CLOSED SUBSETS AND THE LOCAL STRUCTURE OF MONOTONICALLY NORMAL COMPACTA

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ABSTRACT. We prove that in a monotonically normal compactum the character $\tau = \chi(a, X)$ of a point a is determined by a compactum K which is either the ordinal space $[0, \tau]$ or its quotient image obtained from $[0, \tau]$ by identifying a closed unbounded subset of $[0, \tau]$ to a single point. As corollaries, we deduce the fact that every monotonically normal compactum with a small diagonal is metrizable and the existence of a local disjoint well-ordered π-base of the largest possible cardinality (provided each element of the π-base intersects K) $\chi(a, X)$. We also note that a G_δ-homogeneous countably tight monotonically normal compactum is first-countable and homogeneous.

Keywords: Monotonically normal compactum, small diagonal, metrizability, well-ordered π-base, G_δ-homogeneity.

1. INTRODUCTION

It is well-known that if L is a linearly ordered compact topological space and $a \in L$ does not have an immediate predecessor, then L contains a copy of the space of ordinals $[0, \tau]$ for some regular cardinal τ with this inclusion being order-preserving and τ coinciding with a. This τ is the "left-hand character" of a in L. Similarly, if $a \in L$ does not have an immediate successor, then L contains a copy of the space of ordinals $[0, \mu]$ for some regular cardinal μ with this inclusion being order-reversing and μ coinciding with a. This μ is the "right-hand character" of a in L. Consequently, $\chi(a, L) = \max(\tau, \mu)$ and $\chi(L) = \sup\{\tau : [0, \tau] \subset X\}$. So in a compact LOTS it is subsets of the form $[0, \tau]$ that are responsible for the character

at a given point and globally. Moreover, the whole local structure in a compact LOTS is determined by subsets of this form.

The class of the monotonically normal compacta is the closest generalization of compact LOTS as, according to the classical result of M.E. Rudin [Rud01], a compactum is monotonically normal iff it is a continuous image of a compact LOTS. In this note, we study which closed subsets K of monotonically normal compacta are responsible for their character. It turned out, these are not just ordinal spaces $[0, \tau]$, but also quotient spaces obtained from $[0, \tau]$ by identifying a closed unbounded subset of $[0, \tau]$ to a single point (see Theorem 2). As a corollary, we deduce that the character of a monotonically normal compactum X equals to $\sup\{\tau : [0, \tau] \subset X\}$ or to $\sup\{\tau : A(\tau) \subset X\}$ (Corollary 1), where $A(\tau)$ is the Alexandroff compactification (the one-point compactification) of a discrete set of cardinality τ. Yet another corollary is that a monotonically normal compactum of an uncountable character contains a compactum of weight (and cardinality) exactly ω_1, from which we prove that a monotonically normal compactum with a small diagonal is metrizable. By expanding isolated points of K we obtain a disjoint[1] well-ordered local π-base of the largest possible cardinality $\chi(a, X)$ (provided each element of the π-base intersects K, see Theorem 4). This result complements earlier theorem of Williams and Zhou who proved [WiZ98] the existence of a π-base of cardinality $cf(\chi(x, X))$. Finally, we consider a recent concept of G_δ-homogeneity, introduced by A.V. Arhangel'skii [Arh19], and show that each countably tight G_δ-homogeneous monotonically normal compactum is first-countable and homogeneous.

2. The local structure theorems for monotonically normal compacta

Lemma 1. *If β is a regular uncountable cardinal and $f : [0, \beta] \longrightarrow X$ is a continuous mapping into a topological space X such that $f^{-1}(f(\beta)) = \{\beta\}$, then there is a club[2] $B \subset [0, \beta)$ such that the restriction $f_{|B \cup \{\beta\}}$ is a homeomorphism onto a closed subset of X. So X contains a closed copy of $[0, \beta]$.*

Доказательство. For every ordinal $\alpha < \beta$, denote $\max\left(f^{-1}(f([0, \alpha]))\right)$ by $b(\alpha)$. Then $b(\alpha) < \beta$ because $f^{-1}(f([0, \alpha]))$ is a closed subset of $[0, \beta]$ (since mapping f is closed) which does not contain β. Taking into account the regularity of β, it is straightforward to construct by transfinite recursion a cofinal set $B' \subset [0, \beta)$ (for example, containing only successor ordinals) such that

$$\gamma > \sup\{b(\gamma') : \gamma' \in B' \text{ and } \gamma' < \gamma\} \tag{$*$}$$

for every $\gamma \in B'$.

It then follows from $(*)$ that the restriction $f_{|B}$ is one-to-one[3], where $B = \overline{B'}$ is the closure of B' in $[0, \beta)$. Hence the restriction $f_{|B \cup \{\beta\}}$ is one-to-one too. Then $f_{|B \cup \{\beta\}}$ is a desired homeomorphism onto a closed subset of X as $B \cup \{\beta\}$ is a closed subset of $[0, \beta]$. $\qquad\square$

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[1] By a disjoint family we mean a family with pairwise disjoint elements

[2] Which stands for a "closed unbounded set".

