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STABILITY CONDITION AND RIESZ BOUNDS FOR EXPONENTIAL SPLINES

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Abstract. Stability of the family of integer translations of exponential spline $U_{m,p}$ for arbitrary m, p is proven; Riesz bounds are determined. The method presented in the paper allows to calculate Riesz bounds for the convolution of a B-spline and a function with an appropriated Fourier transform.

Keywords: E-spline, Riesz basis, Riesz bounds, functional series.

Mishchenko E.V., Stability condition and Riesz bounds for exponential splines.

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1. Introduction

As known, piece-wise polynomial functions were used by many well-known mathematicians long before the independent apparatus of polynomial splines was introduced by Schoenberg in [1,2]. A large number of works has been devoted to further generalization of the spline concept and study of approximation properties for splines of various nature. The first work in this direction is recognized to be the Schoenberg paper [3], in which trigonometric splines were considered as functions in the kernel of the differential operator $\Delta_m = D(D^2+1)\dots(D^2+m^2)$. Later on, interpolation splines, generated by linear differential operators under various restrictions, were introduced [see, for example, 4,5,6]. With this approach, treatment of L-splines depends heavily on obtaining explicit formulae for a related Green's function and associated divided differences. At the same time, a variational approach to the theory of splines in Hilbert space was developed, in which an interpolation spline is defined as a solution to some extremal problem [7].

The role of splines in various applications is well known. Algorithms based on polynomial splines have proven to be quite efficient for signal processing and imaging in medicine [8], especially for high resolution interpolation, showing the best trade-off between cost and quality among linear techniques. Splines are used as basis functions in numerical methods, for example, in the collocation method [9]. Polynomial splines also play a fundamental role in wavelet theory [10]. Less attention was paid to exponential splines.

In distinction to the approach in [4–6], where exponential splines are defined as functions in the null space of the operator $L_m = (D - pI)(D - 2pI)\dots(D - pmI)$, D is the differentiaton operator, I is the identical operator, $m \in N$, $p \in R$, in the present paper exponential analogues of B-splines, hereinafter called E-splines, are constructed using convolution operators. The class of exponential splines is an extension of polynomial B–splines in the sense that B–splines of order m are limit cases of exponential splines.

It is worth saying that general spline theories are very elegant, but their level of generality often makes it difficult for a researcher to extract information relevant to a particular case, especially while obtaining estimates. These considerations were the motive for studying certain properties of the class of exponential splines, which are fragments of exponentials connected in a smooth way.

The purpose of this work is to study the stability of the system of integer translations of exponential splines or, in other words, to establish the Riesz bounds for this system.

2. Basic concepts and notation

The following concepts and notation are used in the paper.

Values of the functions $\frac{\sin ax}{x}$, $\frac{\sinh bx}{x}$ and similar ones are assumed to be defined at the point $x = 0$ by their limit values:

$$\lim_{x\to 0} \frac{\sin ax}{x} = a, \quad \lim_{x\to 0} \frac{\sinh bx}{x} = b \quad \text{and so on.}$$

.

The Fourier transform of a function $\phi(x)$ is denoted by $\hat{\phi}(\xi)$ and given by

$$\hat{\phi}(\xi) = \frac{1}{\sqrt{2\pi}} \int_R e^{-ix\xi} \phi(x) dx.$$

The convolution $(f * g)(x)$ of two functions $f(x)$ and $g(x)$ is given by the relation

$$(f * g)(x) = f(x) * g(x) = \int_R f(y) g(x - y) dx.$$

The following properties of convolution and Fourier transform will be useful

$$f(x) * g(x) = g(x) * f(x), \tag{1}$$

$$\widehat{f * g}(\xi) = \sqrt{2\pi} \hat{f}(\xi) \cdot \hat{g}(\xi). \tag{2}$$

The symbol $D'(R)$ denotes the space of distributions, $\delta(x)$ stands for the Dirac distribution.

We use the notation $Q_m(x)$ for B-splines of order m which are piecewise polynomial m-times convolutions of the indicator of the interval $[0, 1)$: for $m = 1$

$$Q_1(x) = \chi_{[0,1)}(x),$$

$\chi_{[0,1)}(x)$ is the indicator function of the interval $[0, 1)$:

$$\chi_{[0,1)}(x) = \begin{cases} 1, & \text{if } x \in [0, 1) \\ 0, & \text{otherwise,} \end{cases}$$

for $m > 1$

$$Q_m(x) = (Q_{m-1} * \chi_{[0,1)})(x) = \int_0^1 Q_{m-1}(x - y) dy.$$

The exponential B-splines $U_{m,p}(x)$ are smoothly stitched fragments of exponentials. The order of smoothness depends on the order of the spline.

