GENERALISATION OF COMPANION OF OSTROWSKI'S TYPE INEQUALITY VIA RIEMANN-LIOUVILLE FRACTIONAL INTEGRAL AND APPLICATIONS IN NUMERICAL INTEGRATION, PROBABILITY THEORY AND SPECIAL MEANS

FARAZ MEHMOOD[1,2] AND AKHMADJON SOLEEV[1]

Abstract. We apply Riemann-Liouville fractional integral to get generalisation of companion of Ostrowski's type integral inequality. The present article recaptures the all results of M. W. Alomari's article and also for one more article of different authors. Applications are also deduced for numerical integration, probability theory and special means.

1. Introduction

In the development of mathematics, inequalities are one of the most powerful tools. From two decades back, scholars researched on fractional calculus because of its importance in inequalities.

We quote from [6],

"The subject of fractional calculus (that is, calculus of integrals and derivatives of an arbitrary real or complex order) was planted over 300 years ago. Since that time the fractional calculus has drawn the attention of many researchers in. In recent years, the fractional calculus has played a significant role in many areas of science and engineering."

Due to worth of fractional integral inequalities, many scholars have mentioned certain generalisations of fractional integral inequalities (see [5, 27, 28, 29]).

In 1938, A. M. Ostrowski was a Ukrainian mathematician, who had presented an inequality in his article [26]. Since then this inequality is called an "Ostrowski inequality" and this result had obtained by applying the "Montgomery identity". A number of researches have written their articles [4, 8, 9, 19, 23] about generalisations of "Ostrowski's inequality" in the past some decades. "Ostrowski's inequality" has been proved to be a huge and remarkable tool for the enlargement of various fields of mathematic. Inequalities including integral which create bounds in the physical quantites, are of great significant in the sense that these types of inequalites are not only used in "integral approximation theory, operator theory, nonlinear analysis, numerical integration, stochastic analysis, information theory, statistics and probability theory but we may also see its applications in the several branches

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of physics, engineering and biological sciences". We refer to the readers [14, 20, 21, 22, 25, 30, 31] for some recent contributions to the study of "Ostrowski's inequality".

Here, we present an inequality from article [11] that is given below. Throughout the article $K \subset \mathbb{R}$ and K^o is the interior of the interval K.

Proposition 1.1. *Suppose $g : K \to \mathbb{R}$ is a differentiable mapping in the interval K^o such that $g' \in L[j,k]$, where $j,k \in K$ and $j < k$. If $|g'(\theta)| \leq \mathfrak{M}\ \forall\ \theta \in (j,k)$ where $\mathfrak{M} > 0$ is constant. Then*

$$\left|g(\theta) - \frac{1}{k-j}\int_j^k g(\tau)d\tau\right| \leq \mathfrak{M}(k-j)\left[\frac{1}{4} + \frac{\left(\theta - \frac{j+k}{2}\right)^2}{(k-j)^2}\right]. \tag{1.1}$$

The value $\frac{1}{4}$ is the best possible constant that this can not be replaceed by the smallest one.

For the results related to Ostrowski's inequality (see [1, 2, 10, 31]). Also, the scholar can be refer to the book [11] where several Ostrowski type inequalities are discussed.

In [17], Guessab *et.al.* have derived a companian of Ostrowski's inequality in the below like others.

Proposition 1.2. *Let function $g : [j,k] \to \mathbb{R}$ satisfies the Lipschitz condition, that is $|g(\tau) - g(s)| \leq \mathfrak{M}|\tau - s|$. Then $\forall\ \theta \in [j, \frac{j+k}{2}]$, then*

$$\left|\frac{g(\theta) + g(j+k-\theta)}{2} - \frac{1}{k-j}\int_j^k g(\tau)d\tau\right| \leq \left[\frac{1}{8} + 2\left(\frac{\theta - \frac{3j+k}{4}}{k-j}\right)^2\right]\mathfrak{M}(k-j), \tag{1.2}$$

The value $\frac{1}{8}$ is the best possible constant that this can not be replaceed by the smallest one.

Note that the above inequality is the best due to it gives the trapezoid type inequality for $\theta = \frac{3j+k}{4}$, i.e.,

$$\left|\frac{g(\frac{3j+k}{4}) + g(\frac{j+3k}{4})}{2} - \frac{1}{k-j}\int_j^k g(\tau)d\tau\right| \leq \frac{\mathfrak{M}(k-j)}{8}. \tag{1.3}$$

The constant $\frac{1}{8}$ is the sharp in above the inequality.

In [13], S. S. Dragomir has derived the below companion of Ostrowski inequality.

Proposition 1.3. *Suppose function $g : K \to \mathbb{R}$ is an absolutely continuous in the interval $[j,k]$. Then*

$$\left|\frac{g(\theta)+g(j+k-\theta)}{2}-\frac{1}{k-j}\int_j^k g(\tau)d\tau\right|$$

$$\leq \begin{cases} \left[\frac{1}{8}+2\left(\frac{\theta-\frac{3j+k}{4}}{k-j}\right)^2\right](k-j)\|g'\|_\infty, \quad g'\in L_\infty[j,k], \\ \frac{2^{\frac{1}{q}}}{(q+1)^{\frac{1}{q}}}\left[\left(\frac{\theta-j}{k-j}\right)^{q+1}+\left(\frac{\frac{j+k}{2}-\theta}{k-j}\right)^{q+1}\right]^{\frac{1}{q}}(k-j)^{\frac{1}{q}}\|g'\|_{[j,k],p}\,, \\ \qquad p>1,\ \frac{1}{p}+\frac{1}{q}=1,\ and\, g'\in L_p[j,k], \\ \left[\frac{1}{4}+\left|\frac{\theta-\frac{3j+k}{4}}{k-j}\right|\right]\|g'\|_{[j,k],1} \end{cases} \quad (1.4)$$

hold, $\forall\ \theta\in[j,\frac{j+k}{2}]$.

We need here to define Riemann-Liouville fractional integral(RLFI) (see[15]) for proving our next main result in the second section.

Definition 1.4. The Riemann-Liouville fractional integral operator of order $\gamma>0$ is stated as

$$J_j^\gamma g(\theta)=\frac{1}{\Gamma(\gamma)}\int_j^\theta(\theta-\tau)^{\gamma-1}g(\tau)d\tau,$$
$$J_j^0 g(\theta)=g(\theta),$$

where gamma function $\Gamma(\gamma)$ is defined as

$$\Gamma(\gamma)=\int_0^\infty\theta^{\gamma-1}e^{-\theta}d\theta.$$

In 2002, S. S. Dragomir [12] derived few inequalities for companion for mappings of bounded variation. In 2009, Z. Liu [18] established few companions of the Ostrowski's type inequality for mappings whose second derivatives are absolutely continous. In 2009, Barnett *et. al* [7] derived few companions for Ostrowski's inequalty and the generalised trapezoid inequality. In 2011, M. W. Alomari [3] obtained the companion of inequality of Ostrowski (1.3) for differentiable bounded mappings and also gave the applications. Recently, authors [20] gave a companion of weighted Ostrowski's type inequality for differentiable bounded functions with application.

