7. *the pair $(u,\phi)\in E\times D^{\alpha,2}\left(\mathbb{R}^3\right)$ is a weak solution of (1) if and only if $u\in E$ is a critical point of functional I, where $\phi=\phi_u^\alpha$.*

Since we do not suppose (5), the verification of $(\mathrm{PS})_c$ condition at level $c\in\mathbb{R}$ becomes complicated, thus we introduce the Cerami condition, which was etablished by Cerami [8]. Let E' be dual space of E and assuming that $I\in C^1$.

Definition 1. *The functional I satisfies the Cerami condition at level $c\in\mathbb{R}$, denoted by $(C)_c$, if any sequence $\{u_n\}\subset E$ satisfying*

$$I(u_n)\to c \quad \text{and} \quad \|I'(u_n)\|_{E'}(1+\|u_n\|_E)\to 0 \quad \text{as } n\to+\infty.$$

has a convergence subsequence.

Choosing $\{e\}_i$ an orthonormal basis of E and define $X_i=\mathbb{R}e_i$,

$$Y_k=\oplus_{i=1}^k X_i \quad Z_k=\overline{\oplus_{i=k}^\infty X_i} \quad k\in\mathbb{Z}.$$

Clearly, $E=Y_k\oplus Z_k$.
To prove our results, we need the following symmetric mountain-pass theorem [18].

Theorem 8. *(Symmetric mountain-pass theorem) Let $E=Y_k\oplus Z_k$ be an infinite dimensional Banach space where Y is finite dimensional, let $I\in C^1(E,\mathbb{R})$ be even, satisfies the $(C)_c$ condition and $I(0)=0$, if*
(i) there exist constants $\rho,\delta>0$ satisfying $I\left|_{\partial B_\rho\bigcap Z}\right.=\inf\limits_{u\in Z,\|u\|=\rho}I(u)\geq\delta$;
(ii) for every finite dimensional subspace $\tilde{E}\in E$, there is a constant $C=C(\tilde{E})>0$ such that $\max\limits_{u\in\tilde{E},\|u\|\geq C}I(u)<0$,
then I has an unbounded sequence of critical points.

3. Proof of main result

Before the proof of theorem, the following Lemma plays an important role in obtaining the existence of weak solution for (1).

Lemma 3. *Let $\alpha\in(0,1)$, the functional I satisfies $(C)_c$ condition on E, if (H_1),(H_3),(H_4) and (V) hold.*

Proof. Seeking a contradiction. We assume that $\|u_n\|_E\to\infty$ as $n\to\infty$. Define $\{v_n\}\subset E$ such that $v_n=\frac{u_n}{\|u_n\|_E}$, then clearly $\|v_n\|_E=1$. Hence, there exists a subsequence $\{v_n\}$ such that $v_n\rightharpoonup v$ in E as $n\to\infty$. From Lemma 1 we get, for $2\leq p<2_\alpha^*$

$$v_n\to v \text{ a.e. in } \mathbb{R}^3 \text{ and } v_n\to v \text{ in } L^p\left(\mathbb{R}^3\right) \text{ as } n\to\infty. \tag{14}$$

There are two possible cases.
First, we consider the case $v(x)=0$. From (H_3) we have

$$\begin{aligned} c+1 &\geq I(u_n)-\frac{1}{\mu}\langle I'(u_n),u_n\rangle \\ &= \left(\frac{1}{2}-\frac{1}{\mu}\right)\|u_n\|_E^2+\left(\frac{1}{4}-\frac{1}{\mu}\right)\int\limits_{\mathbb{R}^3}\phi_{u_n}^\alpha u_n^2dx+\int\limits_{\mathbb{R}^3}\left(\frac{g(x,u_n)u_n}{\mu}-G(x,u_n)\right)dx \\ &\geq \left(\frac{1}{2}-\frac{1}{\mu}\right)\|u_n\|_E^2-\frac{\lambda}{\mu}\int\limits_{\mathbb{R}^3}|u_n|^2dx \\ &\geq \left(\frac{1}{2}-\frac{1}{\mu}\right)\|u_n\|_E^2-\frac{\lambda}{\mu}\|v_n\|_{L^2}^2\|u_n\|_E^2, \end{aligned}$$

which implies

$$\frac{c+1}{\|u_n\|_E^2} \geq \frac{1}{2} - \frac{1}{\mu} - \frac{\lambda}{\mu} \|v_n\|_{L^2}^2 .$$