Definition 1. For any non-zero $p \in (-\infty, \infty)$ and natural m we define the function $U_{m,p}(x)$, which we will call the exponential spline, or E-spline, of order m with real parameter p. For $m = 1$

$$U_{1,p}(x) = \varphi_p(x) = \begin{cases} \frac{pe^{px}}{e^p - 1}, & \text{if } x \in [0, 1) \\ 0, & \text{otherwise}, \end{cases}$$

the function $\varphi_p(x)$ is normalized in the sense that

$$\int_R \varphi_p(x)dx = \int_0^1 \varphi_p(x)dx = 1.$$

For $m > 0$ $U_{m+1,p}(x)$ is convolution of the indicator function and E-spline of order m:

$$U_{m+1,p}(x) = (U_{m,p} * \chi_{[0,1)})(x),$$

or, by the convolution property (1),

$$U_{m+1,p}(x) = (Q_{m,p} * \varphi_p)(x).$$

The statement below characterizes the behavior of $U_{1,p}(x)$ for different values of p.

Theorem 1. The following convergence properties take place in the space $D'(R)$

E 1. $\varphi_p(x) \to \delta(x)$ as $p \to -\infty$.
E 2. $\varphi_p(x) \to \chi_{[0,1)}(x)$ for $p \to 0$.
E 3. $\varphi_p(x) \to \delta(1 - x)$ as $p \to \infty$.

Proof. Indeed, by the definition of convergence in the space $D'(R)$, we have to show that $(\varphi_p, f) \to (\delta, f)$ as $p \to -\infty$ for any test function $f \in D(R)$. As $(\delta, f) = f(0)$, we have

$$(\varphi_p, f) - (\delta, f) = \int_R f(x)\varphi_p(x)dx - f(0) \cdot 1 =$$

$$= \int_0^1 f(x)\varphi_p(x)dx - f(0)\int_0^1 \varphi_p(x)dx = \int_0^1 \Big(f(x) - f(0)\Big)\varphi_p(x)dx.$$

The function $f(x)$ is continuous, so
1) $\exists M > 0$ such that

$$\max_{x\in[0,1]} |f(x)| \le M,$$

2) $\forall \varepsilon > 0\ \exists\ \Delta > 0$ such that $|x| < \Delta$ implies $|f(x) - f(0)| < \frac{\varepsilon}{2}$. Then

$$\int_0^1 \Big(f(x) - f(0)\Big)\varphi_p(x)dx =$$

$$= \int_0^\Delta \Big(f(x) - f(0)\Big)\varphi_p(x)dx + \int_\Delta^1 \Big(f(x) - f(0)\Big)\varphi_p(x)dx \le$$

$$\leq \frac{\varepsilon}{2}\int_0^{\Delta}\varphi_p(x)dx + 2M\int_{\Delta}^{1}\varphi_p(x)dx \leq \frac{\varepsilon}{2} + 2M\Big(\frac{e^p - e^{p\Delta}}{e^p - 1}\Big). \quad (3)$$

Fixed Δ, it is possible to specify $P \leq 0$, such that $|\frac{e^p - e^{p\Delta}}{e^p - 1}| < \frac{\varepsilon}{4M}$ $\forall p < P$. Extending the inequality (3), we obtain that for arbitrary $\varepsilon >$, such $P < 0$ can be chosen that $\forall p < P$

$$\int_0^1 \Big(f(x) - f(0)\Big)\varphi_p(x)dx < \varepsilon.$$

Property $E1$ is stated. Similar arguments yields to the property $E3$.

To establish $E2$, we note that the difference $\chi_{[0,1)}(x) - \varphi_p(x)$ outside the interval $[0,1)$ is equal to zero; for $x \in [0,1)$, applying the consequence of the second remarkable limit, we obtain

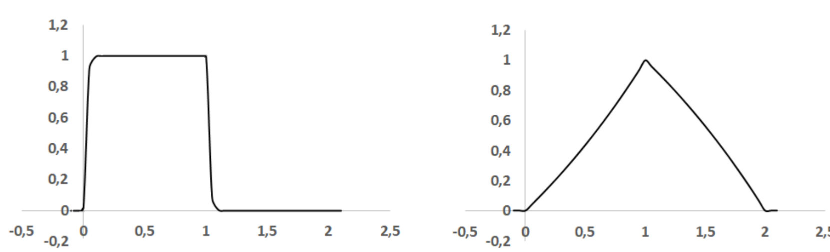
$$\lim_{p\to 0}\chi_{[0,1)}(x) - \varphi_p(x) = \lim_{p\to 0}\frac{e^p - 1 - pe^{px}}{e^p - 1} = 1 - \lim_{p\to 0}\frac{pe^{px}}{e^p - 1}$$
$$= 1 - \lim_{p\to 0}e^{px}\lim_{p\to 0}\frac{p}{e^p - 1} = 1 - 1 = 0$$