In the current article we would derive a companion of weighted Fractional Ostrowski's type inequality for differentiable bounded mappings and then we would give its applications.

2. **Generalisation of Companion of Ostrowski's Type Inequality Via Riemann-Liouville Fractional Integral**

Under present section we would give our results about companion of Ostrowski's type inequality which are as follow:

Theorem 2.1. *Suppose $g : [j,k] \to \mathbb{R}$ is a differentiable mapping in the interval (j,k) and $j < k$ and $w : [j,k] \to \mathbb{R}$ is a probability density function. If $g' \in L^1[j,k]$ and $m_1 \le g'(\tau) \le M_1$, $\forall\ \tau \in [j,k]$, then*

$$\Bigg| g(\theta)\int_j^{\frac{j+k}{2}} w(\tau)d\tau + g(j+k-\theta)(k-\theta)^{1-\gamma}(\theta-j)^{\gamma-1}\int_{\frac{j+k}{2}}^{k} w(\tau)d\tau$$
$$- (k-\theta)^{1-\gamma}\Gamma(\gamma)J_j^{\gamma}(w(k)g(k)) + (\gamma-1)J_j^{\gamma-1}(P(\theta,k)g(k))\Bigg|$$
$$\le (k-\theta)^{1-\gamma}\cdot \max_{\tau\in[j,k]}|(k-\tau)^{\gamma-1}|\cdot\Bigg[\int_j^{\frac{j+\theta}{2}}\left(\int_j^{\tau} w(u)du\right)d\tau$$
$$+\int_{\frac{j+\theta}{2}}^{\theta}\left(\int_j^{\tau} w(u)du\right)d\tau - \int_{\theta}^{\frac{j+k}{2}}\left(\int_{\frac{j+k}{2}}^{\tau} w(u)du\right)d\tau + \int_{\frac{j+k}{2}}^{j+k-\theta}\left(\int_{\frac{j+k}{2}}^{\tau} w(u)du\right)d\tau$$
$$-\int_{j+k-\theta}^{\frac{j-\theta+2k}{2}}\left(\int_k^{\tau} w(u)du\right)d\tau - \int_{\frac{j-\theta+2k}{2}}^{k}\left(\int_k^{\tau} w(u)du\right)d\tau\Bigg]\frac{(M_1+m_1)}{2} \tag{2.1}$$

holds $\forall\ \theta \in [j, \frac{j+k}{2}]$.

Proof. For the sake of proof we state the weighted kernel as;

$$P(\theta,\tau) = (k-\theta)^{1-\gamma}\Gamma(\gamma)\begin{cases} \displaystyle\int_j^{\tau} w(u)du, & \text{if} \quad \tau\in[j,\theta], \\ \displaystyle\int_{\frac{j+k}{2}}^{\tau} w(u)du, & \text{if} \quad \tau\in(\theta, j+k-\theta], \\ \displaystyle\int_k^{\tau} w(u)du, & \text{if} \quad \tau\in(j+k-\theta, k], \end{cases}$$

$\forall\ \theta \in [j, \frac{j+k}{2}]$.

Applying RLFI operator and by parts formula of integration, obtain

$$J_j^{\gamma}(P(\theta,k)g(k)) = \frac{1}{\Gamma(\gamma)}\int_j^k (k-\tau)^{\gamma-1}P(\theta,\tau)g'(\tau)d\tau$$
$$= g(\theta)\int_j^{\frac{j+k}{2}} w(\tau)d\tau + g(j+k-\theta)(k-\theta)^{1-\gamma}(\theta-j)^{\gamma-1}\int_{\frac{j+k}{2}}^{k} w(\tau)d\tau$$
$$-(k-\theta)^{1-\gamma}\Gamma(\gamma)J_j^{\gamma}(w(k)g(k)) + (\gamma-1)J_j^{\gamma-1}(P(\theta,k)g(k)). \tag{2.2}$$

We know that

$$\int_j^k P(\theta,\tau)d\tau = 0. \tag{2.3}$$

Let $C = \frac{M_1+m_1}{2}$. From (2.2) and (2.3) it follows

$$\begin{aligned}
&\frac{1}{\Gamma(\gamma)}\int_j^k (k-\tau)^{\gamma-1}P(\theta,\tau)[g'(\tau)-C]d\tau \\
&= g(\theta)\int_j^{\frac{j+k}{2}} w(\tau)d\tau + g(j+k-\theta)(k-\theta)^{1-\gamma}(\theta-j)^{\gamma-1}\int_{\frac{j+k}{2}}^k w(\tau)d\tau \\
&-(k-\theta)^{1-\gamma}\Gamma(\gamma)J_j^{\gamma}(w(k)g(k)) + (\gamma-1)J_j^{\gamma-1}(P(\theta,k)g(k)). \qquad (2.4)
\end{aligned}$$

Another way we have

$$\begin{aligned}
&\left|\frac{1}{\Gamma(\gamma)}\int_j^k (k-\tau)^{\gamma-1}P(\theta,\tau)[g'(\tau)-C]d\tau\right| \\
&\leq \frac{1}{\Gamma(\gamma)}\max_{\tau\in[j,k]}|(k-\tau)^{\gamma-1}|\cdot\max_{\tau\in[j,k]}|g'(\tau)-C|\cdot\int_j^k|P(\theta,\tau)|d\tau. \qquad (2.5)
\end{aligned}$$

Since

$$\max_{\tau\in[j,k]}|g'(\tau)-C| \leq \frac{M_1+m_1}{2} \qquad (2.6)$$

and

$$\begin{aligned}
\int_j^k |P(\theta,\tau)|d\tau \;=\; &(k-\theta)^{1-\gamma}\Gamma(\gamma)\Bigg[\int_j^{\frac{j+\theta}{2}}\left(\int_j^{\tau} w(u)du\right)d\tau + \int_{\frac{j+\theta}{2}}^{\theta}\left(\int_j^{\tau} w(u)du\right)d\tau \\
&- \int_{\theta}^{\frac{j+k}{2}}\left(\int_{\frac{j+k}{2}}^{\tau} w(u)du\right)d\tau + \int_{\frac{j+k}{2}}^{j+k-\theta}\left(\int_{\frac{j+k}{2}}^{\tau} w(u)du\right)d\tau \\
&- \int_{j+k-\theta}^{\frac{j-\theta+2k}{2}}\left(\int_k^{\tau} w(u)du\right)d\tau - \int_{\frac{j-\theta+2k}{2}}^{k}\left(\int_k^{\tau} w(u)du\right)d\tau\Bigg]. \qquad (2.7)
\end{aligned}$$