Since $\|u_n\|_E \to \infty$ as $n \to \infty$, then

$$\|v_n\|_{L^2}^2 \geq \frac{\mu - 2}{2\lambda},$$

which shows that $v(x) \neq 0$, then we arrive at contradiction. In the second case $v(x) \neq 0$ in $\mathbb{R}^3$, we set $A = \{x \in \mathbb{R}^3 : v(x) \neq 0\}$. Thus meas$(A) > 0$. Since $\|u_n\|_E \to \infty$ as $n \to \infty$ and by Definition 1 and (12), we obtain

$$c = \frac{1}{2} \|u_n\|_E^2 + \frac{1}{4} \int_{\mathbb{R}^3} u^2 \phi_u^\alpha dx - \int_{\mathbb{R}^3} G(x, u_n) dx \to \infty. \tag{15}$$

Combining (10) and (15), we obtain

$$\int_{\mathbb{R}^3} G(x, u_n) dx + c = \frac{1}{2} \|u_n\|_E^2 + \frac{1}{4} \int_{\mathbb{R}^3} u^2 \phi_u^\alpha dx \leq \frac{3}{4} \|u_n\|_E^4 . \tag{16}$$

Moreover, it follows from (H_4) that there exists $s_0 > 1$ such that $G(x,s) > |s|^4$ $\forall\ |s| > s_0,\ x \in \mathbb{R}^3$. Since $G(x,s)$ is continuous. By (H_1), there exists a positive number K such that $|G(x,s)| < K,\ \forall\ (x,s) \in \mathbb{R}^3 \times [-s_0, s_0]$. Then, we can choose $K_0 \in \mathbb{R}$ such that $G(x,s) \geq K_0,\ \forall\ (x,s) \in \mathbb{R}^3 \times \mathbb{R}$, and so

$$\frac{G(x, u_n) - K_0}{\|u_n\|_E^4} \geq 0. \tag{17}$$

It follows from (H_4) that

$$\lim_{x\to\infty} \frac{G(x, u_n)}{\|u_n\|_E^4} = \lim_{x\to\infty} \frac{G(x, u_n)}{|u_n|^4} |v_n|^4 = +\infty, \tag{18}$$

for all $x \in A$. Thus, we see that meas$(A) = 0$. Indeed, if meas$(A) \neq 0$, then it follows from (16)-(18) and Fatou's Lemma that

$$\begin{aligned}
\frac{3}{4} &= \liminf_{n\to\infty} \frac{\frac{3}{4}\int_{\mathbb{R}^3} G(x, u_n) dx}{\int_{\mathbb{R}^3} G(x, u_n) dx + c} \\
&\geq \liminf_{n\to\infty} \int_{\mathbb{R}^3} \frac{\frac{3}{4} G(x, u_n)}{\frac{3}{4} \|u_n\|_E^4} dx \\
&\geq \liminf_{n\to\infty} \int_{\mathbb{R}^3} \frac{G(x, u_n)}{\|u_n\|_E^4} dx - \liminf_{n\to\infty} \int_{\mathbb{R}^3} \frac{K_0}{\|u_n\|_E^4} dx \\
&\geq \liminf_{n\to\infty} \int_{\mathbb{R}^3} \frac{G(x, u_n) - K_0}{\|u_n\|_E^4} dx. \\
&\geq \int_A \liminf_{n\to\infty} \frac{G(x, u_n)}{\|u_n\|_E^4} dx - \int_{\mathbb{R}^3} \limsup_{n\to\infty} \frac{K_0}{\|u_n\|_E^4} dx = +\infty, \qquad (19)
\end{aligned}$$

which is a contradiction, then $v(x) = 0$ a.e $x \in \mathbb{R}^3$. Thus, $\{u_n\}$ is bounded in E. Up to a subsequence, we can assume that $u_n \rightharpoonup u$ in E, from Lemma 1 we conclude $u_n \to u$ in $L^p\left(\mathbb{R}^3\right)$, for all $2 \le p < 2^*_\alpha$. Clearly, we have

$$\langle I'(u_n) - I'(u), u_n - u\rangle \to 0 \text{ and } \|u_n - u\|^2_{L^2} \to 0 \text{ as } n \to \infty. \tag{20}$$