By properties E1 and E2 we have the convergence properties:

$$\lim_{p\to 0}U_{m+1,p}(x) = Q_{m+1}(x),$$
$$\lim_{p\to -\infty}U_{m+1,p}(x) = Q_m(x). \blacktriangleright$$

Example. Figures 1 and 2 illustrate the convergence properties of $U_{m,p}$ for $m = 2$

$$U_{2,p}(x) = \begin{cases} 0, & x \leq 0, \\ \frac{e^{px}-1}{e^p-1}, & 0 \leq x \leq 1, \\ \frac{e^p - e^{p(x-1)}}{e^p-1}, & 1 \leq x \leq 2 \\ 0, & 2 \leq x. \end{cases}$$



a) b)

FIG. 1. E-splines $U_{2,p}$, a) $p = -500$, b) $p = 0,5$

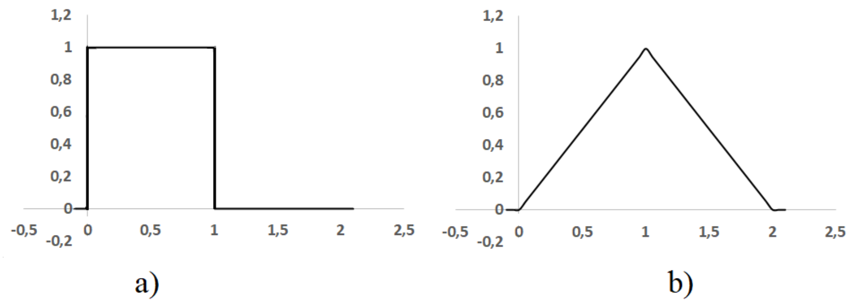


FIG. 2. B–splines Q_m, a) $m = 1$, b) $m = 2$

As well as $Q_m(x)$, $U_{m,p}(x)$ has the compact support (supp $U_{m,p}(x) = [0, m)$) and are continuous for $m > 1$ ($U_{m,p}(x) \in C^{m-1}(R)$) that makes the family $\{U_{m,p}(\cdot - k), k \in Z\}$ very attractive for practical purpose provided that we have an additional information on stability of this family.

3. Stability condition and Riesz bounds

Definition 2. A family of integer translations $\{\varphi(x - n), n \in Z\}$ generated by a function $\varphi(x)$ from a Hilbert space H is said to be stable, if there exist two constants $0 < A, B < \infty$ such that for any sequence $\{c_n\}_{n\in Z} \in l_2$ the following inequalities hold:

$$A \sum_{n=-\infty}^{\infty} |c_n|^2 \leq || \sum_{n=-\infty}^{\infty} c_n\varphi(x - n)||_H^2 \leq B \sum_{n=-\infty}^{\infty} |c_n|^2. \tag{4}$$

Definition 3. If the linear span of the set $\{\varphi(\cdot - n), n \in Z\}$ is stable and, additionally, dense in H, then they say that $\{\varphi(\cdot - n), n \in Z\}$ form an unconditional or Riesz basis in H.

Remark 1. For $A = B = 1$ the Riesz basis turns into an orthonormal basis. For example, $\{\chi_{[0,1)}(\cdot - n), n \in Z\}$ forms the orthonormal basis in the space of functions which are piece-wise constant on the intervals of the form $[k, k + 1)$.

Constants A and B in (4) characterize, in a sense, the "redundancy"of the Riesz basis. They are also called the Riesz constants or the Riesz lower and upper bounds.

The aim of the work is to establish the stability property for the system of integer translations of E-splines for arbitrary $m \in N$ and $p \in (-\infty, \infty)$ and determine its Riesz bounds.

It should be noted that in certain cases determination of Riesz bounds becomes simpler if we pass from the function $\varphi(x)$ to its Fourier transform

$\widehat{\varphi}(\xi)$ and use the theorem which establishes equivalence of two statements [10]:
– the set $\{\phi(\cdot - k), k \in Z\}$ satisfies the Riesz condition with constants $2\pi A$ and $2\pi B$;
– the Fourier transform $\hat{\phi}(\xi)$ satisfies the inequalities

$$A \leq \sum_{k \in Z} |\hat{\phi}((\xi + 2\pi k)|^2 \leq B.$$

By this, to establish stability it suffices to determine the lower and upper estimates, uniform in ξ, for the series

$$\sum_{k \in Z} |\widehat{U}_{m,p}(\xi + 2\pi k)|^2,$$

assuming that the parameter p and the order of the exponential spline m are fixed.