Now from (2.5) to (2.7), it follows that

$$\begin{aligned}
&\Bigg|g(\theta)\int_j^{\frac{j+k}{2}} w(\tau)d\tau + g(j+k-\theta)(k-\theta)^{1-\gamma}(\theta-j)^{\gamma-1}\int_{\frac{j+k}{2}}^k w(\tau)d\tau \\
&-(k-\theta)^{1-\gamma}\Gamma(\gamma)J_j^{\gamma}(w(k)g(k)) + (\gamma-1)J_j^{\gamma-1}(P(\theta,k)g(k))\Bigg| \\
&\leq (k-\theta)^{1-\gamma}\cdot\max_{\tau\in[j,k]}|(k-\tau)^{\gamma-1}|\cdot\Bigg[\int_j^{\frac{j+\theta}{2}}\left(\int_j^{\tau} w(u)du\right)d\tau \\
&+\int_{\frac{j+\theta}{2}}^{\theta}\left(\int_j^{\tau} w(u)du\right)d\tau - \int_{\theta}^{\frac{j+k}{2}}\left(\int_{\frac{j+k}{2}}^{\tau} w(u)du\right)d\tau + \int_{\frac{j+k}{2}}^{j+k-\theta}\left(\int_{\frac{j+k}{2}}^{\tau} w(u)du\right)d\tau \\
&-\int_{j+k-\theta}^{\frac{j-\theta+2k}{2}}\left(\int_k^{\tau} w(u)du\right)d\tau - \int_{\frac{j-\theta+2k}{2}}^{k}\left(\int_k^{\tau} w(u)du\right)d\tau\Bigg]\frac{(M_1+m_1)}{2},
\end{aligned}$$

$\forall\ \theta \in [j, \frac{j+k}{2}]$. □

Remark 2.2. If put $\gamma = 1$ and $w = \frac{1}{k-j}$ in Theorem 2.1, then we recapture the result of Theorem 4 of [3].

Remark 2.3. If put $\gamma = 1$ in Theorem 2.1, then we recapture the result of Theorem 2.1 of [20].

Corollary 2.4. *In the inequality* (2.1), *select*
(*i*) $\theta = \frac{3j+k}{4}$, *obtain*

$$\begin{aligned} &\Bigg| g(\frac{3j+k}{4}) \int_j^{\frac{j+k}{2}} w(\tau)d\tau + 3^{1-\gamma} g(\frac{j+3k}{4}) \int_{\frac{j+k}{2}}^k w(\tau)d\tau \\ &- \left(\frac{3}{4}(k-j)\right)^{1-\gamma} \Gamma(\gamma) J_j^{\gamma}(w(k)g(k)) + (\gamma-1) J_j^{\gamma-1}\left(P\left(\frac{3j+k}{4}, k\right) g(k)\right) \Bigg| \\ &\leq \left(\frac{3}{4}(k-j)\right)^{1-\gamma} \cdot \max_{\tau \in [j,k]} |(k-\tau)^{\gamma-1}| \cdot \Bigg[\int_j^{\frac{7j+k}{8}} \left(\int_j^{\tau} w(u)du\right) d\tau \\ &+ \int_{\frac{7j+k}{8}}^{\frac{3j+k}{4}} \left(\int_j^{\tau} w(u)du\right) d\tau - \int_{\frac{3j+k}{4}}^{\frac{j+k}{2}} \left(\int_{\frac{j+k}{2}}^{\tau} w(u)du\right) d\tau + \int_{\frac{j+k}{2}}^{\frac{j+3k}{4}} \left(\int_{\frac{j+k}{2}}^{\tau} w(u)du\right) d\tau \\ &- \int_{\frac{j+3k}{4}}^{\frac{j+7k}{8}} \left(\int_k^{\tau} w(u)du\right) d\tau - \int_{\frac{j+7k}{8}}^{k} \left(\int_k^{\tau} w(u)du\right) d\tau \Bigg] \frac{(M_1+m_1)}{2}, \end{aligned} \tag{2.8}$$

(*ii*) $\theta = \frac{2j+k}{3}$, *obtain*

$$\begin{aligned} &\Bigg| g(\frac{2j+k}{3}) \int_j^{\frac{j+k}{2}} w(\tau)d\tau + 2^{1-\gamma} g(\frac{j+2k}{3}) \int_{\frac{j+k}{2}}^k w(\tau)d\tau \\ &- \left(\frac{2}{3}(k-j)\right)^{1-\gamma} \Gamma(\gamma) J_j^{\gamma}(w(k)g(k)) + (\gamma-1) J_j^{\gamma-1}\left(P\left(\frac{2j+k}{3}, k\right) g(k)\right) \Bigg| \\ &\leq \left(\frac{2}{3}(k-j)\right)^{1-\gamma} \cdot \max_{\tau \in [j,k]} |(k-\tau)^{\gamma-1}| \cdot \Bigg[\int_j^{\frac{5j+k}{6}} \left(\int_j^{\tau} w(u)du\right) d\tau \\ &+ \int_{\frac{5j+k}{6}}^{\frac{2j+k}{3}} \left(\int_j^{\tau} w(u)du\right) d\tau - \int_{\frac{2j+k}{3}}^{\frac{j+k}{2}} \left(\int_{\frac{j+k}{2}}^{\tau} w(u)du\right) d\tau + \int_{\frac{j+k}{2}}^{\frac{j+2k}{3}} \left(\int_{\frac{j+k}{2}}^{\tau} w(u)du\right) d\tau \\ &- \int_{\frac{j+2k}{3}}^{\frac{j+5k}{6}} \left(\int_k^{\tau} w(u)du\right) d\tau - \int_{\frac{j+5k}{6}}^{k} \left(\int_k^{\tau} w(u)du\right) d\tau \Bigg] \frac{(M_1+m_1)}{2}, \end{aligned} \tag{2.9}$$

(iii) $\theta = \frac{j+k}{2}$, *obtain*

$$\begin{aligned}
&\left| g(\frac{j+k}{2}) \int_j^k w(\tau)d\tau - \left(\frac{k-j}{2}\right)^{1-\gamma} \Gamma(\gamma) J_j^{\gamma}(w(k)g(k)) \right. \\
&\left. + (\gamma-1) J_j^{\gamma-1} \left(P\left(\frac{j+k}{2}, k\right) g(k) \right) \right| \\
&\le \left(\frac{k-j}{2}\right)^{1-\gamma} \cdot \max_{\tau \in [j,k]} |(k-\tau)^{\gamma-1}| \cdot \left[\int_j^{\frac{3j+k}{4}} \left(\int_j^{\tau} w(u)du \right) d\tau \right. \\
&+ \int_{\frac{3j+k}{4}}^{\frac{j+k}{2}} \left(\int_j^{\tau} w(u)du \right) d\tau - \int_{\frac{j+k}{2}}^{\frac{j+3k}{4}} \left(\int_k^{\tau} w(u)du \right) d\tau \\
&\left. - \int_{\frac{j+3k}{4}}^{k} \left(\int_k^{\tau} w(u)du \right) d\tau \right] \frac{(M_1+m_1)}{2}. \qquad (2.10)
\end{aligned}$$

In the following we present special cases of above corollary.

