Stability condition and Riesz bounds for exponential splines

Mishchenko E.V.

January 16, 2023

UDC 517.518.34+517.537.3
MSC 46B15+40A30

Abstract

Stability of the family of integer shifts of exponential spline $U_{m,p}$ for arbitrary m, p is proven; Riesz bounds are determined. The method described in the paper allows to calculate Riesz bounds for the convolution of a B-spline and a function with an appropriated Fourier transform.
Keywords: E-spline, Riesz basis, Fourier transform, functional series.

1 Introduction

The role of splines in various applications is well known. Algorithms based on polynomial splines have proven to be quite efficient for signal processing and imaging in medicine [1], especially for high resolution interpolation, showing the best trade-off between cost and quality among linear techniques. Splines are used as basis functions in numerical methods, for example, in the collocation method [2]. Polynomial splines also play a fundamental role in wavelet theory [3]. Less attention was paid to exponential splines.

It is worth saying that general spline theories are very elegant, but their level of generality is often such that for a researcher it is difficult to extract information relevant to a particular case, especially while obtaining estimates. These considerations were the motive for studying certain properties of the class of exponential splines, which are fragments of exponentials connected in a smooth way.

The class of exponential splines is an extension of polynomial B–splines in the sense that B–splines of order m are limit cases of exponential splines.

The purpose of this work is to study the stability of the system of integer shifts of exponential splines or, in other words, to establish the Riesz bounds for this system.

2 Basic concepts and notation

The following concepts and notation are used in the paper.

The Fourier transform of a function $\phi(x)$ is denoted by $\hat{\phi}(\xi)$ and given by

$$\hat{\phi}(\xi) = \frac{1}{\sqrt{2\pi}} \int_R e^{-ix\xi} \phi(x) dx.$$

The convolution $(f * g)(x)$ of two functions $f(x)$ and $g(x)$ is given by the relation

$$(f * g)(x) = f(x) * g(x) = \int_R f(y) g(x-y) dx.$$

The following properties of convolution and Fourier transform will be useful

$$f(x) * g(x) = g(x) * f(x), \tag{1}$$

$$\widehat{f * g}(\xi) = \sqrt{2\pi} \hat{f}(\xi) \cdot \hat{g}(\xi). \tag{2}$$

The symbol $D'(R)$ denotes the space of distributions.

We use the notation $Q_m(x)$ for B-splines of order m which are piece-wise polynomial functions obtained by m-times convolution of the indicator of the interval $[0,1)$: for $m = 1$

$$Q_1(x) = \chi_{[0,1)}(x),$$

$\chi_{[0,1)}(x)$ is the indicator function of the interval $[0,1)$:

$$\chi_{[0,1)}(x) = \begin{cases} 1, & \text{if } x \in [0,1) \\ 0, & \text{otherwise,} \end{cases}$$

for $m > 1$

$$Q_m(x) = (Q_{m-1} * \chi)(x) = \int_0^1 Q_{m-1}(x-y) dy.$$

The exponential B-splines $U_{m,p}(x)$ are smoothly stitched fragments of exponentials. The order of smoothness depends on the order of the spline.

Definition. For any non-zero $p \in (-\infty, \infty)$ and natural m we define the function $U_{m,p}(x)$, which we will call the exponential spline, or E-spline, of order m with real parameter p. For $m = 1$

$$U_{1,p}(x) = \varphi_p(x) = \begin{cases} \frac{pe^{px}}{e^p - 1}, & \text{if } x \in [0,1) \\ 0, & \text{otherwise,} \end{cases}$$

the function $\varphi_p(x)$ is normalized in the sense that

$$\int_R \varphi_p(x)dx = \int_0^1 \varphi_p(x)dx = 1.$$

E-spline $U_{m+1,p}(x)$ for arbitrary $m > 0$ is convolution of the indicator and E-spline of order m:

$$U_{m+1,p}(x) = (U_{m,p} * \chi_{[0,1)})(x),$$

or, by the convolution property (1),

$$U_{m+1,p}(x) = (Q_{m,p} * \varphi_p)(x).$$

The statement below characterizes the behavior of $U_{1,p}(x)$ for different values of p.