In a view of

$$\widehat{Q}_1(\xi) = \frac{1}{2\pi} e^{-i\xi/2} \frac{\sin(\xi/2)}{\xi/2},$$

by (2) we have

$$\widehat{Q}_m(\xi) = \frac{1}{\sqrt{2\pi}} \Big(e^{-i\xi/2} \frac{\sin(\xi/2)}{\xi/2} \Big)^m$$

and

$$\widehat{U}_{m+1,p}(\xi) = \sqrt{2\pi} \widehat{Q}_m(\xi) \cdot \widehat{\varphi}_p(\xi)$$

The Fourier transform for the function $\varphi_p(x)$ is

$$\widehat{\varphi}_p(\xi) = \frac{1}{\sqrt{2\pi}} \int_0^1 e^{-ix\xi} \frac{pe^{px}}{e^p - 1} dx = \frac{1}{\sqrt{2\pi}} \frac{p}{e^p - 1} \frac{e^{p-i\xi} - 1}{p - i\xi},$$

and then

$$\widehat{U}_{m+1,p}(\xi) = \frac{1}{\sqrt{2\pi}} \Big(e^{-i\frac{\xi}{2}} \frac{\sin \xi/2}{\xi/2} \Big)^m \frac{p}{e^p - 1} \frac{e^{p-i\xi} - 1}{p - i\xi}.$$

Hence

$$|\widehat{U}_{m+1,p}(\xi)|^2 = \frac{1}{2\pi} \cdot \Big(\frac{\sin \xi/2}{\xi/2} \Big)^m \cdot \frac{p^2(e^{2p} - 2e^p \cos \xi + 1)}{(p^2 + \xi^2)(e^p - 1)^2}. \qquad (5)$$

For the further consideration, we rewrite the last factor in (5) as follows:

$$\frac{p^2}{p^2 + \xi^2} \cdot \frac{e^{2p} - 2e^p \cos \xi + 1}{(e^p - 1)^2} = \frac{(p/2)^2}{(p/2)^2 + (\xi/2)^2} \cdot \Big(1 + \frac{\sin^2(\xi/2)}{\sinh^2(p/2)} \Big).$$

Then (5) takes the form

$$|\widehat{U}_{m+1,p}(\xi)|^2 = \frac{1}{2\pi}\frac{(p/2)^2}{(p/2)^2+(\xi/2)^2}\Big(1+\frac{\sin^2(\xi/2)}{\sinh^2(p/2)}\Big)\Big(\frac{\sin\xi/2}{\xi/2}\Big)^m$$

and, by this, we have the series

$$\sum_{k=-\infty}^{\infty}|\widehat{U}_{m+1,p}(\xi+2\pi k)|^2 =$$

$$= \frac{1}{2\pi}\sum_{k=-\infty}^{\infty}\frac{(p/2)^2}{(p/2)^2+(\xi/2+\pi k)^2}\Big(1+\frac{\sin^2(\xi/2+\pi k)}{\sinh^2(p/2)}\Big)\Big(\frac{\sin(\xi/2+\pi k)}{\xi/2+\pi k}\Big)^{2m}. \tag{6}$$

For the sake of convenience we denote

$$a_k(\xi) = \frac{(p/2)^2}{(p/2)^2+(\xi/2+\pi k)^2}\Big(1+\frac{\sin^2(\xi/2+\pi k)}{\sinh^2(p/2)}\Big);$$

$$u_k(\xi) = \frac{1}{2\pi}\Big(\frac{\sin(\xi/2+\pi k)}{\xi/2+\pi k}\Big)^{2m}. \tag{7}$$

Thus, we have the function series

$$\sum_{k=-\infty}^{\infty}|\widehat{U}_{m+1,p}(\xi+2\pi k)|^2 = \sum_{k=-\infty}^{\infty}a_k(\xi)u_k(\xi),$$

which, due to its 2π–periodicity, can be considered only on the interval $[0,2\pi]$.

The following theorem is true.

Theorem 2. The function series (6) converges uniformly on the interval $[0,2\pi]$ for any $p\in(-\infty,\infty)$ and any positive integer m to a continuous function $\mathcal{N}_{m+1,p}(\xi)$.

