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ON THE MODERATE DEVIATION PRINCIPLE FOR m–DEPENDENT RANDOM VARIABLES WITH SUBLINEAR EXPECTATION

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Abstract. In this paper, we obtain the moderate deviation principle for sums of m–dependent strictly stationary random variables in the space with sublinear expectation. Unlike known results, we will require random variables to satisfy a less restrictive Cramer–like condition.

Keywords: large deviation principle, moderate deviation principle, sublinear expectation, m–dependent random variables, stationary sequences.

1. Introduction

Since 1990s the theory of random variables in the sublinear expectation space has been actively developed. This progress can be attributed to two main factors. On the one hand, sublinear expectation retains most of the properties of the regular expectation, which makes it possible to transfer with some changes the main results of classical probability theory to it. On the other hand, the utilization of sublinear expectation spaces allows for statistical inferences to be made even in situations where limited information is available about the assumed distribution of a random elements, which makes this theory highly applicable in solving some practical problems.

Before giving definitions, we briefly review known results. Apparently, the first work in this fields is by Lebedev [3], who investigated a special case of sublinear

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expectation, and gave examples of its applications. Peng [7] introduced the notions of independent random elements, normal distribution and Brownian motion in sublinear expectation space, known as G–normal distribution and G–Brownian motion. Moreover, Peng constructed an analogue of the stochastic integral and investigated both regular and backward stochastic differential equations. In paper [8] Peng extended results of [7] to multidimensional case. Further advancements were made by Peng [9] and Chen, Wu, and Li [10], who derived analogues of the law of large numbers and central limit theorem for random variables in sublinear expectation space. Additionally, they obtained analogues of the main inequalities of classical probability theory for sublinear expectation space. For more detailed information about recent results in the field of theory of sublinear expectation we refer to the book by Peng [6].

Now let us review the results that are directly related to the large deviation principle (LDP). It is important to note that LDP, as well as its special case, the moderate deviation principle (MDP), are valuable tools for estimating the probabilities of rare events. LDP is mostly related to rough exponential asymptotics of probabilities of rare events for sequences of random elements. More detailed information about LDP and MDP in classical probability theory can be found in books by Borovkov [1], Deuschel and Stroock [4] and Dembo and Zeitouni [5].

In the context of sublinear expectation space, one of the earliest results related to LDP for random variables can be attributed to Hu [12], who derived the upper bound of Cramér's theorem for capacities. Gao and Xu [13] obtained LDP for independent and identically distributed random variables in sublinear expectation space. Cao [14] obtained an upper bound of LDP for independent and identically distributed d–dimensional random variables in sublinear expectation space.

Chen and Feng [15] investigated LDP for negatively dependent random variables in sublinear expectation space. They also obtained an upper bound for MDP. Tan and Zong [16] obtained LDP for d–dimensional random variables in sublinear expectation space without requiring them to be independent and identically distributed.

Works by Chen and Xiong [17] and by Gao and Jiang [18] were concerned with LDP for solutions of stochastic differential equations, in which the stochastic integral is constructed with G–Brownian motion.

Zhou and Logachov [19] obtained MDP for weakly dependent random variables in sublinear expectation space, without requiring them to be identically distributed. Guo and Yong [20] obtained MDP for strictly stationary m–dependent random variables in sublinear expectation space.

In this paper we investigate MDP for strictly stationary m–dependent random variables with weaker moment condition than that used in [19] and [20]. Thus, we generalize the results obtained in [19] and [20]. We also note that in case when random variables are independent and considered in classical probability space, the result of this paper directly follows from [2], (see also chapter 5 of [1]).

Now we review needed definitions and related symbols. Let $(\Omega, \mathfrak{F})$ be a sample space and a σ–algebra of its subsets, respectively. We denote by $\mathcal{H}$ a linear space of real-valued functions (random variables) defined on Ω and measurable with respect to $\mathfrak{F}$, such that:

1) $c \in \mathcal{H}$ for any constant $c \in \mathbb{R}$;

2) if $|X| \in \mathcal{H}$ and $|Y| \leqslant |X|$, then $Y \in \mathcal{H}$.
Now we give definition of sublinear expectation defined on $\mathcal{H}$.