[3] Indeed, let $\gamma_1, \gamma_2 \in B$ and $\gamma_1 < \gamma_2$. Then $\gamma_2 \in B'$ or γ_2 is the limit of an infinite subset of B' (or both). Either way there is $\gamma_3 \in B'$ such that $\gamma_1 < \gamma_3 \leqslant \gamma_2$. Then $\gamma_3 > \alpha$ by $(*)$, where $\alpha = \sup\{b(\gamma'') : \gamma'' \in B' \text{ and } \gamma'' \leqslant \gamma_1\}$, so $\gamma_2 > \alpha$ too. $b(\gamma_1)$ belongs to $[0, \alpha]$ because γ_1 is an element or a limit point of the set $\{\gamma'' \in B' : \gamma'' \leqslant \gamma_1\}$, hence $\gamma_2 > b(\gamma_1)$. Thus $f(\gamma_1) \neq f(\gamma_2)$ and $f_{|B}$ is one-to-one.

Lemma 2. *Let $f : L \longrightarrow X$ be an irreducible continuous surjection of a linearly ordered compactum L. Suppose $a \in X$ and the set $\mathcal{U}$ of convex (and open) components of $L \setminus f^{-1}(a)$ has an infinite cardinality μ. Then there is a subset $A \subset L \setminus f^{-1}(a)$ such that each member of $\mathcal{U}$ contains exactly one element of A, the restriction $f_{|A}$ is one-to-one, and $\{a\} \cup f(A)$ is a closed copy of $A(\mu)$ which contains a as the only non-isolated point.*

Доказательство. Denote $f^{-1}(a)$ by D and enumerate the set $\mathcal{U}$ of convex components of $L \setminus D$ by $\mathcal{U} = \{U_\alpha : \alpha < \mu\}$. For every $\alpha < \mu$ let $V_\alpha = X \setminus f(L \setminus U_\alpha)$ and $W_\alpha = f^{-1}(V_\alpha)$. Then each set V_α is not empty due to irreducibility of f, and it is open as the complement to a closed subset $f(L \setminus U_\alpha)$ of X. If $\alpha \neq \beta$, then

$$V_\alpha \cap V_\beta = (X \setminus f(L \setminus U_\alpha)) \cap (X \setminus f(L \setminus U_\beta)) = X \setminus (f(L \setminus U_\alpha) \cup f(L \setminus U_\beta)) =$$
$$= X \setminus f((L \setminus U_\alpha) \cup (L \setminus U_\beta)) = X \setminus f(L) = \emptyset.$$

Therefore $\{V_\alpha : \alpha < \mu\}$ and $\{W_\alpha : \alpha < \mu\}$ are families of pairwise disjoint open subsets of X and L respectively. For every $\alpha < \mu$ pick $x_\alpha \in W_\alpha$ and denote $\{x_\alpha : \alpha < \mu\}$ by A. Then the restriction $f_{|A}$ is one-to-one since $V_\alpha \cap V_\beta = \emptyset$ whenever $\alpha \neq \beta$. The set $f(A)$ is discrete in itself since V_α is an open neighborhood of $f(x_\alpha)$ in X. If U is an open neighborhood of a in X, then $f^{-1}(U)$ is an open neighborhood of D in L, so $f^{-1}(U)$ contains all but finitely many elements of $\mathcal{U}$ (being a locally finite family of subsets of a compactum L, $\{U_\alpha \cap f^{-1}(U) : \alpha < \mu\}$ is finite). Thus every open neighborhood of a in X contains all but finitely many elements of $f(A)$, so $\{a\} \cup f(A)$ is homeomorphic to $A(\mu)$ as required. Lemma 2 is proved. $\square$

Lemma 3. *Let $f : L \longrightarrow X$ be a continuous surjection of a linearly ordered compactum L. Let $a \in X$, $\mathcal{U}$ be the set of convex (open) components of $L \setminus f^{-1}(a)$, and $U = \cup \mathcal{U}$. Then*

   a. *If $Z \subset U$ and Z is not covered by any finite subcollection of $\mathcal{U}$, then there exists a sequence $\{z_n \in Z : n \in \omega\}$ such that distinct z_n belong do distinct elements of $\mathcal{U}$ and $\{z_n\}$ converges to some point of $L \setminus U$.*
   b. *If F is a closed subset of L and $F \subset U$, then F is covered by a finite subcollection of $\mathcal{U}$.*
   c. *If F is a closed subset of L and $F \subset U$, then the set $\{V \in \mathcal{U} : f(F) \cap f(V) \neq \emptyset\}$ is finite.*
   d. *For each $U' \in \mathcal{U}$ the set $\{V \in \mathcal{U} : f(V) \cap f(U') \neq \emptyset\}$ is countable.*

Доказательство. **a.** Assume the contrary, then there exist a countable infinite subfamily $\mathcal{U}' \subset \mathcal{U}$ and a subset $Z' \subset Z$ such that $Z' \subset \cup \mathcal{U}'$ and each element of $\mathcal{U}'$ contains exactly one element of Z'. Then the closure $\overline{Z'}^L$ is a separable LOTS, hence it is first-countable. Since $\overline{Z'}^L$ is compact, it contains an accumulation point z' of Z'. Since $\overline{Z'}^L$ is also first-countable, we can select a sequence $\{z_n \in Z' : n \in \omega\}$ which converges to z'. This is a required sequence as necessarily $z' \in L \setminus U$.
   b. follows directly from **a.**
   c. Assume the contrary, then there exist a countable infinite subfamily $\mathcal{U}' \subset \mathcal{U}$ and a subset $Z' \subset Z$ such that $Z' \subset \cup \mathcal{U}'$, $f(Z') \subset f(F)$, and each element of $\mathcal{U}'$ contains exactly one element of Z'. By item **a**, Z' contains a sequence converging to some $z' \in L \setminus U$. Then $z' \in f^{-1}(f(F))$, hence $f^{-1}(f(F)) \cap (L \setminus U) \neq \emptyset$. This contradiction to the fact that $F \subset U = L \setminus f^{-1}(a)$ completes the proof. $\square$

The following is the main theorem of this paper:

Theorem 1. *Let $f : L \longrightarrow X$ be an irreducible continuous surjection of a linearly ordered compactum L. If $a \in X$ and $\chi(a, X) = \tau > \omega$, then exactly one of the following three conditions is satisfied:*

(1) There is a discrete in itself subspace $A \subset L$ of cardinality τ such that $f_{|\overline{A}}$ is a quotient mapping which identifies the set $\overline{A} \setminus A$ of accumulation points of A to a single point a. Also $f\left(\overline{A}\right) \approx A(\tau)$.