Special Case 1: By putting $w = \frac{1}{k-j}$ and $\gamma = 1$ in *(ii)* of Corollary 2.4, then we get

$$\left| \frac{g(\frac{2j+k}{3}) + g(\frac{j+2k}{3})}{2} - \frac{1}{k-j} \int_j^k g(\tau)d\tau \right| \le \frac{5(k-j)}{72}(M_1+m_1).$$

Special Case 2: By putting $w = \frac{1}{k-j}$ and $\gamma = 1$ in *(iii)* of Corollary 2.4, then we get

$$\left| g(\frac{j+k}{2}) - \frac{1}{k-j} \int_j^k g(\tau)d\tau \right| \le \frac{(k-j)}{4}(M_1+m_1).$$

Remark 2.5.
- (i) First by putting $\gamma = 1$ and $w = \frac{1}{k-j}$ in Theorem 2.1 and then put $\theta = j$ in obtained inequality, we recapture the Corollary 2 of [3].
- (ii) By putting $\gamma = 1$ and $w = \frac{1}{k-j}$ in *(i)* of Corollary 2.4, we recapture the Corollary 1 of [3].

Remark 2.6.
- (i) First by putting $\gamma = 1$ in Theorem 2.1 and then put $\theta = j$ in obtained inequality, we recapture the Corollary 2.3(iv) of [20].
- (ii) By putting $\gamma = 1$ in *(i)* of Corollary 2.4, we recapture the Corollary 2.3(i) of [20].
- (iii) By putting $\gamma = 1$ in *(ii)* of Corollary 2.4, we recapture the Corollary 2.3(ii) of [20].
- (iv) By putting $\gamma = 1$ in *(iii)* of Corollary 2.4, we recapture the Corollary 2.3(iii) of [20].

Ostrowski's type inequality can be defined in the form of following corollary.

Corollary 2.7. *Let the suppositions of Theorem 2.1 be valid. Further, if g is symmetric about the θ-axis, i.e., $g(j+k-\theta)=g(\theta)$, then*

$$\begin{aligned}&\Bigg|g(\theta)\int_j^{\frac{j+k}{2}} w(\tau)d\tau + g(\theta)(k-\theta)^{1-\gamma}(\theta-j)^{\gamma-1}\int_{\frac{j+k}{2}}^k w(\tau)d\tau\\&-(k-\theta)^{1-\gamma}\Gamma(\gamma)J_j^{\gamma}(w(k)g(k))+(\gamma-1)J_j^{\gamma-1}(P(\theta,k)g(k))\Bigg|\\&\le (k-\theta)^{1-\gamma}\cdot\max_{\tau\in[j,k]}|(k-\tau)^{\gamma-1}|\cdot\Bigg[\int_j^{\frac{j+\theta}{2}}\left(\int_j^{\tau} w(u)du\right)d\tau\\&+\int_{\frac{j+\theta}{2}}^{\theta}\left(\int_j^{\tau} w(u)du\right)d\tau-\int_{\theta}^{\frac{j+k}{2}}\left(\int_{\frac{j+k}{2}}^{\tau} w(u)du\right)d\tau+\int_{\frac{j+k}{2}}^{j+k-\theta}\left(\int_{\frac{j+k}{2}}^{\tau} w(u)du\right)d\tau\\&-\int_{j+k-\theta}^{\frac{j-\theta+2k}{2}}\left(\int_k^{\tau} w(u)du\right)d\tau-\int_{\frac{j-\theta+2k}{2}}^{k}\left(\int_k^{\tau} w(u)du\right)d\tau\Bigg]\frac{(M_1+m_1)}{2}\end{aligned}\tag{2.11}$$

holds.

Remark 2.8. First by putting $\gamma=1$ and $w=\frac{1}{k-j}$ in Corollary 2.7 and then put $\theta=j$ in obtained inequality, we recapture the Corollary 3 of [3].

Remark 2.9. First by putting $\gamma=1$ in Corollary 2.7 and then put $\theta=j$ in obtained inequality, we recapture the Corollary 2.5 of [20].

3. Application to Numerical Integration

Let $K_n : j=\theta_0<\theta_1<\cdots<\theta_n=k$ be division of interval $[j,k]$ and $h_i=\theta_{i+1}-\theta_i$, $(i=0,1,2,\cdots,n-1)$.

Consider the quadrature formula

$$\begin{aligned}Q_n(K_n,g) \quad:=\quad &\sum_{i=0}^{n-1}\Bigg[g\Big(\frac{3\theta_i+\theta_{i+1}}{4}\Big)\int_{\theta_i}^{\frac{\theta_i+\theta_{i+1}}{2}} w(\tau)d\tau+3^{1-\gamma}g\Big(\frac{\theta_i+3\theta_{i+1}}{4}\Big)\\&\times\int_{\frac{\theta_i+\theta_{i+1}}{2}}^{\theta_{i+1}} w(\tau)d\tau+(\gamma-1)J_{\theta_i}^{\gamma-1}\left(P\left(\frac{3\theta_i+\theta_{i+1}}{4},\theta_{i+1}\right)g(\theta_{i+1})\right)\Bigg].\end{aligned}\tag{3.1}$$

Here, we prove our result for section.

Theorem 3.1. *Suppose $g: K\to\mathbb{R}$ is a differentiable mapping in the interval K^o and $w:[j,k]\to\mathbb{R}$ is a probability density function, where $j,k\in K$ with $j<k$. If $g'\in L^1[j,k]$ and $m_1\le g'(\theta)\le M_1$, for all $\theta\in[j,k]$, then following holds*

$$\Gamma(\gamma)\sum_{i=0}^{n-1}\left(\frac{3}{4}h_i\right)^{1-\gamma}J_{\theta_i}^{\gamma}(w(\theta_{i+1})g(\theta_{i+1}))=Q_n(K_n,g)+R_n(K_n,g),\tag{3.2}$$

where $Q_n(K_n, g)$ is stated as above and the following remainder $R_n(K_n, g)$ satisfies the estimates

$$\begin{aligned}
|R_n(K_n,g)| \leq \frac{(M_1+m_1)}{2}\sum_{i=0}^{n-1}\Bigg[\left(\frac{3}{4}h_i\right)^{1-\gamma}\max_{\tau\in[\theta_i,\theta_{i+1}]}|(\theta_{i+1}-\tau)^{\gamma-1}|\\
\times\Bigg[\int_{\theta_i}^{\frac{7\theta_i+\theta_{i+1}}{8}}\left(\int_{\theta_i}^{\tau}w(u)du\right)d\tau+\int_{\frac{7\theta_i+\theta_{i+1}}{8}}^{\frac{3\theta_i+\theta_{i+1}}{4}}\left(\int_{\theta_i}^{\tau}w(u)du\right)d\tau\\
-\int_{\frac{3\theta_i+\theta_{i+1}}{4}}^{\frac{\theta_i+\theta_{i+1}}{2}}\left(\int_{\frac{\theta_i+\theta_{i+1}}{2}}^{\tau}w(u)du\right)d\tau+\int_{\frac{\theta_i+\theta_{i+1}}{2}}^{\frac{\theta_i+3\theta_{i+1}}{4}}\left(\int_{\frac{\theta_i+\theta_{i+1}}{2}}^{\tau}w(u)du\right)d\tau\\
-\int_{\frac{\theta_i+3\theta_{i+1}}{4}}^{\frac{\theta_i+7\theta_{i+1}}{8}}\left(\int_{\theta_{i+1}}^{\tau}w(u)du\right)d\tau-\int_{\frac{\theta_i+7\theta_{i+1}}{8}}^{\theta_{i+1}}\left(\int_{\theta_{i+1}}^{\tau}w(u)du\right)d\tau\Bigg]\Bigg]. \qquad (3.3)
\end{aligned}$$