Definition 1. *Sublinear expectation on $\mathcal{H}$ is a functional $\mathbb{E} : \mathcal{H} \to \mathbb{R}$ that satisfies the following properties: for all $X, Y \in \mathcal{H}$*

(i) *monotonicity: if $X \geqslant Y$, then $\mathbb{E}[X] \geqslant \mathbb{E}[Y]$;*
(ii) *constant preserving: $\mathbb{E}[c] = c$, $\forall c \in \mathbb{R}$;*
(iii) *sub–additivity: $\mathbb{E}[X+Y] \leqslant \mathbb{E}[X] + \mathbb{E}[Y]$;*
(iv) *positive homogeneity: $\mathbb{E}[\lambda X] = \lambda \mathbb{E}[X]$, $\forall \lambda \geq 0$.*

The triplet $(\Omega, \mathcal{H}, \mathbb{E})$ is called a sublinear expectation space.

Remark 1 (Peng [6])**.** From properties (ii) and (iii) it follows that for any $X, Y \in \mathcal{H}$

(v) $\mathbb{E}[X + c] = \mathbb{E}[X] + c$, $\forall c \in \mathbb{R}$;
(vi) $\mathbb{E}[X - Y] \geqslant \mathbb{E}[X] - \mathbb{E}[Y]$.

Definition 2. *Let $\boldsymbol{X}_1$ and $\boldsymbol{X}_2$ be two n–dimensional random vectors defined in sublinear expectation spaces $(\Omega_1, \mathcal{H}_1, \mathbb{E}_1)$ and $(\Omega_2, \mathcal{H}_2, \mathbb{E}_2)$ respectively. They are said to be identically distributed, denoted $\boldsymbol{X}_1 \overset{d}{=} \boldsymbol{X}_2$, if*

$$\mathbb{E}_1[\varphi(\boldsymbol{X}_1)] = \mathbb{E}_2[\varphi(\boldsymbol{X}_2)]$$

for any Borel measurable function φ on $\mathbb{R}^n$, such that $\varphi(\boldsymbol{X}_1) \in \mathcal{H}$ and $\varphi(\boldsymbol{X}_2) \in \mathcal{H}$.

Definition 3. *Let $\{X_i\}_{i=1}^n$ be random variables in a sublinear expectation space $(\Omega, \mathcal{H}, \mathbb{E})$. Random variable X_n is said to be independent from $X_1, \ldots, X_{n-1}$, if for any set of nonnegative Borel measurable functions φ_i on $\mathbb{R}$, such that $\varphi_i(X_i) \in \mathcal{H}$, we have*

$$\mathbb{E}\left[\prod_{i=1}^{n} \varphi_i(X_i)\right] = \mathbb{E}\left[\prod_{i=1}^{n-1} \varphi_i(X_i)\right] \mathbb{E}[\varphi_n(X_n)].$$

Definition 4. *A sequence of random variables $\{X_i\}_{i=1}^\infty$ is called m–dependent if there exists $m \geqslant 1$ such that for every $n \geqslant 1$, every $j \geqslant m+1$ and every pair of nonnegative Borel measurable functions φ_1 and φ_2, such that $\varphi_1(X_1, \ldots, X_n) \in \mathcal{H}$ and $\varphi_2(X_{n+m+1}, \ldots, X_{n+j}) \in \mathcal{H}$, we have*

$$\mathbb{E}\left[\varphi_1(X_1, \ldots, X_n)\varphi_2(X_{n+m+1}, \ldots, X_{n+j})\right] = \mathbb{E}\left[\varphi_1(X_1, \ldots, X_n)\right] \mathbb{E}\left[\varphi_2(X_{n+m+1}, \ldots, X_{n+j})\right].$$

Definition 5. *A sequence of random variables $\{X_i\}_{i=1}^\infty$ on $(\Omega, \mathcal{H}, \mathbb{E})$ is called strictly stationary, if for every $n \geqslant 1$, every $k \geqslant 1$ and every Borel measurable function φ on $\mathbb{R}^n$, such that $\varphi(X_1, X_2, \ldots, X_n) \in \mathcal{H}$ and $\varphi(X_{1+k}, X_{2+k}, \ldots, X_{n+k}) \in \mathcal{H}$, we have*

$$\mathbb{E}[\varphi(X_1, X_2, \ldots, X_n)] = \mathbb{E}[\varphi(X_{1+k}, X_{2+k}, \ldots, X_{n+k})].$$