(2) τ is a regular cardinal, a is not a unique accumulation point of any subset of $X \setminus \{a\}$ of cardinality τ, and there is $[0, \tau] \subset L$ such that $f(\tau) = a$ and the restriction $f_{|[0, \tau]}$ is one-to-one.

(3) τ is a singular cardinal and a is not a unique accumulation point of any subset of $X \setminus \{a\}$ of cardinality τ. Furthermore, for $\nu = cf(\tau)$ there exist an increasing transfinite sequence $\{\tau_\alpha < \tau : \alpha < \nu\}$ of regular uncountable cardinals which is cofinal in τ, a family of pairwise disjoint intervals $\{[y_\alpha, y'_\alpha) \subset L : \alpha < \nu\}$ (or $\{(y'_\alpha, y_\alpha] \subset L : \alpha < \nu\}$), and a family of order-reversing homeomorphisms ϕ_α of $[0, \tau_\alpha]$ onto a closed subset of $[y_\alpha, y'_\alpha)$ (respectively, order-preserving homeomorphisms ϕ_α of $[0, \tau_\alpha]$ onto a closed subset of $(y'_\alpha, y_\alpha]$) with $\phi_\alpha(\tau_\alpha) = y_\alpha$, and such that for $T = \cup\{\phi_\alpha([0, \tau_\alpha)) : \alpha < \nu\}$, $f_{|\overline{T}}$ is a quotient mapping which identifies the set $\overline{T} \setminus T$ to a single point a.

Доказательство. Denote $f^{-1}(a)$ by D and enumerate the set $\mathcal{U}$ of convex (and open) components of $L \setminus D$ by $\mathcal{U} = \{U_\alpha : \alpha < \mu\}$. Note[4] that $\mu \leqslant \tau$.

If $\mu = \tau$, apply Lemma 2. Use D and A as defined in Lemma 2, then $f\left(\overline{A} \setminus A\right) = f(D) = \{a\}$ as obviously $\overline{A} \setminus A \subset D$; also $a \notin f(A)$. This means that condition (1) is satisfied.

Assume from now on that $\mu < \tau$. Note that $\chi(D, L) = \chi(a, X) = \tau$ since f is a closed mapping. Thus D is of type G_τ in L (i.e., the intersection of τ open subsets of L).

For every subset M of $X \setminus \{a\}$ of cardinality μ^+ there is a subset M' of $f^{-1}(M)$ of the same cardinality such that $f_{|M'}$ is one-to one. Then there exist $U'' \in \mathcal{U}$ and $M'' = M' \cap U''$ with $|M''| = \mu^+$. Since M'' does have some accumulation points in U'', M has some accumulation points in $X \setminus \{a\}$. Therefore, a is not a unique accumulation point of any subset M of $X \setminus \{a\}$ of cardinality μ^+, let alone of cardinality τ.

Case I. τ is a regular cardinal.

Since D is the intersection of μ closed subsets $L \setminus U_\alpha$, at least one of them, say $L \setminus U_{\alpha'}$, will be of type G_τ in this case. Hence at least one of points y', y'', where $U_{\alpha'} = (y', y'')$, is of character τ in the segment $[y', y'']$. If $\chi(y'', [y', y'']) = \tau$, then the space of ordinals $[0, \tau]$ can be embedded in $(y', y'']$ with the preservation of order so that τ coincides with y''. If $\chi(y', [y', y'']) = \tau$, then the space of ordinals $[0, \tau]$ can be embedded in $[y', y'')$ in the reversed order so that τ coincides with y'. Either way, Lemma 1 implies that condition (2) is satisfied.

Case II. τ is a singular cardinal.

We denote the cofinality of τ by ν; let $\{\tau_\alpha < \nu : \alpha < \nu\}$ be an increasing transfinite sequence of regular uncountable cardinals which is cofinal in τ. If there

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[4]$X \setminus \{a\}$ is the union of a family $\{F_\alpha \subset X \setminus \{a\} : \alpha < \tau\}$ of closed subsets of X because $\chi(a, X) = \tau$. Since $f^{-1}(F_\alpha)$ is covered by a finite subcollection of $\mathcal{U}$ for every $\alpha < \tau$ by Lemma 3b and $L \setminus f^{-1}(a) = \cup \mathcal{U} = \cup \{f^{-1}(F_\alpha) : \alpha < \tau\}$, we have that $\mu = |\mathcal{U}| \leqslant \tau$.