Theorem. The following convergence properties take place in the space $D'(R)$

E 1. $\varphi_p(x) \to \delta(x)$ as $p \to -\infty$.

E 2. $\varphi_p(x) \to \chi_{[0,1)}(x)$ for $p \to 0$.

E 3. $\varphi_p(x) \to \delta(1-x)$ as $p \to \infty$.

Indeed, by the definition of convergence in the space $D'(R)$, we have to show that $(\varphi_p, f) \to (\delta, f)$ as $p \to -\infty$ for any test function $f \in D(R)$. As $(\delta, f) = f(0)$, we have

$$(\varphi_p, f) - (\delta, f) = \int_R f(x)\varphi_p(x)dx - f(0) \cdot 1 =$$

$$= \int_0^1 f(x)\varphi_p(x)dx - f(0)\int_0^1 \varphi_p(x)dx = \int_0^1 \Big(f(x) - f(0)\Big)\varphi_p(x)dx.$$

The function $f(x)$ is continuous, so

1) $\exists M > 0$ such that

$$\max_{x \in [0,1]} |f(x)| \leq M,$$

2) $\forall \varepsilon > 0 \; \exists \; \Delta > 0$ such that $|x| < \Delta$ implies $|f(x) - f(0)| < \frac{\varepsilon}{2}$. Then

$$\int_0^1 \Big(f(x) - f(0)\Big)\varphi_p(x)dx =$$

$$= \int_0^{\Delta} \Big(f(x) - f(0)\Big)\varphi_p(x)dx + \int_{\Delta}^1 \Big(f(x) - f(0)\Big)\varphi_p(x)dx \leq$$

$$\leq \frac{\varepsilon}{2}\int_0^{\Delta} \varphi_p(x)dx + 2M\int_{\Delta}^1 \varphi_p(x)dx \leq \frac{\varepsilon}{2} + 2M\Big(\frac{e^p - e^{p\Delta}}{e^p - 1}\Big). \qquad (3)$$

Fixed Δ, it is possible to specify $P \leq 0$, such that $|\frac{e^p - e^{p\Delta}}{e^p - 1}| < \frac{\varepsilon}{4M} \; \forall p < P$. Extending the inequality (3), we obtain that for arbitrary $\varepsilon >$, such $P < 0$ can be chosen that $\forall p < P$

$$\int_0^1 \Big(f(x) - f(0)\Big)\varphi_p(x)dx < \varepsilon.$$

Property $E1$ is stated. Similar arguments yields to the property $E3$.

To establish $E2$, we note that the difference $\chi_{[0,1)}(x) - \varphi_p(x)$ outside the interval $[0, 1)$ is equal to zero; for $x \in [0, 1)$, applying the mean value theorem and removing the uncertainty by L'Hopital's rule, we obtain

$$\chi_{[0,1)}(x) - \varphi_p(x) = \frac{e^p - 1 - pe^{px}}{e^p - 1} = \frac{p(e^{p\xi} - e^{px})}{e^p - 1} = \frac{p^2 e^{p\xi^*}}{e^p - 1},$$

here $\xi, \xi^* \in [0, 1]$ and

$$\lim_{p\to\pm 0} (\chi_{[0,1)}(x) - \varphi_p(x), f(x)) = 0.$$

By properties E1 and E2 we have the convergence properties:

$$\lim_{p\to 0} U_{m+1,p}(x) = Q_{m+1}(x),$$

$$\lim_{p\to -\infty} U_{m+1,p}(x) = Q_m(x).$$

Example. Figures 1 and 2 illustrate the convergence properties of $U_{m,p}$ for $m = 2$

$$U_{2,p}(x) = \begin{cases} 0, & x \leq 0, \\ \frac{e^{px} - 1}{e^p - 1}, & 0 \leq x \leq 1, \\ \frac{e^p - e^{p(x-1)}}{e^p - 1}, & 1 \leq x \leq 2 \\ 0, & 2 \leq x. \end{cases}$$