Combining the Hölder inequality, Lemma 2 and (10), we obtain

$$\begin{aligned} \left|\int_{\mathbb{R}^3} \phi^\alpha_{u_n} u_n (u_n - u)\, dx\right| &\le \left\|\phi^\alpha_{u_n}\right\|_{L^{2^*_\alpha}} \|u_n\|_{L^{\frac{12}{3+2\alpha}}} \|u_n - u\|_{L^{\frac{12}{3+2\alpha}}} \\ &\le C\left\|\phi^\alpha_{u_n}\right\|_{D^{\alpha,2}} \|u_n\|_{L^{\frac{12}{3+2\alpha}}} \|u_n - u\|_{L^{\frac{12}{3+2\alpha}}} \\ &\le C\|u_n\|^3_E \|u_n - u\|_{L^{\frac{12}{3+2\alpha}}}. \end{aligned}$$

Similarly, we derive that

$$\left|\int_{\mathbb{R}^3} \phi^\alpha_u u (u_n - u)\, dx\right| \le C\|u\|^3_E \|u_n - u\|_{L^{\frac{12}{3+2\alpha}}}.$$

We have

$$\begin{aligned} &\left|\int_{\mathbb{R}^3} \left(\phi^\alpha_{u_n} u_n - \phi^\alpha_u u\right)(u_n - u)\, dx\right| \\ &\le \left|\int_{\mathbb{R}^3} \phi^\alpha_{u_n} u_n (u_n - u)\, dx\right| + \left|\int_{\mathbb{R}^3} \phi^\alpha_u u (u_n - u)\, dx\right| \to 0 \quad \text{as} \quad n \to \infty. \end{aligned} \tag{21}$$

According to (H_1) and the Hölder inequality, we obtain

$$\begin{aligned} &\left|\int_{\mathbb{R}^3} \left(g(x, u_n) - g(x, u)\right)(u_n - u)\, dx\right| \\ &\le C_1 \int_{\mathbb{R}^3} \left(|u_n| + |u|\right)|u_n - u|\, dx + C_1 \int_{\mathbb{R}^3} \left(|u_n|^{p-1} + |u|^{p-1}\right)|u_n - u|\, dx \\ &\le C_1\left(\|u_n\|_{L^2} + \|u\|_{L^2}\right)\|u_n - u\|_{L^2} + C_1\left(\|u_n\|^{p-1}_{L^p} + \|u\|^{p-1}_{L^p}\right)\|u_n - u\|_{L^p} \\ &\le C\left(\|u_n\|_E + \|u\|_E\right)\|u_n - u\|_{L^2} + C\left(\|u_n\|^{p-1}_E + \|u\|^{p-1}_E\right)\|u_n - u\|_{L^p} \to 0 \end{aligned}$$

as $n \to \infty$. Thus

$$\begin{aligned} \|u_n - u\|^2_E &= \langle I'(u_n) - I'(u), u_n - u\rangle - \int_{\mathbb{R}^3} \left(V(x)u_n(u_n - u) - V(x)u(u_n - u)\right) dx \\ &- \int_{\mathbb{R}^3} \left(\phi^\alpha_{u_n} u_n - \phi^\alpha_u u\right)(u_n - u)\, dx + \int_{\mathbb{R}^3} \left(f(x, u_n) - f(x, u)\right)(u_n - u) dx \to 0 \end{aligned}$$

as $n \to \infty$. Therefore, $\{u_n\}$ converges strongly in E. □

Lemma 4. *Suppose that (H_1),(H_4) and (V) are satisfied. Then, for each $\tilde{E} \subset E$, we have*

$$I(u) \to -\infty, \qquad \|u\|_E \to \infty, \quad u \in \tilde{E}.$$

Proof. Arguing indirectly, suppose that there exists $M > 0$ for some $\{u_n\} \subset \tilde{E}$ and all $n \in \mathbb{N}$, such that $I(u_n) \geq -M$ with $\|u_n\|_E \to \infty$. Set $v_n = \frac{u_n}{\|u_n\|_E}$, then $\|v_n\|_E = 1$, up to subsequence we may suppose that $v_n \rightharpoonup v$ in E. Since $\dim(\tilde{E}) < \infty$, then $v_n \to v \in \tilde{E}$ in E, $v(x)_n \to v(x)$ a.e. in $\mathbb{R}^3$ and so $\|v\|_E = 1$. Hence, by similar argument in (17)-(19) with (H_1),(H_4), we get