Proof. Applying inequality (2.8) on the intervals $[\theta_i, \theta_{i+1}]$, we get

$$\begin{aligned}
R_i(K_i,g) &= \Gamma(\gamma)\left(\frac{3}{4}h_i\right)^{1-\gamma}J_{\theta_i}^{\gamma}(w(\theta_{i+1})g(\theta_{i+1}))\\
&- \Bigg[g(\frac{3\theta_i+\theta_{i+1}}{4})\int_{\theta_i}^{\frac{\theta_i+\theta_{i+1}}{2}}w(\tau)d\tau+3^{1-\gamma}g(\frac{\theta_i+3\theta_{i+1}}{4})\int_{\frac{\theta_i+\theta_{i+1}}{2}}^{\theta_{i+1}}w(\tau)d\tau\\
&+ (\gamma-1)J_{\theta_i}^{\gamma-1}\left(P\left(\frac{3\theta_i+\theta_{i+1}}{4},\theta_{i+1}\right)g(\theta_{i+1})\right)\Bigg]. \qquad (3.4)
\end{aligned}$$

Summing (3.4) over i from 0 to $n-1$, then

$$\begin{aligned}
R_n(K_n,g) &= \Gamma(\gamma)\sum_{i=0}^{n-1}\left(\frac{3}{4}h_i\right)^{1-\gamma}J_{\theta_i}^{\gamma}(w(\theta_{i+1})g(\theta_{i+1}))\\
&- \sum_{i=0}^{n-1}\Bigg[g(\frac{3\theta_i+\theta_{i+1}}{4})\int_{\theta_i}^{\frac{\theta_i+\theta_{i+1}}{2}}w(\tau)d\tau+3^{1-\gamma}g(\frac{\theta_i+3\theta_{i+1}}{4})\\
&\times \int_{\frac{\theta_i+\theta_{i+1}}{2}}^{\theta_{i+1}}w(\tau)d\tau+(\gamma-1)J_{\theta_i}^{\gamma-1}\left(P\left(\frac{3\theta_i+\theta_{i+1}}{4},\theta_{i+1}\right)g(\theta_{i+1})\right)\Bigg],
\end{aligned}$$

which follows the form of (2.8), i.e.

$$\begin{aligned}
|R_n(K_n,g)| = \Bigg|\Gamma(\gamma)\sum_{i=0}^{n-1}\left(\frac{3}{4}h_i\right)^{1-\gamma}J_{\theta_i}^{\gamma}(w(\theta_{i+1})g(\theta_{i+1}))\\
-\sum_{i=0}^{n-1}\Bigg[g(\frac{3\theta_i+\theta_{i+1}}{4})\int_{\theta_i}^{\frac{\theta_i+\theta_{i+1}}{2}}w(\tau)d\tau+3^{1-\gamma}g(\frac{\theta_i+3\theta_{i+1}}{4})\int_{\frac{\theta_i+\theta_{i+1}}{2}}^{\theta_{i+1}}w(\tau)d\tau\\
+(\gamma-1)J_{\theta_i}^{\gamma-1}\left(P\left(\frac{3\theta_i+\theta_{i+1}}{4},\theta_{i+1}\right)g(\theta_{i+1})\right)\Bigg]\Bigg|
\end{aligned}$$

$$\leq \frac{(M_1+m_1)}{2}\sum_{i=0}^{n-1}\Bigg[\left(\frac{3}{4}h_i\right)^{1-\gamma}\max_{\tau\in[\theta_i,\theta_{i+1}]}|(\theta_{i+1}-\tau)^{\gamma-1}|$$
$$\times\Bigg[\int_{\theta_i}^{\frac{7\theta_i+\theta_{i+1}}{8}}\left(\int_{\theta_i}^{\tau}w(u)du\right)d\tau+\int_{\frac{7\theta_i+\theta_{i+1}}{8}}^{\frac{3\theta_i+\theta_{i+1}}{4}}\left(\int_{\theta_i}^{\tau}w(u)du\right)d\tau$$
$$-\int_{\frac{3\theta_i+\theta_{i+1}}{4}}^{\frac{\theta_i+\theta_{i+1}}{2}}\left(\int_{\frac{\theta_i+\theta_{i+1}}{2}}^{\tau}w(u)du\right)d\tau+\int_{\frac{\theta_i+\theta_{i+1}}{2}}^{\frac{\theta_i+3\theta_{i+1}}{4}}\left(\int_{\frac{\theta_i+\theta_{i+1}}{2}}^{\tau}w(u)du\right)d\tau$$
$$-\int_{\frac{\theta_i+3\theta_{i+1}}{4}}^{\frac{\theta_i+7\theta_{i+1}}{8}}\left(\int_{\theta_{i+1}}^{\tau}w(u)du\right)d\tau-\int_{\frac{\theta_i+7\theta_{i+1}}{8}}^{\theta_{i+1}}\left(\int_{\theta_{i+1}}^{\tau}w(u)du\right)d\tau\Bigg]\Bigg].$$

This completes the required proof. □

Remark 3.2. If put $\gamma = 1$ and $w = \frac{1}{k-j}$ in Theorem 3.1, we can get the result of Theorem 5 of [3].

Remark 3.3. By putting $\gamma = 1$ in Theorem 3.1, we recapture the result of Theorem 3.1 of [20].

4. **Applications to Probability Theory**

Throughout this section we consider $w : [j,k] \to [0,1]$. Suppose Y is a random variable taking values in the finite interval $[j,k]$ with probability density function $g : [j,k] \to [0,1]$ and with cumulative distribution function $G : [j,k] \to [0,1]$ is introduced and defined by us, i.e,

$$G(\theta) = P(Y \leq \theta) = \Gamma(\gamma)J_j^{\gamma}(w(\theta)g(\theta)) = \int_j^{\theta}(\theta-\tau)^{\gamma-1}w(\tau)g(\tau)d\tau,\;\; j \leq \theta \leq \frac{j+k}{2},$$

and

$$E(Y) = \int_j^k \tau g(\tau)d\tau\ ,\ E_w(Y) = \int_j^k \tau w(\tau)g(\tau)d\tau$$
$$E_{wf}(Y) = \Gamma(\gamma)J_j^{\gamma}(kw(k)g(k)) = \int_j^k \tau(k-\tau)^{\gamma-1}w(\tau)g(\tau)d\tau$$
$$E_{wf1}(Y) = \Gamma(\gamma)J_j^{\gamma-1}(kw(k)g(k)) = \int_j^k \tau(k-\tau)^{\gamma-2}w(\tau)g(\tau)d\tau,$$
$$E_{wf2}(Y) = \Gamma(\gamma)J_j^{\gamma}(kw'(k)g(k)) = \int_j^k \tau(k-\tau)^{\gamma-1}w'(\tau)g(\tau)d\tau,$$
$$E_{wf3}(Y) = \Gamma(\gamma)J_j^{\gamma}(kw(k)g'(k)) = \int_j^k \tau(k-\tau)^{\gamma-1}w(\tau)g'(\tau)d\tau,$$

are the expectation, weighted expectation and weighted fractional expectation of random variable 'Y' in interval $[j,k]$. Then we can write the following theorem as:

Theorem 4.1. *Suppose $g : [j,k] \to \mathbb{R}$ is differentiable mapping in interval (j,k) and $j < k$. If $g' \in L^1[j,k]$ and $m_1 \leq g'(\tau) \leq M_1$, for all $\tau \in [j,k]$. Further, suppose* *that function w is differentiable, then*

$$\Bigg|G(\theta)\int_j^{\frac{j+k}{2}} w(\tau)d\tau + G(j+k-\theta)(k-\theta)^{1-\gamma}(\theta-j)^{\gamma-1}\int_{\frac{j+k}{2}}^{k} w(\tau)d\tau$$
$$-(k-\theta)^{1-\gamma}\Big((\gamma-1)E_{wf1}(Y) - E_{wf2}(Y) - E_{wf3}(Y)\Big) + (\gamma-1)J_j^{\gamma-1}(P(\theta,k)G(k))\Bigg|$$
$$\leq (k-\theta)^{1-\gamma}\cdot \max_{\tau\in[j,k]}|(k-\tau)^{\gamma-1}|\cdot\Bigg[\int_j^{\frac{j+\theta}{2}}\left(\int_j^{\tau} w(u)du\right)d\tau$$
$$+\int_{\frac{j+\theta}{2}}^{\theta}\left(\int_j^{\tau} w(u)du\right)d\tau - \int_{\theta}^{\frac{j+k}{2}}\left(\int_{\frac{j+k}{2}}^{\tau} w(u)du\right)d\tau + \int_{\frac{j+k}{2}}^{j+k-\theta}\left(\int_{\frac{j+k}{2}}^{\tau} w(u)du\right)d\tau$$
$$-\int_{j+k-\theta}^{\frac{j-\theta+2k}{2}}\left(\int_k^{\tau} w(u)du\right)d\tau - \int_{\frac{j-\theta+2k}{2}}^{k}\left(\int_k^{\tau} w(u)du\right)d\tau\Bigg]\frac{(M_1+m_1)}{2} \tag{4.1}$$

holds $\forall\ \theta\in[j,\frac{j+k}{2}]$.

Proof. Select $g = G$, we obtain (4.1), by applying the identity

$$\Gamma(\gamma)J_j^{\gamma}(w(k)g(k)) = \int_j^k (k-\tau)^{\gamma-1}w(\tau)g(\tau)d\tau = (\gamma-1)E_{wf1}(Y) - E_{wf2}(Y) - E_{wf3}(Y).$$

Since $G(j) = 0$ and $G(k) = 1$.
We left the details to research scholars. □

Corollary 4.2. *Select $\gamma = 1$ in Theorem 4.1. Then get the following*

$$\Bigg|G(\theta)\int_j^{\frac{j+k}{2}} w(\tau)d\tau + G(j+k-\theta)\int_{\frac{j+k}{2}}^{k} w(\tau)d\tau + E_w(Y) + \int_j^k \tau w'(\tau)G(\tau)d\tau$$
$$-kw(k)\Bigg| \leq \Bigg[\int_j^{\frac{j+\theta}{2}}\left(\int_j^{\tau} w(u)du\right)d\tau + \int_{\frac{j+\theta}{2}}^{\theta}\left(\int_j^{\tau} w(u)du\right)d\tau$$
$$-\int_{\theta}^{\frac{j+k}{2}}\left(\int_{\frac{j+k}{2}}^{\tau} w(u)du\right)d\tau + \int_{\frac{j+k}{2}}^{j+k-\theta}\left(\int_{\frac{j+k}{2}}^{\tau} w(u)du\right)d\tau$$
$$-\int_{j+k-\theta}^{\frac{j-\theta+2k}{2}}\left(\int_k^{\tau} w(u)du\right)d\tau - \int_{\frac{j-\theta+2k}{2}}^{k}\left(\int_k^{\tau} w(u)du\right)d\tau\Bigg]\frac{(M_1+m_1)}{2}$$

holds $\forall\ \theta\in[j,\frac{j+k}{2}]$, *where $E_w(Y)$ is the weighted expectation of Y.*

Remark 4.3. If we put $w = \frac{1}{k-j}$ in Corollary 4.2 and taking the expectation $E(Y) = \int_j^k \tau G(\tau)d\tau = k - \int_j^k G(\tau)d\tau$, we recapture Theorem 6 of [3].

Corollary 4.4. *Select* $\theta = \frac{3j+k}{4}$ *in Theorem* 4.1*, we get*

$$\Bigg|G(\frac{3j+k}{4})\int_j^{\frac{j+k}{2}} w(\tau)d\tau + G(\frac{j+3k}{4})\left(\frac{3}{4}(k-j)\right)^{1-\gamma}\left(\frac{k-j}{4}\right)^{\gamma-1}\int_{\frac{j+k}{2}}^{k} w(\tau)d\tau$$
$$-\left(\frac{3}{4}(k-j)\right)^{1-\gamma}\Big((\gamma-1)E_{wf1}(Y) - E_{wf2}(Y) - E_{wf3}(Y)\Big)$$
$$+(\gamma-1)J_j^{\gamma-1}(P(\frac{3j+k}{4},k)G(k))\Bigg| \le \left(\frac{3}{4}(k-j)\right)^{1-\gamma}\cdot \max_{\tau\in[j,k]}|(k-\tau)^{\gamma-1}|\cdot$$
$$\Bigg[\int_j^{\frac{7j+k}{8}}\left(\int_j^{\tau} w(u)du\right)d\tau + \int_{\frac{7j+k}{8}}^{\frac{3j+k}{4}}\left(\int_j^{\tau} w(u)du\right)d\tau - \int_{\frac{3j+k}{4}}^{\frac{j+k}{2}}\left(\int_{\frac{j+k}{2}}^{\tau} w(u)du\right)d\tau$$
$$+\int_{\frac{j+k}{2}}^{\frac{j+3k}{4}}\left(\int_{\frac{j+k}{2}}^{\tau} w(u)du\right)d\tau - \int_{\frac{j+3k}{4}}^{\frac{j+7k}{8}}\left(\int_k^{\tau} w(u)du\right)d\tau - \int_{\frac{j+7k}{8}}^{k}\left(\int_k^{\tau} w(u)du\right)d\tau\Bigg]$$
$$\times\frac{(M_1+m_1)}{2}.$$

Remark 4.5. By putting $\gamma = 1$ and $w = \frac{1}{k-j}$ in Corollary 4.4, we recapture Corollary 4 of [3].