is a cardinal $\tau' < \tau$ such that each $U_\alpha \in \mathcal{U}$ is a union of $\leqslant \tau'$ closed subsets of L, then $L \setminus D$ is a union of no more than $\max(\tau', \mu) < \tau$ closed subsets of L, and this contradicts the fact that D is of type G_τ in L. Hence there is a subfamily $\mathcal{U}' = \{U'_\alpha : \alpha < \nu\} \subset \mathcal{U}$ of cardinality ν and a set $\{y_\alpha : \alpha < \nu\}$ of the corresponding endpoints (each y_α is the left endpoint of U'_α or each y_α is the right endpoint of U'_α) such that the character of y_α in $\{y_\alpha\} \cup U'_\alpha$ is greater than or equal to τ_α for each $\alpha < \nu$. Without loss of generality we can assume that $\chi(y_\alpha, \{y_\alpha\} \cup U'_\alpha) = \tau_\alpha$ for each $\alpha < \nu$ (it is known that the character of any point of a LOTS is a regular cardinal). Also, we can assume that y_α is the left endpoint of $\{y_\alpha\} \cup U'_\alpha$ for every $\alpha < \nu$. Let y'_α denote the right endpoint of U'_α, then $\{y_\alpha\} \cup U'_\alpha = [y_\alpha, y'_\alpha)$.

Due to $\chi(y_\alpha, [y_\alpha, y'_\alpha)) = \tau_\alpha$, each $[y_\alpha, y'_\alpha)$ contains a closed copy of the space of ordinals $[0, \tau_\alpha]$ in which τ_α corresponds to y_α. In other words, there is an order-reversing homeomorphism ϕ_α of $[0, \tau_\alpha]$ onto a closed subset of $[y_\alpha, y'_\alpha) \subset L$ with $\phi_\alpha(\tau_\alpha) = y_\alpha$.

Denote the composition mapping $f(\phi_\alpha(\cdot))$ by ψ_α. Since $\phi_\alpha(\tau_\alpha) = y_\alpha \in D$ and $\phi_\alpha((0, \tau_\alpha)) \subset U'_\alpha$, which is disjoint from D, we have that ψ_α is a continuous mapping of $[0, \tau_\alpha]$ into X such that $\psi_\alpha^{-1}(\psi_\alpha(\tau_\alpha)) = \{\tau_\alpha\}$. Therefore, by Lemma 1, there is a club $B_\alpha \subset [0, \tau_\alpha)$ such that the restriction $\psi_{\alpha | B_\alpha \cup \{\tau_\alpha\}}$ is one-to-one. Without loss of generality we can assume that $B_\alpha = [0, \tau_\alpha)$. From now on we denote $\phi_\alpha([0, \tau_\alpha))$ by T_α and $\phi_\alpha([0, \tau_\alpha])$ by $\overline{T_\alpha}$. The latter reasoning means that

$$f_{|\overline{T_\alpha}} \text{ is one-to-one} \tag{$**$}$$

for every $\alpha < \nu$. Fix an ordinal $\beta < \nu$. It follows from Lemma 3c and uncountability of β that the index set

$$\{\alpha < \nu : \alpha \neq \beta \text{ and } f(T_\beta) \cap f(U'_\alpha) \neq \emptyset\}$$

is finite. Hence for every $\beta < \nu$ there is an ordinal $i_\beta < \tau_\beta$ such that

$$\{\alpha < \nu : \alpha \neq \beta \text{ and } f(\phi_\beta([i_\beta, \tau_\beta])) \cap f(U'_\alpha) \neq \emptyset\} = \emptyset.$$

Since both $[i_\beta, \tau_\beta]$ and $\phi_\beta([i_\beta, \tau_\beta])$ are homeomorphic to $[0, \tau_\beta]$, we can assume that

$$\{\alpha < \nu : \alpha \neq \beta \text{ and } f(T_\beta) \cap f(U'_\alpha) \neq \emptyset\} = \emptyset$$

to begin with. Therefore,

$$f(T_\beta) \cap f(T_\alpha) = \emptyset \tag{$***$}$$

whenever $\alpha, \beta < \nu$ and $\alpha \neq \beta$.

Now let $T = \cup\{T_\alpha : \alpha < \nu\}$, then $T \subset L \setminus D$ since $T_\alpha \subset U'_\alpha$ for every $\alpha < \nu$ and further $U'_\alpha \subset \cup \mathcal{U}' \subset \cup \mathcal{U} = L \setminus D$. We claim that T is closed in $L \setminus D$. Indeed, if x is an arbitrary point of $L \setminus D$, then there exists a unique element U_x of $\mathcal{U}$ which contains x. If $U_x \in \mathcal{U} \setminus \mathcal{U}'$, then $U_x \cap T = \emptyset$. If $U_x \in \mathcal{U}'$, then $U_x = U'_\gamma$ for some $\gamma < \nu$, hence $U_x \cap T = U'_\gamma \cap T = T_\gamma$, which is closed in U'_γ (since the unique point of $\overline{T_\gamma}^L \setminus T_\gamma$ is $y_\gamma \in D$). In both cases U_x is such an open neighborhood of x in $L \setminus D$ that $U_x \cap T$ is closed in U_x. Thus T is closed in $L \setminus D$. This means that $\overline{T}^L \setminus T \subset D$, hence $\overline{T} = (\overline{T} \cap D) \cup (\cup\{T_\alpha : \alpha < \nu\})$. Now $(**)$ and $(***)$ imply that the only non-singleton fiber of the restriction $f_{|\overline{T}}$ is $\overline{T} \cap D = \overline{T} \setminus T$. The mapping

$f_{|\overline{T}}$ is quotient because it is closed as a mapping of a compactum, so condition (3) is satisfied. The proof is complete. $\qquad\square$

Conditions of Theorem 1 characterize a closed subset (respectively, $f\left(\overline{A}\right) \approx A(\tau)$, $f([0,\tau]) \approx [0,\tau]$, or $f\left(\overline{T}\right)$) of a monotonically normal compactum X which is responsible for the character of a given $a \in X$. Next, we note that this closed subset can be obtained from $[0,\tau]$ in a rather uniform way: either it is homeomorphic to $[0,\tau]$ itself, or it is homeomorphic to the space, obtained from $[0,\tau)$ by identifying its club (complement to which is cofinal in τ) to a point.