Definition 6. *We define an upper probability (a capacity) $\mathbb{V}$ as follows*

$$\mathbb{V}(A) := \mathbb{E}[\mathbf{I}(A)], \quad A \in \mathfrak{F}.$$

Definition 7. *A sequence of random variables s_n in $(\Omega, \mathcal{H}, \mathbb{E})$ is said to satisfy LDP in $\mathbb{R}$ with rate function I and speed $1/\psi(n) : \lim\limits_{n\to\infty} \psi(n) = \infty$, if for any Borel set $B \in \mathfrak{B}(\mathbb{R})$*

$$\limsup_{n\to\infty} \frac{1}{\psi(n)} \ln \mathbb{V}(s_n \in B) \leqslant - \inf_{y \in [B]} I(y), \tag{1}$$

$$\liminf_{n\to\infty} \frac{1}{\psi(n)} \ln \mathbb{V}(s_n \in B) \geqslant - \inf_{y\in(B)} I(y), \tag{2}$$

where $[B]$ is the closure of set B, and (B) is the interior of the set B. Here we put $I(\varnothing) := \infty$.

Definition 8. *A sequence of random variables s_n in $(\Omega, \mathcal{H}, \mathbb{E})$ is said to satisfy MDP if it satisfies LDP with the same parameters as LDP for a sequence of independent and identically distributed Gaussian random variables with zero mean in classical theory.*

We will consider a sequence of strictly stationary m–dependent random variables $\{X_i\}_{i=1}^{\infty}$ in a sublinear expectation space $(\Omega, \mathcal{H}, \mathbb{E})$ with

$$\mathbb{E}[X_i] = \mathbb{E}[-X_i] = 0, \quad i \in \mathbb{N}, \tag{3}$$

and

$$\lim_{n\to\infty} \frac{1}{n}\mathbb{E}\left[\sum_{i=1}^{n} X_i\right]^2 = \sigma^2 < \infty. \tag{4}$$

In addition, we require the following moment condition to be satisfied. For some $q > 0, \alpha \in (0, 1], M > 0$

$$\mathbb{E}[e^{q|X_1|^{\alpha}}] < M. \tag{5}$$

Let $x = x(n)$ be a sequence of positive real numbers, such that

$$\lim_{n\to\infty} \frac{x(n)}{\sqrt{n}} = \infty, \quad \lim_{n\to\infty} \frac{x^{2-\alpha}(n)}{n} = 0. \tag{6}$$

Denote

$$S_k := \sum_{i=1}^{k} X_i, \quad k \in \mathbb{N}.$$

The following theorem is the main result of the paper.

Theorem 1 (**MDP for sums of m–dependent r.v.s**). *Let conditions (3)– (6) be satisfied. Then the sequence*

$$s_n = \frac{S_n}{x(n)}$$

satisfies LDP in $\mathbb{R}$ with speed $1/\psi(n) := \frac{n}{x^2(n)}$ and rate function $I(y) := \frac{y^2}{2\sigma^2}$, $y \in \mathbb{R}$.

Consider an example.

Example. Let $(\Omega, \mathfrak{F})$ be a measurable space, and let $\mathbf{P}_1$ and $\mathbf{P}_2$ be a pair of probability measures defined on that space. We denote expectations corresponding to $\mathbf{P}_1$ and $\mathbf{P}_2$ as $\mathbf{E}_1$ and $\mathbf{E}_2$ respectively. Let $\mathcal{H}$ be a linear space of all real-valued functions X defined on Ω, such that they are measurable with respect to $\mathfrak{F}$, and

$$\max_{i\in\{1,2\}} \mathbf{E}_i|X| < \infty.$$

We define

$$\mathbb{E}[X] = \max_{i\in\{1,2\}} \mathbf{E}_i X,$$

for $X \in \mathcal{H}$. The triplet $(\Omega, \mathcal{H}, \mathbb{E})$ is a sublinear expectation space.