Proof. We fix a real $p\neq 0$ and a non negative integer m and present the series (6) as a sum of three terms

$$\sum_{k=-\infty}^{\infty}a_k(\xi)u_k(\xi) = \sum_{k=-\infty}^{-1}a_k(\xi)u_k(\xi) + \sum_{k=1}^{\infty}a_k(\xi)u_k(\xi) +$$

$$+\frac{1}{2\pi}\frac{(p/2)^2}{(p/2)^2+(\xi/2)^2}\Big(1+\frac{\sin^2(\xi/2)}{\sinh^2(p/2)}\Big)\Big(\frac{\sin\xi/2}{\xi/2}\Big)^{2m}. \tag{8}$$

We prove convergence of the series in the first and second terms using the Dirichlet's test: a series $\sum_{k=1}^{\infty}a_k(\xi)u_k(\xi)$ converges uniformly on a set Ξ if $a_k(\xi)$ and $u_k(\xi)$ satisfy the conditions

1. the sequence $\{a_k(\xi)\}$ is monotone for any ξ and converges uniformly to 0 on Ξ as $k\to\infty$;
2. partial sums $S_n(\xi)=\sum_{k=1}^{\infty}u_k(\xi)$ are uniformly bounded on Ξ.

In our case $\Xi=[0,2\pi]$ and $a_k(\xi)$ and $u_k(\xi)$ in $\sum_{k=1}^{\infty} a_k(\xi)u_k(\xi)$ are given by (7).

1. As

$$a_k(\xi)=\frac{(p/2)^2}{(p/2)^2+(\xi/2+\pi k)^2}\Big(1+\frac{\sin^2(\xi/2+\pi k)}{\sinh^2(p/2)}\Big)=$$

$$=\frac{(p/2)^2}{(p/2)^2+(\xi/2+\pi k)^2}\Big(1+\frac{\sin^2(\xi/2)}{\sinh^2(p/2)}\Big),$$

obviously, for any fixed $\xi\in\Xi$ the sequence $\{a_k(\xi)\}$ decreases monotonically as $k\to\infty$. Moreover,

$$a_k(\xi)\le\frac{(p/2)^2}{(p/2)^2+(\pi k)^2}\Big(1+\frac{1}{\sinh^2(p/2)}\Big).$$

Hence for any given $\varepsilon>0$ we can specify $K>1$ such that $a_k(\xi)<\varepsilon$ for any $\xi\in\Xi$ as soon as $k>K$. The uniform convergence to zero of the sequence $\{a_k(\xi)\}$ is shown. Thus, condition 1 is fulfilled.

2. Next, we show that the partial sums $S_n(\xi)=\sum_{k=1}^{n} u_k(\xi)$ are uniformly bounded.

In [11, Th.1] it was shown that

$$\sum_{k=-\infty}^{\infty}\frac{1}{2\pi}\Big(\frac{\sin(\xi/2+\pi k)}{\xi/2+\pi k}\Big)^{2m}=\frac{1}{2\pi}\frac{C_{m-1}(\cos^2(\xi/2))}{(2m-1)!},$$

for every fixed $m\ge 1$. Trigonometric polynomials $C_m(\cos^2\xi)$ were proven [11, Pr.1] to be defined on the whole R, continuous, positive, axially symmetric with the axis $\xi=0$, π–periodic functions. They reach their maximum values at the points $\xi=k\pi$, and their minimum values at the points $\xi=\frac{(2k+1)\pi}{2}$, $k\in Z$. Besides,

$$\frac{1}{2\pi}\frac{C_m(\cos^2(\pi/2))}{(2m+1)!}\le\sum_{k=-\infty}^{\infty}|\widehat{Q}_{m+1}(\xi+2\pi k)|^2\le\frac{1}{2\pi}\frac{C_m(\cos^2(0))}{(2m+1)!}.$$

$$\frac{C_m(\cos^2(0))}{(2m+1)!}=1,\quad\frac{C_m(\cos^2(\pi/2))}{(2m+1)!}=\frac{2^{2m+3}}{\pi^{2(m+1)}}\sum_{k=0}^{\infty}\frac{1}{(1+2k)^{2(m+1)}}\qquad(9)$$

and the numerical sequence $\{C_m(\cos^2(\pi/2))/(2m+1)!\}$ tends monotonically to 0 as $m\to\infty$.