$$\lim_{n\to\infty} \int\limits_{\mathbb{R}^3} \frac{4G(x,u_n)}{\|u_n\|_E^4} dx \geq \lim_{n\to\infty} \int\limits_{A} \frac{4G(x,u_n)}{\|u_n\|_E^4} dx = +\infty. \tag{22}$$

which is a contradiction. □

Corollary 1. *Under assumptions (H_1),(H_4) and (V), for every finite dimensional subspace $\tilde{E} \subset E$, there is a constant $C = C(\tilde{E}) > 0$ such that*

$$I(u) \leq 0 \quad \text{for all} \ \ u \in \tilde{E} \quad \text{with } \|u\|_E \geq C.$$

Lemma 5. *For $2 \leq p < 2^*_\alpha$, we have that*

$$\Gamma_k := \sup_{u \in Z_k, \|u\|=1} \|u\|_{L^p} \to 0 \quad as \ \ k \to \infty$$

Proof. Since the embedding from E into L^p is compact, then we can prove Lemma 5 by a similar way as Lemma 2.10 in [9]. □

By Lemma 5, we can choose an integer $m \geq 1$ such that

$$\|u\|_{L^2}^2 \leq \frac{1}{2C_1} \|u\|_E^2, \quad \|u\|_{L^p}^p \leq \frac{p}{4C_1} \|u\|_E^p \quad \forall u \in Z_m. \tag{23}$$

Lemma 6. *Suppose that (H_1),(H_4) and (V) are satisfied, there exist constants $\rho, \delta > 0$ satisfying $I\big|_{\partial B_\rho \bigcap Z_m} \geq \delta > 0$.*

Proof. From (H_1) and (23), for $u \in Z_m$, choosing $\rho := \|u\|_E = \frac{1}{2}$, we derive

$$\begin{aligned} I(u) &= \frac{1}{2} \int\limits_{\mathbb{R}^3} \left(|D^\alpha u|^2 + V(x)u^2\right) dx + \frac{1}{4} \int\limits_{\mathbb{R}^3} \phi_u^\alpha u^2 dx - \int\limits_{\mathbb{R}^3} G(x,u)\, dx \\ &\geq \frac{1}{2} \|u\|_E^2 - \frac{C_1}{2} \|u\|_{L^2}^2 dx - \frac{C_1}{p} \|u\|_{L^p}^p dx \\ &\geq \frac{1}{4} (\|u\|_E^2 - \|u\|_E^p) \\ &= \frac{2^{p-2} - 1}{2^{p+2}} := \delta > 0. \end{aligned}$$

This completes the proof. □

Proof of Theorem1.1 Let $Y = Y_m$ and $Z = Z_m$. Clearly, $I(u)$ is even due to (H_2). By Lemma 3, Lemma 6 and Corollary 1, all conditions of Theorem 8 are satisfied. Thus, the problem (1) has infinitely many solutions.

4. Conclusion

The aim of this paper is the study the existence of distributional solutions in the Bessel potential space for fractional Schrödinger-Maxwell system involving distributional Riesz derivative without (AR)-condition, by exploiting the symmetric mountain pass theorem. From our perspective, this paper seems to enrich the related results of this class of systems that involves this kind of fractional operator.

References

[1] Ambrosetti, Antonio, and Paul H. Rabinowitz. "Dual variational methods in critical point theory and applications."Journal of functional Analysis 14.4 (1973): 349-381. Zbl 0273.49063

[2] Bertoin, Jean. Lévy processes. Vol. 121. Cambridge: Cambridge university press, 1996. Zbl 0861.60003

[3] Benci, Vieri, and Donato Fortunato. "An eigenvalue problem for the Schrödinger-Maxwell equations."Topological Methods in Nonlinear Analysis 11.2 (1998): 283-293. Zbl 0926.35125

[4] Bisci, Giovanni Molica, Vicenţiu D. Rădulescu, and Raffaella Servadei. Variational methods for nonlocal fractional problems. Vol. 162. Cambridge University Press, 2016. Zbl 1356.49003

[5] Boutebba, Hamza, et al. "The nontrivial solutions for nonlinear fractional Schrödinger-Poisson system involving new fractional operator."Advances in the Theory of Nonlinear Analysis and its Application 7.1: 121-132.