Let $\{X_i\}_{i=1}^{\infty}$ be a sequence of independent random variables, such that every random variable from a sequence $\{X_{2i+1}\}_{i=0}^{\infty}$ has absolutely continuous probability distribution with pdf

$$f_X(x) = \frac{1}{2}\frac{\beta}{\theta}\left(\frac{|x|}{\theta}\right)^{\beta-1} e^{-\left(\frac{|x|}{\theta}\right)^{\beta}}$$

in $(\Omega, \mathfrak{F}, \mathbf{P}_1)$, where

$$\theta = \sqrt{\frac{1}{\Gamma\left(1+\frac{2}{\beta}\right)}},$$

and $\beta \in (0,1]$, and Rademacher distribution (takes values -1 and 1 with the same probability) in $(\Omega, \mathfrak{F}, \mathbf{P}_2)$, and every random variable from a sequence $\{X_{2i}\}_{i=1}^{\infty}$ has Rademacher distribution in $(\Omega, \mathfrak{F}, \mathbf{P}_1)$, and absolutely continuous probability distribution with pdf $f_X(x)$ in $(\Omega, \mathfrak{F}, \mathbf{P}_2)$. Thus for all random variables of the sequence $\{X_i\}_{i=1}^{\infty}$ we have $\mathbf{E}_1 X_i = \mathbf{E}_2 X_i = 0$ and $\mathbf{E}_1 X_i^2 = \mathbf{E}_2 X_i^2 = 1$.

We define a sequence of random variables Y_n in $(\Omega, \mathcal{H}, \mathbb{E})$ as follows: $Y_n := X_n + X_{n+1}$.

Using remark 1, we have

$$\mathbb{E}[Y_n] = \mathbb{E}[X_n + X_{n+1}] \leqslant \mathbb{E}[X_n] + \mathbb{E}[X_{n+1}] = 0,$$
$$\mathbb{E}[Y_n] = \mathbb{E}[X_n - (-X_{n+1})] \geqslant \mathbb{E}[X_n] - \mathbb{E}[-X_{n+1}] = 0,$$

which implies

$$\mathbb{E}[Y_n] = \mathbb{E}[-Y_n] = 0,$$

for all n.

Due to independence of $\{X_i\}_{i=1}^{\infty}$, for any fixed n we have

$$\mathbf{E}_j[Y_1 + \ldots + Y_n]^2 = \mathbf{E}_j[X_1 + 2X_2 \ldots + 2X_n + X_{n+1}]^2$$
$$= \mathbf{E}_j[X_1^2] + \mathbf{E}_j[X_{n+1}^2] + 4\sum_{i=2}^{n}\mathbf{E}_j[X_i^2] = 2 + 4(n-1), \quad j = 1, 2,$$

which implies

$$\lim_{n\to\infty}\frac{1}{n}\mathbb{E}\left[\sum_{i=1}^{n} Y_i\right]^2 = 4.$$

It is easy to see that $\{Y_i\}_{i=1}^{\infty}$ is a sequence of m–dependent random variables with $m = 1$.

Let φ be a Borel measurable function on $\mathbb{R}^n$, such that $\varphi(Y_1, \ldots, Y_n) \in \mathcal{H}$. It is easy to see that for any k we have

$$\max_{i\in\{1,2\}} \mathbf{E}_i\varphi(Y_1, \ldots, Y_n) = \max_{i\in\{1,2\}} \mathbf{E}_i\varphi(Y_{1+k}, \ldots, Y_{n+k}).$$

Thus, the sequence $\{Y_i\}_{i=1}^{\infty}$ is strictly stationary.

Every random variable of the sequence $\{Y_i\}_{i=1}^{\infty}$ has absolutely continuous distribution in probability spaces $(\Omega, \mathfrak{F}, \mathbf{P}_1)$ and $(\Omega, \mathfrak{F}, \mathbf{P}_2)$ with the following pdf

$$f_Y(x) = \frac{1}{2}\Big(f_X(x-1) + f_X(x+1)\Big),$$

from which it follows that for any $q > 0$ and $\alpha < \beta$ we have

$$\mathbb{E}[e^{q|Y_n|^{\alpha}}] \leqslant C, \quad n \in \mathbb{N}.$$

Thus the theorem 1 is applicable to the sequence $\{Y_i\}_{i=1}^{\infty}$ with the rate function

$$I(y) = \frac{y^2}{8}.$$

The rest of this paper consists of sections 2 and 3. In section 2 we prove theorem 1, and in section 3 we formulate and prove auxiliary results.

2. Proof of theorem 1

In this section we prove the main result of the paper. The proof is devided into two steps: first we prove inequality (1), then we prove inequality (2) (see definition 7).