Corollary 4.6. *If* G *is symmetric about the* $\theta-$*axis in the Theorem* 4.1*, i.e.,* $G(j+k-\theta) = G(\theta)$*, then*

$$\Bigg|G(\theta)\int_j^{\frac{j+k}{2}} w(\tau)d\tau + G(\theta)(k-\theta)^{1-\gamma}(\theta-j)^{\gamma-1}\int_{\frac{j+k}{2}}^{k} w(\tau)d\tau$$
$$-(k-\theta)^{1-\gamma}\Big((\gamma-1)E_{wf1}(Y) - E_{wf2}(Y) - E_{wf3}(Y)\Big) + (\gamma-1)J_j^{\gamma-1}(P(\theta,k)G(k))\Bigg|$$
$$\le (k-\theta)^{1-\gamma}\cdot\max_{\tau\in[j,k]}|(k-\tau)^{\gamma-1}|\cdot\Bigg[\int_j^{\frac{j+\theta}{2}}\left(\int_j^{\tau} w(u)du\right)d\tau$$
$$+\int_{\frac{j+\theta}{2}}^{\theta}\left(\int_j^{\tau} w(u)du\right)d\tau - \int_{\theta}^{\frac{j+k}{2}}\left(\int_{\frac{j+k}{2}}^{\tau} w(u)du\right)d\tau + \int_{\frac{j+k}{2}}^{j+k-\theta}\left(\int_{\frac{j+k}{2}}^{\tau} w(u)du\right)d\tau$$
$$-\int_{j+k-\theta}^{\frac{j-\theta+2k}{2}}\left(\int_k^{\tau} w(u)du\right)d\tau - \int_{\frac{j-\theta+2k}{2}}^{k}\left(\int_k^{\tau} w(u)du\right)d\tau\Bigg]\frac{(M_1+m_1)}{2}$$

holds $\forall\ \theta \in [j, \frac{j+k}{2}]$.

Remark 4.7. If put $\gamma = 1$ and $w = \frac{1}{k-j}$ in Corollary 4.6, we recapture Corollary 5 of [3].

Before application to special means, we would present some special means and these means will apply in the 5th section.

Special Means: These means can be found in [32].
(a) The Arithmetic Mean

$$A = \frac{j+k}{2}; \quad j,k \ge 0.$$

(b) The Geometric Mean

$$G = G(j,k) = \sqrt{jk}; \quad j,k \geq 0.$$

(c) The Harmonic Mean

$$H = H(j,k) = \frac{2}{\frac{1}{j}+\frac{1}{k}}; \quad j,k > 0.$$

(d) The Logarithmic Mean

$$L = L(j,k) = \begin{cases} j, & \text{if} \quad j = k \\ \dfrac{k-j}{\ln k - \ln j}, & \text{if} \quad j \neq k; \end{cases} \quad j,k > 0.$$

(e) Identric Mean

$$I = I(j,k) = \begin{cases} j, & \text{if} \quad j = k \\ \ln\left(\dfrac{\left(\frac{k^k}{j^j}\right)^{\frac{1}{k-j}}}{e}\right), & \text{if} \quad j \neq k; \end{cases} \quad j,k > 0.$$

(f) $p-$Logarithmic Mean

$$L_p = L_p(j,k) = \begin{cases} j, & \text{if} \quad j = k \\ \left(\dfrac{k^{p+1}-j^{p+1}}{(p+1)(k-j)}\right)^{\frac{1}{p}}, & \text{if} \quad j \neq k, \end{cases}$$

where $p \in \mathbb{R}\backslash\{-1,0\}, j,k > 0$. It is known that L_p monotonically increasing over $p \in \mathbb{R}$, $L_0 = I$ and $L_{-1} = L$.

5. Application to Special Means

Example no 1: Consider

$$\begin{aligned} \gamma &= 1, \\ g(\theta) &= \theta^p,\ p \in \mathbb{R}\backslash\{-1,0\},\ then\, for\ j<k, \\ then \qquad \frac{1}{(k-j)}\int_j^k g(\tau)d\tau &= L_p^p(j,k), \\ \frac{g(j)+g(k)}{2} &= A(j^p,k^p), \\ and \qquad \frac{j+k}{2} &= A, \end{aligned}$$

where $\theta \in [j, \frac{j+k}{2}]$.
Therefore, (2.1) becomes

$$\begin{aligned} &\left|\frac{\theta^p + (2A-\theta)^p}{2} - L_p^p(j,k)\right| \leq \left[\int_j^{\frac{j+\theta}{2}}\left(\int_j^\tau w(u)du\right)d\tau \right. \\ &+ \int_{\frac{j+\theta}{2}}^{\theta}\left(\int_j^\tau w(u)du\right)d\tau - \int_\theta^A\left(\int_A^\tau w(u)du\right)d\tau + \int_A^{2A-\theta}\left(\int_A^\tau w(u)du\right)d\tau \\ &\left. - \int_{2A-\theta}^{A+\frac{k-\theta}{2}}\left(\int_k^\tau w(u)du\right)d\tau - \int_{A+\frac{k-\theta}{2}}^{k}\left(\int_k^\tau w(u)du\right)d\tau\right]\frac{(M_1+m_1)}{2}. \end{aligned}$$

If we choose $\theta = j$ in (2.1), we get

$$\left|A(j^p, k^p) - L_p^p(j,k)\right| \leq \left[\int_A^k \left(\int_A^\tau w(u)du\right)d\tau - \int_j^A \left(\int_A^\tau w(u)du\right)d\tau\right]\frac{(M_1+m_1)}{2}.$$

Example no 2: Consider

$$\begin{aligned}
\gamma &= 1,\\
g(\theta) &= \frac{1}{\theta}, \quad \theta \neq 0\\
\textit{then} \qquad \frac{1}{k-j}\int_j^k g(\tau)d\tau &= L^{-1}(j,k),\\
\frac{g(j)+g(k)}{2} &= \frac{A}{G^2},\\
\textit{and} \qquad \frac{j+k}{2} &= A,
\end{aligned}$$

where $\theta \in [j, \frac{j+k}{2}] \subset (0,\infty)$.
Therefore, (2.1) becomes

$$\begin{aligned}
\left|\frac{A}{\theta(j+k-\theta)} - L^{-1}(j,k)\right| &\leq \left[\int_j^{\frac{j+\theta}{2}} \left(\int_j^\tau w(u)du\right)d\tau\right.\\
&+ \int_{\frac{j+\theta}{2}}^{\theta} \left(\int_j^\tau w(u)du\right)d\tau - \int_\theta^A \left(\int_A^\tau w(u)du\right)d\tau + \int_A^{2A-\theta} \left(\int_A^\tau w(u)du\right)d\tau\\
&\left.- \int_{2A-\theta}^{A+\frac{k-\theta}{2}} \left(\int_k^\tau w(u)du\right)d\tau - \int_{A+\frac{k-\theta}{2}}^{k} \left(\int_k^\tau w(u)du\right)d\tau\right]\frac{(M_1+m_1)}{2}.
\end{aligned}$$