Thus, partial sums $S_n(\xi)$ are uniformly bounded on $[0,2\pi]$:

$$S_n(\xi)=\sum_{k=1}^{n}\frac{1}{2\pi}\Big(\frac{\sin(\xi/2+\pi k)}{\xi/2+\pi k}\Big)^{2m}<\sum_{k=-\infty}^{\infty}\frac{1}{2\pi}\Big(\frac{\sin(\xi/2+\pi k)}{\xi/2+\pi k}\Big)^{2m}=$$

$$= \frac{1}{2\pi}\frac{C_{m-1}(\cos^2(\xi/2))}{(2m-1)!} \leq \frac{1}{2\pi}.$$

It is evident that $a_k(\xi) = a_{-k}(-\xi)$, $u_k(\xi) = u_{-k}(-\xi)$, and the first term and the second term in (8) are related by the formula

$$\sum_{k=-\infty}^{-1} a_k(\xi)u_k\xi = \sum_{k=1}^{\infty} a_k(-\xi)u_k(-\xi),$$

whence convergence of the series in the first term is also shown.

Finally, the function

$$\mu(p,\xi) = \frac{1}{2\pi}\frac{(p/2)^2}{(p/2)^2+(\xi/2)^2}\Big(1+\frac{\sin^2(\xi/2)}{\sinh^2(p/2)}\Big)\Big(\frac{\sin\xi/2}{\xi/2}\Big)^{2m}.$$

is defined on the whole Ξ, including $\xi = 0$, and continuous:

$$\lim_{\xi\to 0}\Big(\frac{\sin\xi/2}{\xi/2}\Big)^{2m} = 1 \quad \Rightarrow \quad \lim_{\xi\to 0}\mu(p,\xi) = \frac{1}{2\pi}.$$

If $p = 0$, then in accordance with item $E2$ and the properties of the Fourier transform

$$\widehat{U}_{m,0}(\xi) = \widehat{Q}_m(\xi),$$

and convergence of series (6) for $p = 0$ is proven. ▶

Corollary. For every $p \in [0,\infty)$ and integer $m \geq 1$ there exist constants $A_{m,p}, B_{m,p} \geq 0$ such that

$$A_{m,p} \leq \sum_{k\in Z}|\widehat{U}_{m,p}(\xi+2\pi k)|^2 \leq B_{m,p}.$$

Proof. Summarizing all the above, we conclude that the series (6) for fixed m and p converges uniformly to a continuous 2π-periodic function $\mathcal{N}_{m+1,p}(\xi)$. The continuous function $\mathcal{N}_{m,p}(\xi)$ on the interval $[0,2\pi]$ has a minimum and a maximum

$$A_{m,p} = \min_{\xi\in[0,2\pi]}\mathcal{N}_{m,p}(\xi);$$

$$B_{m,p} = \max_{\xi\in[0,2\pi]}\mathcal{N}_{m,p}(\xi).$$ ▶

Next, we have to estimate their values. If it occurs that $0 < A_{m,p}, B_{m,p}$, then existence of Riesz bounds becomes to be proven.

Lemma 1. The function

$$f(p) = \frac{1}{p^2} - \frac{1}{\sinh^2 p}$$

reaches its maximum value $1/3$ at $p = 0$.

Proof. Indeed, by the symmetry of $f(p)$, it suffices to consider behavior of $f(p)$ only for $p \in [0, \infty)$. Its first derivative equals

$$f'(p) = -2\Big(\frac{1}{p^3} - \frac{\cosh p}{\sinh^3 p}\Big) = -2 \cdot \frac{\sinh^3 p - \cosh p \cdot p^3}{\sinh^3 p \cdot p^3}.$$

Next we show that the function $h(p) = \sinh^3 p - \cosh p \cdot p^3$ is positive for $p > 0$ and $f'(0) = 0$. If so, $f(p)$ monotonically decreases on $[0, \infty)$ and monotonically increases on $(-\infty, 0]$. Consequently, $p = 0$ is its maximum. Using the Taylor series expansion, we obtain

$$\sinh^2 p - p^2 = \frac{1}{3} p^4 + O(p^6),$$

$$\sinh^2 p \cdot p^2 = p^4 + O(p^6).$$

We define $f(0)$ using the limit passage and have

$$f(0) = \lim_{p \to 0} f(p) = \lim_{p \to 0} \frac{1}{p^2} - \frac{1}{\sinh^2 p} = \lim_{p \to 0} \frac{\sinh^p - p^2}{\sinh^2 p \cdot p^2} = \frac{1}{3}.$$