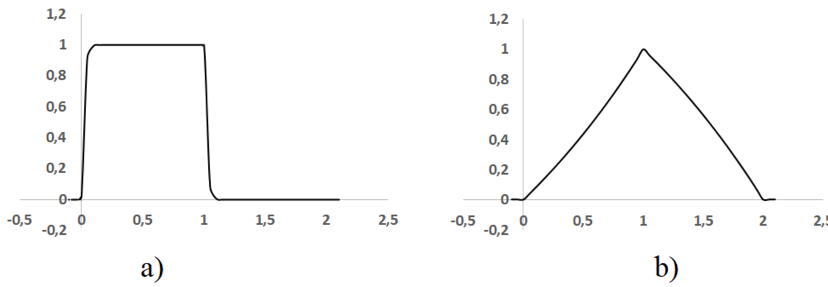

Figure 1: E-splines $U_{2,p}$, a) $p = -500$, b) $p = 0,5$

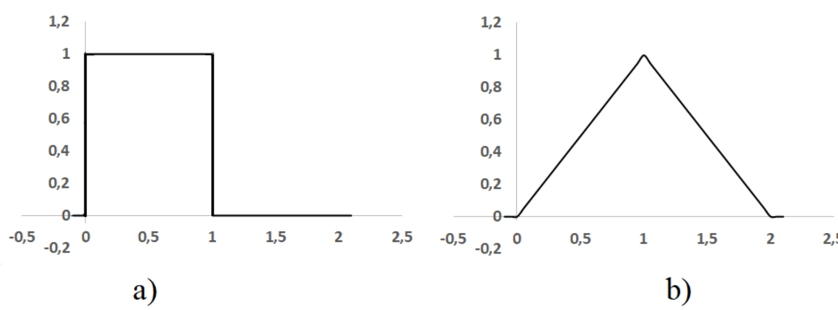

Figure 2: B–splines Q_m, a) $m = 1$, b) $m = 2$

As well as $Q_m(x)$, $U_{m,p}(x)$ has the compact support (supp $U_{m,p}(x) = [0, m))$ and are continuous for $m > 1$ ($U_{m,p}(x) \in C^{m-1}(R)$) that makes the family $\{U_{m,p}(\cdot - k), k \in Z\}$ very attractive for practical purpose provided that we have an additional information on stability of this family.

3 Stability condition and Riesz bounds

Definition 1. A family of integer shifts $\{\varphi(x - n), n \in Z\}$ generated by a function $\varphi(x)$ from a Hilbert space H is said to be stable, if there exist two constants $0 < A, B < \infty$ such that for any sequence $\{c_n\}_{n \in Z} \in l_2$ the following inequalities hold:

$$A \sum_{n=-\infty}^{\infty} |c_n|^2 \leq || \sum_{n=-\infty}^{\infty} c_n \varphi(x - n)||_H^2 \leq B \sum_{n=-\infty}^{\infty} |c_n|^2. \qquad ()$$

Definition 2. If the linear span of the set $\{\varphi(\cdot - n), n \in Z\}$ is stable and, additionally, dense in H, then they say that $\{\varphi(\cdot - n), n \in Z\}$ form an unconditional or Riesz basis in H.

Remark. For $A = B = 1$ the Riesz basis turns into an orthonormal basis. For example, $\{\chi_{[0,1)}(\cdot - n), n \in Z\}$ forms the orthonormal basis in the space of functions which are piece-wise constant on the intervals of the form $[k, k + 1)$.