Theorem 2. *Let X be a monotonically normal compactum. If $a \in X$ and $\chi(a,X) = \tau > \omega$, then X contains a closed subspace K for which a is the unique complete accumulation point, and K is homeomorphic either to $[0,\tau]$ or to the space, obtained from $[0,\tau)$ by identifying a club (complement to which is cofinal in τ) to a point. Depending on the suitable condition of Theorem 1, we have the following cases:*

(a) τ is a regular cardinal and K is homeomorphic to $[0,\tau]$ (condition (2) of Theorem 1).

*(b) K is homeomorphic to the space, obtained from $[0,\tau)$ by identifying to a point a club, which consists of **all** the limit ordinals of τ (condition (1) of Theorem 1).*

(c) τ is a singular cardinal and K is homeomorphic to the space, obtained from $[0,\tau)$ by identifying to a point a club C of cardinality $cf(\tau)$ (condition (3) of Theorem 1). Moreover, for every $\tau' < \tau$ there is an element $\alpha_{\tau'} \in C$ which is a successor in C and such that $cf\left(\alpha_{\tau'}\right) \geqslant \tau'$.

In any case, $\chi(a,K) = |K| = \tau$.

Доказательство. We use the notation of Theorem 1.

In Theorem 1, condition (1) is satisfied when $\{f(a)\} \cup f(A)$ is the one-point compactification of a discrete in itself subspace $f(A)$ and $|f(A)| = \tau$, i.e. $\{f(a)\} \cup f(A) = A(\tau)$. It is well-known that $A(\tau)$ can be obtained from $[0,\tau)$ by identifying the set of all its limit ordinals to a point and that the character of the unique accumulation point of $A(\tau)$ equals τ.

Condition (2) implies Theorem 2 trivially with $K \approx [0,\tau]$; $\chi(a,K) = \tau$ because τ is a regular cardinal in this case.

If condition (3) is satisfied, let L' be the product $\nu \times \tau$ equipped with the lexicographic order. Consider $L'' \subset L'$, $L'' = \{(\alpha,\beta) \in L' : \beta < \tau_\alpha\}$ and $L''' \subset L''$, $L''' = \{(\alpha,0) \in L' : 1 \leqslant \alpha < \nu\}$. Denote by h the natural order isomorphism (which is also a homeomorphism) from L'' onto $[0,\tau)$. Then $h\left(L'''\right)$ is a club in $[0,\tau)$ which we further denote by C. We claim that $f\left(\overline{T}\right)$ is homeomorphic to $K = \left(h\left(L''\right)\right)/C$ (the latter is the space obtained from $h\left(L''\right)$ by identifying C to a point). Indeed, $f\left(T_\alpha\right) \approx T_\alpha \approx [0,\tau_\alpha)$ for every $\alpha < \nu$ in Case II of Theorem 1. The corresponding subset of K, $q\left(h\left(\{(\alpha,\beta) \in L'' : \beta < \tau_\alpha\}\right)\right)$ (where q is that quotient mapping of $h\left(L''\right)$ which identifies C to a point), is trivially homeomorphic to $[0,\tau_\alpha)$. It follows from $(**)$ and $(***)$ that the only non-singleton fiber of $f_{|\overline{T}}$ is $\overline{T} \setminus T$. Therefore, a typical open neighborhood of a in $f\left(\overline{T}\right)$ is a complement to the union of proper initial segments of finitely many $f\left(T_\alpha\right)$.

In the same way, a typical neighborhood of $q(C)$ in $q([0,\tau))$ is a complement to the union of finitely many sets $q\left(h\left(\{(\alpha_i,\beta) \in L'' : 1 \leqslant \beta \leqslant \beta_i\}\right)\right)$ with each $\alpha_i < \nu$ and $\beta_i < \tau_i$. Thus $f\left(\overline{T}\right) \approx [0,\tau)/C$ as required.

Finally, $\chi(a, K) = \tau$ because also $f\left(\overline{T}\right) \approx [0, \tau]/\left(C \cup \{\tau\}\right)$ and the closed set $C \cup \{\tau\}$, which is identified to a, is of character τ in $[0, \tau]$ since the sequence of cardinals $\{\tau_\alpha < \nu : \alpha < \nu\}$ (from Case II of the proof of Theorem 1) is cofinal in τ.

The proof of Theorem 2 is complete. $\qquad\square$

Note that condition (1) of Theorem 1 does not imply that L itself contains a subset homeomorphic to a club of a regular uncountable cardinal. In fact, X may even be first-countable while $\chi(a, X) > \omega$. Indeed, let S be the double arrow space. Insert an isolated point into each jump of S and denote the resulting linearly ordered space by L. Then L naturally contains S as a closed subset. Denote the quotient mapping which identifies S to a point by q. Then $q(L)$ is homeomorphic to the Alexandroff compactification of the discrete space of cardinality continuum because there are continuum many jumps in S. But being first-countable, L does not contain a subset homeomorphic to a club of a regular uncountable cardinal.

Corollary 1. *Let X be a monotonically normal compactum. Then $\chi(X)$ is equal to the maximum of* $\sup\{\tau : [0, \tau] \subset X\}$, $\sup\{\tau : A(\tau) \subset X\}$.