Now turn back to $h(p)$. As known,

$$\sinh^3 p = \frac{1}{4}(\sinh 3 - 3 \sinh p),$$

then

$$h(p) = \sinh^3 p - \cosh p \cdot p^3 = \sum_{n=0}^{\infty} \Big[\frac{1}{4}\Big(\frac{(3p)^{2n+1}}{(2n+1)!} - \frac{3p^{2n+1}}{(2n+1)!}\Big) - \frac{p^{2n+3}}{(2n)!}\Big] =$$

$$= \sum_{n=1}^{\infty} \Big[\frac{1}{4}\Big(\frac{(3p)^{2n+1}}{(2n+1)!} - \frac{3p^{2n+1}}{(2n+1)!}\Big) - \sum_{n=0}^{\infty} \frac{p^{2n+3}}{(2n)!}\Big] =$$

$$= \sum_{n=0}^{\infty} \Big[\frac{1}{4}\Big(\frac{(3p)^{2n+3}}{(2n+3)!} - \frac{3p^{2n+3}}{(2n+3)!}\Big) - \frac{p^{2n+3}}{(2n)!}\Big] =$$

$$= \frac{1}{4} \sum_{n=0}^{\infty} p^{2n+3} \frac{3^{2n+3} - 3 - 4(2n+1)(2n+2)(2n+3)}{(2n+3)!}$$

The problem reduces to studying signs of the coefficients

$$A_n = 3^{2n+3} - 3 - 4(2n+1)(2n+2)(2n+3), \quad n = 0, 1, ...,$$

or behavior of the function

$$A(x) = 9 \cdot 3^x - 3 - 4x(x+1)(x+2),$$

because $A_n = A(2n+1)$. As easily seen, $A_0 = A(1) = 0$, $A_1 = A(3) = 0$. Hence $h(p) = A_2 p^7 + O(p^9)$ in the neighborhood of $p = 0$. Then

$$f'(0) = \lim_{p \to 0} f'(p) = \lim_{p \to 0} \frac{h(p)}{\sinh^3 p^3} = 0.$$

For any $\alpha, B, a, b, c, d > 0$ there exists $x^* > 0$ such that $Be^{\alpha x} > a + bx + cx^2 + dx^3$ for every $x > x^*$. Let x^* be determined for $\alpha = \ln 3$, $B = 9$, $a = 3$, $b = 8$, $c = 12$, $d = 4$. This proves that $A(x) > 0$ for any integer x such that $x > [\frac{x^*-1}{2}] + 1$ and, generally speaking, it is left to check by direct calculations whether $A_n \geq 0$ for a finite number of n such that $2 \leq n \leq [\frac{x^*-1}{2}] + 1$, but we were lucky: in our case $x^* = 4$, for example, and no additional calculations were needed. ▶

Lemma 2. The function

$$g(\xi) = \frac{1}{\sin^2 \xi} - \frac{1}{\xi^2}$$

reaches its minimum value 1/3 at the point $\xi = 0$.

Proof. Let

$$G(x) = \frac{1}{x} - \operatorname{ctg} x. \tag{10}$$

For $|x| < \pi$

$$x \operatorname{ctg} x = \sum_{n=0} (-1)^n B_{2n} \frac{2^{2n}}{(2n)!} x^{2n} = x + \sum_{n=1}^{\infty} (-1)^{2n+1} \frac{2^{2n+1}\zeta(2n)}{(2\pi)^{2k}} x^{2n},$$

B_{2n} are the Bernoulli numbers

$$B_{2n} = 2(-1)^{n+1} \frac{\zeta(2n)(2n)!}{(2\pi)^{2k}}, n > 1; B_0 = 1; B_2 = \frac{1}{6}, \tag{11}$$

$\zeta(s)$ is the Riemann zeta function

$$\zeta(2n) = \sum_{k=1}^{\infty} \frac{1}{k^{2n}},\ n \geq 1. \tag{12}$$

$$G(x) = \frac{1}{x} - \operatorname{ctg} x = \sum_{n=1}^{\infty} (-1)^{2n+2} \frac{2^{2n+1}\zeta(2n)}{(2\pi)^{2n}} x^{2n-1} =$$

$$= \sum_{n=1}^{\infty} \frac{2^{2n+1}\zeta(2n)}{(2\pi)^{2n}} x^{2n-1} = \frac{x}{3} + \sum_{n=1}^{\infty} x^{2n+1} a_{n+1}, \tag{13}$$

$$a_n = \frac{2^{2n+1}\zeta(2n)}{(2\pi)^{2n}} > 0, \quad a_1 = 1/3 \ \text{(see } (11), (12)), \tag{14}$$

$$g(x) = G'(x) = \frac{1}{\sin^2 x} - \frac{1}{x^2},\ g(0) = 1/3. \tag{15}$$

In a view of (13), (14)

$$g'(x) = \frac{-2\cos x}{\sin^3 x} + \frac{2}{x^3} = G''(x) \geq 0, \ \text{for } x \geq 0, \tag{16}$$

moreover,

$$g'(x) > 0,\ 0 < x < \pi,\ g'(x) = 0 \text{ at } x = 0 \text{ only}.$$

If $\pi \le x$, obviously, $g(x) \ge 1 - \frac{1}{x^2} > 1/3$ for every x.
Hence

$$g(\xi) = \frac{1}{\sin^2 \xi} - \frac{1}{\xi^2} > 1/3, \;\; \xi > 0, \;\; g(0) = 1/3. \qquad \blacktriangleright$$