The constants A and B in the general formula characterize, in a sense, the "redundancy" of the Riesz basis. They are also called the Riesz constants or the Riesz lower and upper bounds.

The aim of the work is to establish the stability property for the system of integer shifts of E-splines for arbitrary $m \in N$ and $p \in (-\infty, \infty)$ and determine its Riesz bounds.

It should be noted that in certain cases determination of Riesz bounds becomes simpler if we pass from the function $\varphi(x)$ to its Fourier transform $\hat{\varphi}(\xi)$ and use the theorem which establish equivalence of two statements [3]:
– the set $\{\phi(\cdot - k), k \in Z\}$ satisfies the Riesz condition with constants $2\pi A$ and $2\pi B$;
– the Fourier transform $\hat{\phi}(\xi)$ satisfies the inequalities

$$A \leq \sum_{k \in Z} |\hat{\phi}((\xi + 2\pi k)|^2 \leq B,$$

By this, it suffices to determine the lower and upper estimates, uniform in ξ, for the series

$$\sum_{k\in Z} |\widehat{U}_{m,p}(\xi + 2\pi k)|^2, \tag{5}$$

assuming that the parameter p and the order of the exponential spline m are fixed.

In a view of

$$\widehat{Q}_1(\xi) = \frac{1}{2\pi} e^{-i\xi/2} \frac{\sin(\xi/2)}{\xi/2},$$

we have

$$\widehat{Q}_m(\xi) = \frac{1}{\sqrt{2\pi}} \Big(e^{-i\xi/2} \frac{\sin(\xi/2)}{\xi/2} \Big)^m.$$

According to (2)

$$\widehat{U}_{m+1,p}(\xi) = \sqrt{2\pi}\widehat{Q}_m(\xi) \cdot \widehat{\varphi}_p(\xi),$$

the Fourier transform for the function $\varphi_p(x)$ is

$$\widehat{\varphi}_p(\xi) = \frac{1}{\sqrt{2\pi}} \int_0^1 e^{-ix\xi} \frac{pe^{px}}{e^p - 1} dx = \frac{1}{\sqrt{2\pi}} \frac{p}{e^p - 1} \frac{e^{p-i\xi} - 1}{p - i\xi},$$

and then

$$\widehat{U}_{m+1,p}(\xi) = \frac{1}{\sqrt{2\pi}} \Big(e^{-i\frac{\xi}{2}} \frac{\sin \xi/2}{\xi/2} \Big)^m \frac{p}{e^p - 1} \frac{e^{p-i\xi} - 1}{p - i\xi}.$$

Hence

$$|\widehat{U}_{m+1,p}(\xi)|^2 = \frac{1}{2\pi} \cdot \Big(\frac{\sin \xi/2}{\xi/2} \Big)^m \cdot \frac{p^2(e^{2p} - 2e^p \cos \xi + 1)}{(p^2 + \xi^2)(e^p - 1)^2}. \tag{6}$$

For the further consideration, we rewrite the last factor in (6) in the following form:

$$\frac{p^2}{p^2 + \xi^2} \cdot \frac{e^{2p} - 2e^p \cos \xi + 1}{(e^p - 1)^2} = \frac{(p/2)^2}{(p/2)^2 + (\xi/2)^2} \cdot \Big(1 + \frac{\sin^2(\xi/2)}{\sinh^2(p/2)} \Big).$$

Then (6) takes the form

$$|\widehat{U}_{m+1,p}(\xi)|^2 = \frac{1}{2\pi} \frac{(p/2)^2}{(p/2)^2 + (\xi/2)^2} \Big(1 + \frac{\sin^2(\xi/2)}{\sinh^2(p/2)} \Big) \Big(\frac{\sin \xi/2}{\xi/2} \Big)^m$$

and, by this, we have the series

$$\sum_{k=-\infty}^{\infty} |\widehat{U}_{m+1,p}(\xi + 2\pi k)|^2 =$$

$$= \frac{1}{2\pi} \sum_{k=-\infty}^{\infty} \frac{(p/2)^2}{(p/2)^2 + (\xi/2 + \pi k)^2} \Big(1 + \frac{\sin^2(\xi/2 + \pi k)}{\sinh^2(p/2)}\Big) \Big(\frac{\sin(\xi/2 + \pi k)}{\xi/2 + \pi k}\Big)^{2m}. \tag{7}$$