Step 1. Let F be any given closed set. If $F = \varnothing$ then the result is obvious. Suppose that $F \neq \varnothing$. Denote

$$y_- := \sup\{y \in F : y < 0\} \leqslant 0, \quad y_+ := \inf\{y \in F : y \geqslant 0\} \geqslant 0.$$

It is easy to see that $F \subseteq (-\infty, y_-] \cup [y_+, +\infty)$. Here we assume that $y_- = -\infty$, if $F \cap (-\infty, 0] = \varnothing$ and $y_+ = +\infty$, if $F \cap [0, +\infty) = \varnothing$. It is easy to see that if $F \neq \varnothing$, then at least one of the values y_- and y_+ is finite. We have that

$$\begin{aligned}
\ln \mathbb{V}(s_n \in F) &= \ln \mathbb{E}[\mathbf{I}(s_n \in F)] \\
&\leqslant \ln\Big(\mathbb{E}\big[\mathbf{I}(s_n \in (-\infty, y_-])\big] + \mathbb{E}\big[\mathbf{I}(s_n \in [y_+, +\infty))\big]\Big) \\
&\leqslant \ln\Big(2\max\big(\mathbb{E}\big[\mathbf{I}(s_n \in (-\infty, y_-])\big], \mathbb{E}\big[\mathbf{I}(s_n \in [y_+, +\infty))\big]\big)\Big) \qquad (7)
\end{aligned}$$

Denote

$$A_n := \left\{\omega : \max_{1 \leqslant i \leqslant n} |X_i| \leqslant x(n)\right\}.$$

Now we find the upper bound for $\mathbb{E}\big[\mathbf{I}(s_n \in (-\infty, y_-])\big]$. Using Chebyshev inequality (see lemma 1, (ii)) with the function $f(x) = x$, $x > 0$, for any $\lambda > 0$ we have that

$$\begin{aligned}
\mathbb{E}[\mathbf{I}(s_n \in (-\infty, y_-])] &\leqslant \mathbb{E}\big[\mathbf{I}(s_n \in (-\infty, y_-])\mathbf{I}(A_n)\big] + \mathbb{E}[\mathbf{I}(\overline{A}_n)] \\
&\leqslant \mathbb{E}\left[\mathbf{I}\left(-\lambda\frac{x(n)}{n}S_n \geqslant -\lambda\frac{x^2(n)}{n}y_-\right)\mathbf{I}(A_n)\right] + \mathbb{E}[\mathbf{I}(\overline{A}_n)] \\
&= \mathbb{E}\left[\mathbf{I}\left(e^{-\lambda\frac{x(n)}{n}S_n}\mathbf{I}(A_n) \geqslant e^{-\lambda\frac{x^2(n)}{n}y_-}\right)\mathbf{I}(A_n)\right] + \mathbb{E}[\mathbf{I}(\overline{A}_n)] \\
&\leqslant \mathbb{E}\left[\mathbf{I}\left(e^{-\lambda\frac{x(n)}{n}S_n}\mathbf{I}(A_n) \geqslant e^{-\lambda\frac{x^2(n)}{n}y_-}\right)\right] + \mathbb{E}[\mathbf{I}(\overline{A}_n)] \\
&\leqslant \frac{\mathbb{E}\left[e^{-\lambda\frac{x(n)}{n}S_n}\mathbf{I}(A_n)\right]}{e^{-\lambda\frac{x^2(n)}{n}y_-}} + n\mathbb{E}\big[\mathbf{I}(|X_1| > x(n))\big].
\end{aligned}$$

Using Chebyshev inequality (see lemma 1, (ii)) with the function $f(x) = e^{qx^{\alpha}}$ and moment condition (5), we have that

$$n\mathbb{E}\big[\mathbf{I}(|X_1| > x(n))\big] \leqslant n\frac{\mathbb{E}[e^{q|X_1|^{\alpha}}]}{e^{qx(n)^{\alpha}}} \leqslant Mne^{-qx^{\alpha}(n)}. \qquad (8)$$

From lemma 3 it follows that

$$\mathbb{E}\left[e^{\lambda\frac{x(n)}{n}S_n}\mathbf{I}(A_n)\right] = e^{\frac{x^2(n)}{n}\left(\frac{\lambda^2}{2}\sigma^2 + o(1)\right)}, \qquad (9)$$

as $n \to \infty$. From (8) and (9) it follows that

$$\mathbb{E}\big[\mathbf{I}(s_n \in (-\infty, y_-])\big] \leqslant e^{\frac{x^2(n)}{n}\left(\frac{\lambda^2}{2}\sigma^2+\lambda y_- + o(1)\right)} + Mne^{-qx(n)^\alpha}.$$