[6] Bucur, Claudia, and Enrico Valdinoci. Nonlocal diffusion and applications. Vol. 20. Cham: Springer, 2016. Zbl 1377.35002

[7] Caffarelli, Luis, and Luis Silvestre. "An extension problem related to the fractional Laplacian."Communications in partial differential equations 32.8 (2007): 1245-1260. Zbl 1143.26002

[8] Cerami, Giovanna. "An existence criterion for the critical points on unbounded manifolds."Istit. Lombardo Accad. Sci. Lett. Rend. A 112.2 (1978): 332-336. Zbl 0436.58006

[9] Chen, Jianhua, Xianhua Tang, and Huxiao Luo. "Infinitely many solutions for fractional Schrödinger-Poisson systems with sign-changing potential."(2017). Zbl 1370.35116

[10] D'Elia, Marta, et al. "Towards a unified theory of fractional and nonlocal vector calculus."Fractional Calculus and Applied Analysis 24.5 (2021): 1301-1355. Zbl 1498.26008

[11] Di Nezza, Eleonora, Giampiero Palatucci, and Enrico Valdinoci. "Hitchhiker's guide to the fractional Sobolev spaces."Bulletin des sciences math?matiques 136.5 (2012): 521-573. Zbl 1252.46023

[12] Gao, Zu, Xianhua Tang, and Sitong Chen. "Existence of ground state solutions for a class of nonlinear fractional Schrödinger-Poisson systems with super-quadratic nonlinearity."Complex Variables and Elliptic Equations 63.6 (2018): 802-814. Zbl 1390.35397

[13] Kim, Jae-Myoung, and Jung-Hyun Bae. "Infinitely many solutions of fractional Schrödinger-Maxwell equations."Journal of Mathematical Physics 62.3 (2021): 031508. Zbl 1461.81034

[14] Jin, Tiankun. "Multiplicity of solutions for a fractional Schrödinger-Poisson system without (PS) condition."AIMS Mathematics 6.8 (2021): 9048-9058. Zbl 1485.35157

[15] Laskin, Nikolai. "Fractional quantum mechanics and Lévy path integrals."Physics Letters A 268.4-6 (2000): 298-305. Zbl 0948.81595

[16] Laskin, Nick. "Fractional schrödinger equation."Physical Review E 66.5 (2002): 056108.

[17] Lo, C. W., and J. F. Rodrigues. "On a class of fractional obstacle type problems related to the distributional Riesz derivative."arXiv preprint arXiv:2101.06863 (2021).

[18] Rabinowitz, Paul H., ed. Minimax methods in critical point theory with applications to differential equations. No. 65. American Mathematical Soc., 1986. Zbl 0609.58002

[19] Stein, Elias M. Singular Integrals and Differentiability Properties of Functions (PMS-30), Volume 30. Princeton university press, 2016. Zbl 0207.13501

[20] Shieh, Tien-Tsan, and Daniel E. Spector. "On a new class of fractional partial differential equations."Advances in Calculus of Variations 8.4 (2015): 321-336. Zbl 1330.35510

[21] Shieh, Tien-Tsan, and Daniel E. Spector. "On a new class of fractional partial differential equations II."Advances in Calculus of Variations 11.3 (2018): 289-307. Zbl 1451.35257

[22] Šilhavý, Miroslav. "Fractional vector analysis based on invariance requirements (critique of coordinate approaches)."Continuum Mechanics and Thermodynamics 32.1 (2020): 207-228. Zbl 1443.26004

[23] Teng, Kaimin. "Existence of ground state solutions for the nonlinear fractional Schrödinger-Poisson system with critical Sobolev exponent."Journal of Differential Equations 261.6 (2016): 3061-3106. Zbl 1386.35458

[24] Willem, Michel. Minimax theorems. Vol. 24. Springer Science and Business Media, 1997.

[25] Wei, Zhongli. "Existence of infinitely many solutions for the fractional Schrödinger-Maxwell equations."arXiv preprint arXiv:1508.03088 (2015).

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