Remark 2. We find $G(0)$, $g(0)$, $g'(0)$ putting $x = 0$ in the series (13) and its derivatives. Although they could be defined also by the limit passage as $x \to 0$ in explicit expressions (10), (15), (16).

Lemma 3. For any $p \in (-\infty, \infty)$ and $\xi \in (-\infty, \infty)$ the following estimates hold:

$$\frac{\sin^2 \xi}{\xi^2} \le \frac{p^2}{p^2 + \xi^2} \cdot \frac{\sinh^2 p + \sin^2 \xi}{\sinh^2 p} \le 1. \tag{17}$$

Proof. As for any $\xi, p \in (-\infty, \infty)$

$$\sin^2 \xi \le \xi^2, \qquad p^2 \le \sinh^2 p$$

we have

$$p^2 \cdot \sinh^2 p + p^2 \cdot \sin^2 \xi \le p^2 \cdot \sinh^2 p + \sinh^2 p \cdot \xi^2,$$

and

$$\frac{p^2}{p^2 + \xi^2}\Big(1 + \frac{\sin^2 \xi}{\sinh^2 p}\Big) \le 1.$$

The right inequality in (17) is proven.

Next, for any $p, \xi \in (-\infty, \infty)$

$$\frac{1}{p^2} - \frac{1}{\sinh^2 p} \le \sup_{p \in R}\Big(\frac{1}{p^2} - \frac{1}{\sinh^2 p}\Big) = \frac{1}{3} =$$

$$= \inf_{\xi \in R}\Big(\frac{1}{\sin^2 \xi} - \frac{1}{\xi^2}\Big) \le \frac{1}{\sin^2 \xi} - \frac{1}{\xi^2}. \tag{18}$$

Hence, for any $p \in (-\infty, \infty)$ and $\xi \in [0, 2\pi]$

$$\frac{\sin^2 \xi}{\xi^2} \le \frac{p^2}{p^2 + \xi^2} \cdot \frac{\sinh^2 p + \sin^2 \xi}{\sinh^2 p}. \blacktriangleright$$

Lemma 3 together with (6) and (9) allows us to assert that for a fixed $m \in N$ and any $p, \xi \in (-\infty, \infty)$

$$\sum_{k=-\infty}^{\infty} |\widehat{Q}_{m+1}(\xi + 2\pi k)|^2 \le \sum_{k=-\infty}^{\infty} |\widehat{U}_{m+1,p}(\xi + 2\pi k)|^2 \le \sum_{k=-\infty}^{\infty} |\widehat{Q}_{m}(\xi + 2\pi k)|^2.$$

It is easy to see that $\mathcal{N}_{m+1,p}(\xi) = \mathcal{N}_{m+1,-p}(\xi)$. Therefore $A_{m,p} = A_{m,-p}$, $B_{m,p} = B_{m,-p}$.

Thus, we proved the following

Theorem 3. The system of integer translations $\{U_{m,p}(x - k)\}_{k \in Z}$, where $U_{m,p}(x)$ is the exponential spline of order m, is stable for arbitrary

$m \in N$ and $p \in (-\infty, \infty)$. Its lower and upper Riesz bounds $2\pi A_{m,p}$, $2\pi B_{m,p}$ satisfies inequalities

$$\frac{2^{2m+3}}{\pi^{2(m+1)}} \sum_{k=0}^{\infty} \frac{1}{(1+2k)^{2(m+1)}} \leq 2\pi A_{m,p} < 2\pi B_{m,p} \leq 1. \qquad (19)$$

(19) turns into equality at $p = 0$.

4. Conclusion

We proved that the family of integer translations of the exponential spline $U_{m,p}$ is stable for arbitrary $p \in R$, $m \geq 1$; its Riesz bounds were found.

As follows from the above, in a similar way one can obtain Riesz bases with required properties considering integer translations of convolution of $Q_m(x)$ and an appropriate function $\varphi(x)$. Here the order m provides the smoothness property, whereas the function $\varphi(x)$ guarantees such properties, for example, as compactness of the support or symmetry and affects the value of Riesz bounds.

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