By choosing $\lambda = -\frac{y_-}{\sigma^2}$, we get

$$\mathbb{E}\big[\mathbf{I}(s_n \in (-\infty, y_-])\big] \leqslant e^{-\frac{x^2(n)}{n}\left(\frac{y_-^2}{2\sigma^2}+o(1)\right)} + Mne^{-qx(n)^\alpha}. \tag{10}$$

By using the similar method we can get

$$\mathbb{E}\big[\mathbf{I}(s_n \in [y_+, +\infty))\big] \leqslant e^{-\frac{x^2(n)}{n}\left(\frac{y_+^2}{2\sigma^2}+o(1)\right)} + Mne^{-qx(n)^\alpha}. \tag{11}$$

The maximum of right-hand sides of inequalities (10) and (11) is determined by the minimum of y_-^2 and y_+^2. Suppose that $y_-^2 \leqslant y_+^2$. We have

$$\begin{aligned}
&\limsup_{n\to\infty} \frac{n}{x^2(n)} \ln\left(e^{-\frac{x^2(n)}{n}\left(\frac{y_-^2}{2\sigma^2}+o(1)\right)} + Mne^{-qx(n)^\alpha}\right)\\
&= \limsup_{n\to\infty} \frac{n}{x^2(n)} \ln\left(\left(e^{-\frac{x^2(n)}{n}\left(\frac{y_-^2}{2\sigma^2}+o(1)\right)}\right)\left(1+e^{\ln(Mn)-qx(n)^\alpha+\frac{x^2(n)}{n}\left(\frac{y_-^2}{2\sigma^2}+o(1)\right)}\right)\right)\\
&= \limsup_{n\to\infty} \left(-\left(\frac{y_-^2}{2\sigma^2}+o(1)\right) + \frac{n}{x^2(n)}\ln\left(1+e^{\ln(Mn)-qx(n)^\alpha+\frac{x^2(n)}{n}\left(\frac{y_-^2}{2\sigma^2}+o(1)\right)}\right)\right)\\
&= \limsup_{n\to\infty} \left(-\left(\frac{y_-^2}{2\sigma^2}+o(1)\right)\right.\\
&\qquad\left. + \frac{n}{x^2(n)}\ln\left(1+e^{-qx(n)^\alpha\left(1-\frac{\ln(Mn)}{qx(n)^\alpha}-\frac{x^{2-\alpha}(n)}{qn}\left(\frac{y_-^2}{2\sigma^2}+o(1)\right)\right)}\right)\right)\\
&= -\frac{y_-^2}{2\sigma^2}.
\end{aligned}$$

Similarly by assuming $y_+^2 \leqslant y_-^2$ we get

$$\limsup_{n\to\infty} \frac{n}{x^2(n)} \ln\left(e^{-\frac{x^2(n)}{n}\left(\frac{y_+^2}{2\sigma^2}+o(1)\right)} + Mne^{-qx(n)^\alpha}\right) = -\frac{y_+^2}{2\sigma^2}.$$

Thus we get

$$\limsup_{n\to\infty} \frac{n}{x^2(n)} \ln\big(\mathbb{V}(s_n \in F)\big) \leqslant -\min\left(\frac{y_-^2}{2\sigma^2}, \frac{y_+^2}{2\sigma^2}\right).$$

It remains to notice, that

$$\inf_{y\in F} I(y) = \min(I(y_-), I(y_+)) = \min\left(\frac{y_-^2}{2\sigma^2}, \frac{y_+^2}{2\sigma^2}\right).$$

Step 2. Let G be any given open set. If $G = \varnothing$, then the result is obvious. Suppose that $G \neq \varnothing$. It is easy to see, that for any $l \geqslant 0$ the set

$$K_l := \{y : I(y) \leqslant l\}$$

is compact. Since $G \neq \varnothing$, there exists $l_G > 0$ such that $G \cap K_{l_G} \neq \varnothing$.