For the sake of convenience we denote

$$a_k(\xi) = \frac{(p/2)^2}{(p/2)^2 + (\xi/2 + \pi k)^2} \Big(1 + \frac{\sin^2(\xi/2 + \pi k)}{\sinh^2(p/2)}\Big);$$

$$u_k(\xi) = \frac{1}{2\pi} \Big(\frac{\sin(\xi/2 + \pi k)}{\xi/2 + \pi k}\Big)^{2m}. \tag{8}$$

Thus, we have the functional series

$$\sum_{k=-\infty}^{\infty} |\widehat{U}_{m+1,p}(\xi + 2\pi k)|^2 = \sum_{k=-\infty}^{\infty} a_k(\xi) u_k(\xi),$$

which, due to its 2π–periodicity, can be considered only on the interval $[0, 2\pi]$.

Let us prove the following theorem:

Theorem. The function series (7) converges uniformly on the interval $[0, 2\pi]$ for any $p \in (-\infty, \infty)$ and any positive integer m to a continuous function $\mathcal{N}_{m+1,p}(\xi)$.

Proof. We fix a real $p \neq 0$ and a non negative integer m and present the series (7) as a sum of three terms

$$\sum_{k=-\infty}^{\infty} a_k(\xi) u_k(\xi) = \sum_{k=-\infty}^{-1} a_k(\xi) u_k(\xi) + \sum_{k=1}^{\infty} a_k(\xi) u_k(\xi) +$$

$$+ \frac{1}{2\pi} \frac{(p/2)^2}{(p/2)^2 + (\xi/2)^2} \Big(1 + \frac{\sin^2(\xi/2)}{\sinh^2(p/2)}\Big) \Big(\frac{\sin \xi/2}{\xi/2}\Big)^{2m}.$$

We prove convergence of the series in the first and second terms using the Dirichlet's test: a series $\sum_{k=1}^{\infty} a_k(\xi) u_k(\xi)$ converges uniformly on a set Ξ if $a_k(\xi)$ and $u_k(\xi)$ satisfy the conditions

1. the sequence $\{a_k(\xi)\}$ is monotone for any ξ and converges uniformly to 0 on Ξ as $k \to \infty$;
2. partial sums $S_n(\xi) = \sum_{k=1}^{\infty} u_k(\xi)$ are uniformly bounded on Ξ.

In our case $\Xi = [0, 2\pi]$ and $a_k(\xi)$ and $u_k(\xi)$ in $\sum_{k=1}^{\infty} a_k(\xi) u_k(\xi)$ are given by (8).

1. As

$$a_k(\xi) = \frac{(p/2)^2}{(p/2)^2 + (\xi/2 + \pi k)^2}\Big(1 + \frac{\sin^2(\xi/2 + \pi k)}{\sinh^2(p/2)}\Big) =$$

$$= \frac{(p/2)^2}{(p/2)^2 + (\xi/2 + \pi k)^2}\Big(1 + \frac{\sin^2(\xi/2)}{\sinh^2(p/2)}\Big),$$

obviously, for any fixed $\xi \in \Xi$ the sequence $\{a_k(\xi)\}$ decreases monotonically as $k \to \infty$. Moreover,

$$a_k(\xi) \leq \frac{(p/2)^2}{(p/2)^2 + (\pi k)^2}\Big(1 + \frac{1}{\sinh^2(p/2)}\Big).$$

Hence for any given $\varepsilon > 0$ we can specify $K > 1$ such that $a_k(\xi) < \varepsilon$ for any $\xi \in \Xi$ as soon as $k > K$. The uniform convergence to zero of the sequence $\{a_k(\xi)\}$ is shown. Thus, condition 1 is fulfilled.