Доказательство. $\chi(X) \geqslant \sup\{\tau : [0, \tau] \subset X\}$ and $\chi(X) \geqslant \sup\{\tau : A(\tau) \subset X\}$ since $\chi([0, \tau]) = \tau$ and $\chi(A(\tau)) = \tau$.

In order to prove the reverse inequality, consider an irreducible continuous mapping $f : L \longrightarrow X$ of a linearly ordered compactum L onto X.

If $\chi(X)$ is a successor cardinal, then $\chi(X) = \chi(a', X)$ for some $a' \in X$. Thereby Corollary 1 directly follows from Theorem 2 and conditions (1) and (2) (but not condition (3) as every successor ordinal is a regular one) of Theorem 1.

If $\chi(X)$ is a limit cardinal, then for every $\chi' < \chi(X)$ there exist an ordinal χ and a point $x_\chi \in X$ such that $\chi' < \chi \leqslant \chi(X)$ and $\chi(x_\chi, X) \geqslant \chi$. If χ is a regular cardinal, then an argument similar to the successor case applies. If χ is a singular cardinal, then we have two subcases. If condition (1) of Theorem 1 is satisfied, then X contains a copy of $A(\chi)$, and hence of $A(\chi')$, with the only non-isolated point being x_χ. Otherwise, condition (3) of Theorem 1 must be satisfied, so that L contains a subset $\overline{T_\alpha}$ of cardinality $\tau_\alpha \geqslant \chi'$ and such that $f_{|\overline{T_\alpha}}$ is one-to-one. Moreover, we have $f\left(\phi_\alpha(\tau_\alpha)\right) = x_\chi$ for the composition homeomorphism $f\left(\phi_\alpha(.)\right) : [0, \tau_\alpha] \to X$. Thus X contains a copy of $[0, \chi']$. So X contains a copy either of $A(\chi')$ or of $[0, \chi']$ for arbitrary large $\chi' < \chi(X)$. The proof is complete. $\quad\square$

Corollary 2. *A monotonically normal compactum of an uncountable character contains a compactum of cardinality and weight ω_1.*

Доказательство. By Corollary 1, a monotonically normal compactum X of an uncountable character contains either $[0, \tau]$ or $A(\tau)$ for an uncountable cardinal τ. Hence X contains either $[0, \omega_1]$ or $A(\omega_1)$ respectively which are both compacta of cardinality and weight ω_1. $\qquad\square$

3. Small diagonal in monotonically normal compacta

Recall that $\Delta(X) = \{(x, x) \in X^2 : x \in X\}$ is called the diagonal of a topological space X. A space X has a *small diagonal* if each uncountable subset of X^2 disjoint from the diagonal has an uncountable subset whose closure is disjoint from the diagonal. It is well-known (and easily seen) that a metrizable space has a G_δ diagonal and that a space with a G_δ diagonal has a small diagonal. So a small

diagonal in a space is a generalization of metrizability. In some classes of spaces a small diagonal is equivalent to metrizability. For example, a space with a small diagonal is metrizable if it is a Lindelöf LOTS [BeL97, Theorem 4.1], or if it contains a compact subspace of weight ω_1 [ArB92]. The main problem in the area, posed by M. Hušek, is whether every compactum with a small diagonal is metrizable [Hus77]. A consistent positive answer to this problem was given by Zhou in [Zho82] (see [Gru02] for details), it is unknown whether the positive answer holds in ZFC. We prove a theorem in the direction of the mentioned result from [BeL97].

Theorem 3. *A monotonically normal compactum X with a small diagonal is metrizable.*

Доказательство. If X is not first-countable then it contains a compactum of weight ω_1 by Corollary 2, so it does not have a small diagonal at the first place by [ArB92, Corollary 3]. Thus X is first-countable. Assume towards contradiction that X is not metrizable, then X contains a separable non-metrizable subspace Y by [DoP06, Corollary 6]. Then Y is hereditarily Lindelöf by [Ost78]. (Or use a folklore result that a separable monotonically normal compactum is a continuous image of the double arrow space, hence it is hereditarily Lindelöf.) But according to [Grue02, Theorem 3.6], a first-countable hereditarily Lindelöf space with a small diagonal does have a G_δ-diagonal, a contradiction. Theorem 3 is proved.   □

4. Well-ordered π-bases

A family $\mathcal{B}$ of open sets of a topological space X is called a (neighborhood) π-base at some $x \in X$ if every open neighborhood of x contains an element of $\mathcal{B}$. A π-base is well-ordered if its elements can be well-ordered so that every open neighborhood of x contains a terminal segment of $\mathcal{B}$. Williams and Zhou proved in [WiZ98, Theorem 3.1] that each point x of a compact monotonically normal space X has a well-ordered neighborhood π-base of cardinality $cf(\chi(x, X))$. We complement this result in case when τ is singular by replacing $cf(\chi(x, X))$ with $\chi(x, X)$.

Theorem 4. *The isolated points of the compactum K from Theorem 2 could be simultaneously expanded to their open neighborhoods in X to form at a a disjoint well-ordered π-base $\mathcal{B}$ of cardinality $\tau = \chi(a, X)$.*

First, we prove an auxiliary lemma.