If we choose $\theta = j$ in (2.1), we get

$$\left|\frac{A}{G^2} - L^{-1}(j,k)\right| \leq \left[\int_A^k \left(\int_A^\tau w(u)du\right)d\tau - \int_j^A \left(\int_A^\tau w(u)du\right)d\tau\right]\frac{(M_1+m_1)}{2}.$$

Example no 3: Consider

$$\begin{aligned}
\gamma &= 1,\\
g(\theta) &= ln\theta, \quad \theta \in (0,\infty)\\
\textit{then} \qquad \frac{1}{k-j}\int_j^k g(\tau)d\tau &= ln(I(j,k)),\\
\frac{g(j)+g(k)}{2} &= lnG,\\
\textit{and} \qquad \frac{j+k}{2} &= A,
\end{aligned}$$

where $\theta \in [j, \frac{j+k}{2}] \subset (0, \infty)$.
Therefore, (2.1) becomes

$$\left| ln\left[\frac{[\theta(2A-\theta)]^{\frac{1}{2}}}{I(j,k)}\right]\right| \leq \left[\int_j^{\frac{j+\theta}{2}} \left(\int_j^\tau w(u)du\right) d\tau \right.$$
$$+ \int_{\frac{j+\theta}{2}}^{\theta} \left(\int_j^\tau w(u)du\right) d\tau - \int_\theta^A \left(\int_A^\tau w(u)du\right) d\tau + \int_A^{2A-\theta} \left(\int_A^\tau w(u)du\right) d\tau$$
$$\left. - \int_{2A-\theta}^{A+\frac{k-\theta}{2}} \left(\int_k^\tau w(u)du\right) d\tau - \int_{A+\frac{k-\theta}{2}}^{k} \left(\int_k^\tau w(u)du\right) d\tau \right] \frac{(M_1+m_1)}{2}.$$

If we choose $\theta = j$ in (2.1), we get

$$\left| ln\left[\frac{G}{I(j,k)}\right]\right| \leq \left[\int_A^k \left(\int_A^\tau w(u)du\right) d\tau - \int_j^A \left(\int_A^\tau w(u)du\right) d\tau\right] \frac{(M_1+m_1)}{2}.$$

Example no 4: Consider

$$\begin{aligned} \gamma &= 1, \\ g(\theta) &= e^\theta, \quad \theta \in (-\infty, \infty) \\ \textit{then} \qquad \frac{1}{k-j}\int_j^k g(\tau)d\tau &= \frac{e^k - e^j}{k-j}, \\ \frac{g(j)+g(k)}{2} &= A(e^j, e^k), \\ \textit{and} \qquad \frac{j+k}{2} &= A, \end{aligned}$$

where $\theta \in [j, \frac{j+k}{2}]$.
Therefore, (2.1) becomes

$$\left| \frac{e^\theta + e^{(2A-\theta)}}{2} - \frac{e^k - e^j}{k-j}\right| \leq \left[\int_j^{\frac{j+\theta}{2}} \left(\int_j^\tau w(u)du\right) d\tau \right.$$
$$+ \int_{\frac{j+\theta}{2}}^{\theta} \left(\int_j^\tau w(u)du\right) d\tau - \int_\theta^A \left(\int_A^\tau w(u)du\right) d\tau + \int_A^{2A-\theta} \left(\int_A^\tau w(u)du\right) d\tau$$
$$\left. - \int_{2A-\theta}^{A+\frac{k-\theta}{2}} \left(\int_k^\tau w(u)du\right) d\tau - \int_{A+\frac{k-\theta}{2}}^{k} \left(\int_k^\tau w(u)du\right) d\tau \right] \frac{(M_1+m_1)}{2}.$$

If we choose $\theta = j$ in (2.1), we get

$$\left| A(e^j, e^k) - \frac{e^k - e^j}{k-j}\right|$$
$$\leq \left[\int_A^k \left(\int_A^\tau w(u)du\right) d\tau - \int_j^A \left(\int_A^\tau w(u)du\right) d\tau\right] \frac{(M_1+m_1)}{2}.$$

Example no 5: Consider

$$\begin{aligned} \gamma &= 1, \\ g(\theta) &= \tan\theta, \quad \theta \neq \frac{\pi}{2} \pm n\pi \\ \textit{then} \qquad \frac{1}{k-j}\int_j^k g(\tau)d\tau &= \ln\left[\frac{\sec k}{\sec j}\right]^{k-j}, \\ \frac{g(j)+g(k)}{2} &= A(\tan j, \tan k), \\ \textit{and} \qquad \frac{j+k}{2} &= A, \end{aligned}$$

where $\theta \in [j, \frac{j+k}{2}]$.
Therefore, (2.1) becomes

$$\left|\frac{\tan\theta + \tan(2A-\theta)}{2} - \ln\left[\frac{\sec k}{\sec j}\right]^{k-j}\right| \leq \left[\int_j^{\frac{j+\theta}{2}}\left(\int_j^\tau w(u)du\right)d\tau + \int_{\frac{j+\theta}{2}}^{\theta}\left(\int_j^\tau w(u)du\right)d\tau - \int_\theta^A\left(\int_A^\tau w(u)du\right)d\tau + \int_A^{2A-\theta}\left(\int_A^\tau w(u)du\right)d\tau - \int_{2A-\theta}^{A+\frac{k-\theta}{2}}\left(\int_k^\tau w(u)du\right)d\tau - \int_{A+\frac{k-\theta}{2}}^{k}\left(\int_k^\tau w(u)du\right)d\tau\right]\frac{(M_1+m_1)}{2}.$$

If we choose $\theta = j$ in (2.1), we get

$$\left|A(\tan j, \tan k) - \ln\left[\frac{\sec k}{\sec j}\right]^{k-j}\right| \leq \left[\int_A^k\left(\int_A^\tau w(u)du\right)d\tau - \int_j^A\left(\int_A^\tau w(u)du\right)d\tau\right]\frac{(M_1+m_1)}{2}.$$

6. Conclusion

In this article our aim was to generalise the results of [3] and [20]. We have obtained generalisation of companion of Ostrowski's type integral inequality by applying the Riemann-Liouville fractional integral. By using suitable substitutions we have recaptured the all results of M. W. Alomari's article [3] and given some special cases and also recaptured the all results of one more article [20] of different authors. Further, we have deduced applications to numerical integration, probability theory and special means.

Declarations

Conflict of interest
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Authors' contributions
Faraz Mehmood made the main contribution in conceiving the presented research. Akhmadjon Soleev and Faraz Mehmood worked jointly on each section while Faraz Mehmood drafted the whole manuscript. Both authors read and approved the final manuscript.

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1-Department of Mathematics, Samarkand State University, University boulevard 15, Samarkand 140104, Uzbekistan
Email address: faraz.mehmood@duet.edu.pk
Email address: asoleev@yandex.com/asoleev@yandex.ru

2-Department of Mathematics, Dawood University of Engineering and Technology, New M. A. Jinnah Road, Karachi-74800, Pakistan