2. Let us show that the partial sums $S_n(\xi) = \sum_{k=1}^{n} u_k(\xi)$ are uniformly bounded.

In [] it was shown among others that

$$\sum_{k=-\infty}^{\infty} \frac{1}{2\pi}\Big(\frac{\sin(\xi/2 + \pi k)}{\xi/2 + \pi k}\Big)^{2m} = \frac{1}{2\pi}\frac{C_{m-1}(\cos^2(\xi/2))}{(2m-1)!)},$$

for every fixed $m \geq 1$. Trigonometric polynomials $C_m(\cos^2 \xi)$ were shown to be continuous, positive, symmetric with respect to $\xi = 0$, π–periodic functions which are defined on R. They reach their maximum values at the points $\xi = k\pi$, and their minimum values at the points $\xi = \frac{(2k+1)\pi}{2}$, $k \in Z$. At the same time

$$\frac{C_m(\cos^2(k\pi))}{(2m+1)!} = 1, \tag{3}$$

and the numerical sequence $\{C_m(\cos^2(\pi/2))\}$ tends monotonically to 0 as $m \to \infty$.

Thus, partial sums $S_n(\xi)$ are uniformly bounded on $[0, 2\pi]$:

$$S_n(\xi) = \sum_{k=1}^{n} \frac{1}{2\pi}\Big(\frac{\sin(\xi/2 + \pi k)}{\xi/2 + \pi k}\Big)^{2m} < \sum_{k=-\infty}^{\infty} \frac{1}{2\pi}\Big(\frac{\sin(\xi/2 + \pi k)}{\xi/2 + \pi k}\Big)^{2m} =$$

$$= \frac{1}{2\pi}\frac{C_{m-1}(\cos^2(\xi/2))}{(2m-1)!)} \leq \frac{1}{2\pi}.$$

It is evident that $a_k(\xi) = a_{-k}(-\xi)$, $u_k(\xi) = u_{-k}(-\xi)$, and the first term and the second term in (5) are related by the formula

$$\sum_{k=-\infty}^{-1} a_k(\xi) u_k \xi = \sum_{k=1}^{\infty} a_k(-\xi) u_k(-\xi),$$

whence convergence of the series in the first term is also shown.

Finally, the function

$$\mu(p,\xi) = \frac{1}{2\pi} \frac{(p/2)^2}{(p/2)^2 + (\xi/2)^2} \Big(1 + \frac{\sin^2(\xi/2)}{\sinh^2(p/2)}\Big) \Big(\frac{\sin \xi/2}{\xi/2}\Big)^{2m}.$$

is continuous and defined at all points $\xi \in \Xi$, including $\xi = 0$:

$$\lim_{\xi \to 0} \Big(\frac{\sin \xi/2}{\xi/2}\Big)^{2m} = 1 \quad \Rightarrow \quad \lim_{\xi \to 0} \mu(p,\xi) = \frac{1}{2\pi}.$$

If $p = 0$, then in accordance with item $E2$ and the properties of the Fourier transform

$$\widehat{U}_{m,0}(\xi) = \widehat{Q}_m(\xi),$$

the convergence of series (5) for $p = 0$ is proven.

Summarizing all the above, we conclude that the series (7) for fixed m and p converges uniformly to a continuous 2π-periodic function $\mathcal{N}_{m+1,p}(\xi)$. The continuous function $\mathcal{N}_{m,p}(\xi)$ on the interval $[0, 2\pi]$ has a minimum and a maximum, which we can take as the lower and upper Riesz bounds:

$$A_{m,p} = \min_{\xi \in [0,2\pi]} \mathcal{N}_{m,p}(\xi);$$

$$B_{m,p} = \max_{\xi \in [0,2\pi]} \mathcal{N}_{m,p}(\xi).$$

Thus, the existence of Riesz boundaries is proven. Next, we derive their values.