Lemma 4. *Let $f : L \longrightarrow X$ be an irreducible perfect continuous mapping of a GO-space L onto a regular space X. If an infinite set $B \subset X$ is discrete in itself, then there exists a disjoint family of open neighborhoods $\{O_x : x \in B\}$ such that the sets of accumulation points (in X) of C and of $\{O_x : x \in C\}$ coincide for every $C \subset B$.*

Доказательство. The set B is discrete and closed in $X \setminus B'$, where $B' = \overline{B}^X \setminus B$. Since $X \setminus B'$ is collectionwise Hausdorff (and even collectionwise normal as monotone normality is a hereditary property and it implies collectionwise normality), there exists a disjoint family $\{V_x \subset X : x \in B\}$ of open neighborhoods in X of the elements of B. We can assume that each V_x is disjoint from B'. It then follows from the normality of $X \setminus B'$ that B and $(X \setminus B') \setminus (\cup\{V_x : x \in B\})$ have disjoint open

neighborhoods W and V respectively. For every $x \in B$ let $W_x = W \cap V_x$, then the set of accumulation points of the family $\{W_x : x \in B\}$ coincides with B'.

We denote the set of convex (and open) components of $L \setminus f^{-1}(B')$ by $\mathcal{U}$. For each $x \in B$, the preimage $f^{-1}(x)$ is compact, so it is covered by a finite subfamily of $\mathcal{U}$ (so that each member of this subfamily intersects $f^{-1}(x)$) which we denote by $\mathcal{U}_x$. For every $x \in B$ let $U_x = \cup \mathcal{U}_x$ and $O_x = W_x \cap \widetilde{U}_x$, where $\widetilde{U}_x = X \setminus f(L \setminus U_x)$. Fix $C \subset B$ and an accumulation point $y \in X$ of $\{O_x : x \in C\}$. Then $y \in B'$ because $\overline{\cup\{O_x : x \in C\}} \subset \overline{\cup\{W_x : x \in C\}} = B'$. Clearly y is also an accumulation point of $\{\widetilde{U}_x : x \in C\}$. Consequently, $F = f^{-1}(y)$ contains an accumulation point of $\{U_x : x \in C\}$, so that every open neighborhood of F intersects infinitely many U_x. Let $U(F)$ be an open neighborhood of F in L, then $U(F)$ is the union of pairwise disjoint open intervals (convex components of $U(F)$ in L). Since F is compact, we can assume without loss of generality that $U(F)$ consists of finitely many such intervals. Then at least one of these intervals, which we further denote by U', intersects infinitely many U_x. If U' intersects a member $U^*_{x'}$ of $\mathcal{U}_{x'}$ for some $x' \in C$ and at the same time does not intersect $f^{-1}(x')$, then at least one point of $U^*_{x'}$ (which is also a point of $f^{-1}(x')$) does not belong to U'. But in a GO-space L an interval U' may properly partially contain at most two disjoint intervals[5]. Since all members of $\mathcal{U}$ are pairwise disjoint, we conclude that U' intersects $f^{-1}(x)$ for infinitely many $x \in C$, thus $y \in \overline{C}$ as required. Lemma 4 is proved. $\qquad\square$

Now we prove theorem 4. Let B be the set of isolated points of compactum K from Theorem 2 and let O be an open neighborhood of a in X. Then $|B| = \tau = \chi(a, X)$. Also $|B \setminus O| < \tau$ as a is the unique complete accumulation point of K. Denote by $\mathcal{B}$ a disjoint family $\{O_x : x \in B\}$ of open neighborhoods of the elements of B obtained in Lemma 4. According to this lemma, each $z \in X \setminus O$ has an open neighborhood which intersects less than τ elements of $\mathcal{B}$. Due to compactness of X, and hence of $X \setminus O$, some open neighborhood in X of $X \setminus O$ intersects less than τ elements of $\mathcal{B}$. Therefore O may not contain only $< \tau$ elements of $\mathcal{B}$. This means that $\mathcal{B}$ is the required disjoint well-ordered π-base at a of cardinality $\chi(a, X)$. The proof is complete. $\qquad\square$

Note that since $|K| = \tau$, no disjoint family of sets intersecting K can be of cardinality $> \tau$. In some cases Theorem 4 can be strengthened.

Theorem 5. *Let X be a monotonically normal compactum and $a \in X$ with $\chi(a, X) = \tau > \omega$.*

In item (a) of Theorem 2, any $G_{<\tau}$-set[6] containing a also contains all the elements of the π-base $\mathcal{B}$ constructed in Theorem 4 except perhaps a subfamily of cardinality $< \tau$.

In item (b) of Theorem 2, if $\omega < \mu \leqslant \tau$, then any $G_{<\mu}$-set containing a also contains all the elements of the π-base $\mathcal{B}$ constructed in Theorem 4 except perhaps a subfamily of cardinality $< \mu$.

In item (c) of Theorem 2, for arbitrary large regular cardinals $\mu \leqslant \tau$ there exist disjoint well-ordered π-bases $\mathcal{B}_\mu$ at a of cardinality μ such that any $G_{<\mu}$-set containing a also contains all the elements of $\mathcal{B}_\mu$ except perhaps a subfamily of cardinality $< \mu$.

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[5]Consider a compact linearly ordered extension $\widetilde{L}$ of L. A convex open extension $\widetilde{U'}$ of U' in $\widetilde{L}$ has exactly two endpoints, and one of them must belong to a convex open extension of $U^*_{x'}$.

[6]I.e., the intersection of $< \tau$ open sets.

Доказательство. We use the notation of Theorem 2. Let $\eta < \tau$ and suppose $W' = \cap_{\alpha < \eta} W_\alpha$, where each W_α is an open neighborhood of a in X. The same argument as in the proof of Theorem 4 shows that for every $\alpha < \eta$, $|\{x \in B : O_x \not\subset W_\alpha\}| < \tau$ in item (a) of Theorem 2. Then $\{x \in B : O_x \not\subset W'\} = \cup_{\alpha < \eta}\{x \in B : O_x \not\subset W_\alpha\}$, so $|\{x \in B : O_x \not\subset W'\}| < \tau$ due to regularity of τ.

Let $\eta < \mu$ and suppose $W'' = \cap_{\alpha < \eta} W_\alpha$, where each W_α is an open neighborhood of a in X. In item (b) of Theorem 2 we have that $K \setminus W_\alpha$ is finite for every $\alpha < \eta$. Therefore $|\{x \in B : O_x \not\subset W_\alpha\}| \leqslant \omega$ by Lemma 4 as we can assume that each W_α is of type F_σ. Then $|\{x \in B : O_x \not\subset W''\}| \leqslant \eta \cdot \omega = \eta < \mu$.

In item (c) of Theorem 2, for every $\eta' < \tau$ there is a regular cardinal μ with $\eta' \leqslant \mu \leqslant \tau$ such that K contains a closed copy of $[0, \mu]$ so that μ coincides with a. Let $\eta < \mu$ and suppose $W''' = \cap_{\alpha < \eta} W_\alpha$, where each W_α is an open neighborhood of a in X. The same argument as in the proof of Theorem 4 (and as in the beginning of the proof of Theorem 5) shows that for every $\alpha < \eta$, $|\{x \in B : O_x \not\subset W_\alpha\}| < \mu$ in item (c) of Theorem 2. Then $\{x \in B : O_x \not\subset W'''\} = \cup_{\alpha < \eta}\{x \in B : O_x \not\subset W_\alpha\}$, so $|\{x \in B : O_x \not\subset W'''\}| < \mu$ due to regularity of μ. Theorem 5 is proved. $\qquad\square$

5. Homogeneity in monotonically normal compacta

A topological space X is homogeneous if for every $x, y \in X$ there exists a homeomorphism h from X onto itself such that $h(x) = y$. Homogeneity has been studied extensively in various classes of spaces. One of considered problems here is to find additional conditions on homogeneous spaces which imply first-countability. (It is enough to find a single point of first-countability, then the definition of homogeneity guaranties that considered space is first-countable at every point.) For example, Alan Dow [Dow88] proved that consistently every nonempty countably tight compactum is first-countable at some point. Hence, consistently, every homogeneous countably tight compactum is first-countable. A.V. Arhangel'skii [Arh19] introduced the concept of G_δ-homogeneity. He called a topological space X G_δ-homogeneous if, for every $x \in X$ and every nonempty G_δ set $P \subset X$, there exists a homeomorphism h of X onto itself such that $h(x) \in P$. Clearly, every homogeneous space is G_δ-homogeneous. The converse is not true since the Čech-Stone remainder ω^* of ω appeared to be G_δ-homogeneous [Arh19, Theorem 2.10] (which is a bit unexpected since ω^* itself is *very* far from being homogeneous). On the other hand, Arhangel'skii noted that if X is G_δ-homogeneous and contains a point of countable pseudocharacter, then X is homogeneous [Arh19, Lemma 2.9].

A space is called isocompact [Bac70] if every closed countably compact subspace of X is compact. It is known that (see [Bac70] and [Arh19]) that for a compactum X

X is first-countable at $x \Rightarrow X \setminus \{x\}$ is isocompact $\Rightarrow X$ is countably tight at x

A space is ω-Lindelöf if every closed separable subset of X is Lindelöf. It is proved in [Arh19], among many other results, that if X is a G_δ-homogeneous compactum such that $X \setminus \{x\}$ is both isocompact and ω-Lindelöf for every $x \in X$, then X is first-countable and homogeneous. We note that in the presence of monotone normality, isocompactness of every $X \setminus \{x\}$ alone, and even countable tightness of X, implies both the first countability and homogeneity of X.

Theorem 6. *If X is a G_δ-homogeneous monotonically normal compactum then the following conditions are equivalent:*

(a) $t(X) = \omega$.

(b) $\chi(X) = \omega$.

(c) $X \setminus \{x\}$ is isocompact for every $x \in X$.

Either of these conditions implies that X is homogeneous.

Proof of Theorem 6. The implication (b) $\Rightarrow$ (c) holds because the complement to a point in the first-countable space X is of type F_σ, and since every F_σ subset of an isocompact space is isocompact [Bac70, Theorem 2.2]. The implication (c) $\Rightarrow$ (a) is proved in [Arh19, Proposition 2.1]. The implication (a) $\Rightarrow$ (b) follows from Lemma 5.

Lemma 5. *If X is a monotonically normal compactum of countable tightness, then the set of points of first-countability is everywhere dense in X.*

Proof of Lemma 5. Lemma 5 is a direct consequence of [Arh78, (2.2.8)] which states that in a countably tight compactum the set of points of first-countability is everywhere dense if the closure in X of every countable subset of X is first-countable (in itself). In Lemma 5 the closure $\overline{A}^X$ of a countable subset of X is a separable monotonically normal compactum, which is hereditarily Lindelöf, hence first-countable. Lemma 5 is proved, thus implying in the presence of G_δ-homogeneity both the implication (a) $\Rightarrow$ (b) and homogeneity of X by virtue of [Arh, Lemma 2.9]. Therefore, in the presence of G_δ-homogeneity either of conditions (a), (b), (c) in Theorem 6 implies the others and homogeneity of X. The proof of Theorem 6 is complete. $\square$

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