Lemma. For any $p \in (-\infty, \infty)$ and $\xi \in (-\infty, \infty)$ the following estimates hold:

$$\frac{\sin^2 \xi}{\xi^2} \le \frac{p^2}{p^2 + \xi^2} \cdot \frac{\sinh^2 p + \sin^2 \xi}{\sinh^2 p} \le 1.$$

Proof.

Due to inequalities $\sin^2 \xi \le \xi^2$ and $p^2 \le \sinh^2 p$ for any $\xi, p \in (-\infty, \infty)$, we have

$$0 < \xi^2 \cdot (\sinh^2 p - p^2) \le \sinh^2 p \cdot \xi^2 - \sin^2 \xi \cdot p^2,$$

or

$$p^2 \sinh^2 p + p^2 \sin^2 \xi \leq p^2 \sinh^2 p + \xi^2 \sinh^2 p,$$

and

$$\frac{p^2}{p^2+\xi^2}\Big(1+\frac{\sin^2 \xi}{\sinh^2 p}\Big) \leq 1.$$

Further, the function $\frac{1}{p^2} - \frac{1}{\sinh^2 p}$ reaches its maximum value $1/3$ at the point $p = 0$; the function $\frac{1}{\sin^2 \xi} - \frac{1}{\xi^2}$ reaches its minimum value at the point $\xi = 0$. Then for any pair $p, \xi \in (-\infty, \infty)$

$$\frac{1}{p^2} - \frac{1}{\sinh^2 p} \leq \frac{1}{\sin^2 \xi} - \frac{1}{\xi^2},$$

and the equality holds for the pair $(0, 0)$. Hence, for any $p \in (-\infty, \infty)$ and $\xi \in [0, 2\pi]$

$$\frac{\sin^2 \xi}{\xi^2} \leq \frac{p^2}{p^2+\xi^2} \cdot \frac{\sinh^2 p + \sin^2 \xi}{\sinh^2 p}.$$

This allows us to assert that for a fixed $m \in N$ and any $p, \xi \in (-\infty, \infty)$

$$\sum_{k=-\infty}^{\infty} |\widehat{Q}_{m+1}(\xi+2\pi k)|^2 \leq \sum_{k=-\infty}^{\infty} |\widehat{U}_{m+1,p}(\xi+2\pi k)|^2 \leq \sum_{k=-\infty}^{\infty} |\widehat{Q}_{m}(\xi+2\pi k)|^2$$

Thus, for the set of integer shifts of an exponential spline of any order the stability condition with constant $2\pi A_{m,p}$ and $2\pi B_{m,p}$ is fulfilled. The constant $B_{m,p}$ does not depend on the values of m and p and equals 1. $A_{m,p}$ satisfies the inequality

$$0 < \frac{C_{m-1}(\cos^2(\pi/2))}{2\pi(2m-1)!} \leq A_{m,p}$$

for any $m \in N$, $p \in (-\infty, \infty)$; equality holds at $p = 0$.

It is easy to see that $\mathcal{N}_{m+1,p}(\xi) = \mathcal{N}_{m+1,-p}(\xi)$. Therefore $A_{m,p} = A_{m,-p}$, $B_{m,p} = B_{m,-p}$.

4 Conclusion

Thus, the stability condition for the family of integer shifts $\{U_{m,p}(\cdot - k), k \in Z\}$ is proven; Riesz bounds are found.

As follows from the above, in a similar way one can obtain Riesz bases with required properties by considering integer shifts of a function which is convolution of $Q_m(x)$ and an appropriate function $\varphi(x)$. Here the order m provides the smoothness property, whereas the function $\varphi(x)$ guarantees such properties, for example, as compactness of the support or symmetry and affects the value of Riesz bounds.

5 References

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