Since G is an open set and $\inf_{y\in G\cap K_{l_G}} I(y) = \inf_{y\in G} I(y)$, for any $\varepsilon > 0$ there exists $y \in G\cap K_{l_G}$ such that

$$\inf_{x\in G} I(x) \geqslant I(y) - \varepsilon. \tag{12}$$

For any $\delta > 0$ denote

$$y^{(\delta)} := (y-\delta, y+\delta).$$

Since G is an open set, for small enough δ we have

$$\mathbb{V}(s_n \in G) \geqslant \mathbb{E}\left[\mathbf{I}(s_n \in G)\mathbf{I}(A_n)\right] \geqslant \mathbb{E}\left[\mathbf{I}(s_n \in y^{(\delta)})\mathbf{I}(A_n)\right]. \tag{13}$$

For $\lambda \in \mathbb{R}$ and $n \in \mathbb{N}$ we denote

$$A(\lambda, n) := \ln \mathbb{E}\left[e^{\lambda \frac{x(n)}{n} S_n}\mathbf{I}(A_n)\right].$$

From definition of $y^{(\delta)}$ it follows that

$$\begin{aligned}\mathbf{I}(s_n \in y^{(\delta)})\mathbf{I}(A_n)e^{\lambda \frac{x(n)}{n} S_n} &= \mathbf{I}(s_n \in y^{(\delta)})\mathbf{I}(A_n)e^{\lambda \frac{x^2(n)}{n} s_n} \\ &\leqslant \mathbf{I}(s_n \in y^{(\delta)})\mathbf{I}(A_n)e^{\frac{x^2(n)}{n}(\lambda y + |\lambda|\delta)}.\end{aligned} \tag{14}$$

Using remark 1 and inequality (14), for any $\lambda \in \mathbb{R}$ and small enough $\delta > 0$ we have that

$$\begin{aligned}&\ln \mathbb{E}\left[\mathbf{I}(s_n \in y^{(\delta)})\mathbf{I}(A_n)\right] \\ &\geqslant \ln \mathbb{E}\left[\mathbf{I}(s_n \in y^{(\delta)})\mathbf{I}(A_n)e^{\lambda \frac{x(n)}{n} S_n - A(\lambda,n)} e^{-\lambda \frac{x^2(n)}{n} y + A(\lambda,n)} e^{-|\lambda| \frac{x^2(n)}{n}\delta}\right] \\ &= -\lambda \frac{x^2(n)}{n} y + A(\lambda, n) - |\lambda| \frac{x^2(n)}{n}\delta \\ &\qquad + \ln \mathbb{E}\left[\left(1 - \mathbf{I}(s_n \notin y^{(\delta)})\right)\mathbf{I}(A_n)e^{\lambda \frac{x(n)}{n} S_n - A(\lambda,n)}\right] \\ &\geqslant -\lambda \frac{x^2(n)}{n} y + A(\lambda, n) - |\lambda| \frac{x^2(n)}{n}\delta \\ &\qquad + \ln\left(1 - \mathbb{E}\left[\mathbf{I}(s_n \notin y^{(\delta)})\mathbf{I}(A_n)e^{\lambda \frac{x(n)}{n} S_n - A(\lambda,n)}\right]\right).\end{aligned} \tag{15}$$

For any $r > 0$ we have

$$\begin{aligned}&\mathbb{E}\left[\mathbf{I}(s_n \notin y^{(\delta)})\mathbf{I}(A_n)e^{\lambda \frac{x(n)}{n} S_n - A(\lambda,n)}\right] \\ &\leqslant \mathbb{E}\left[\mathbf{I}(s_n \geqslant y+\delta)\mathbf{I}(A_n)e^{\lambda \frac{x(n)}{n} S_n - A(\lambda,n)}\right] + \mathbb{E}\left[\mathbf{I}(s_n \leqslant y-\delta)\mathbf{I}(A_n)e^{\lambda \frac{x(n)}{n} S_n - A(\lambda,n)}\right] \\ &\leqslant \mathbb{E}\left[\frac{e^{\frac{rx^2(n)}{n\sigma^2}(s_n-(y+\delta))}\mathbf{I}(A_n)e^{\lambda \frac{x(n)}{n} S_n}}{e^{A(\lambda,n)}}\right] + \mathbb{E}\left[\frac{e^{\frac{rx^2(n)}{n\sigma^2}((y-\delta)-s_n)}\mathbf{I}(A_n)e^{\lambda \frac{x(n)}{n} S_n}}{e^{A(\lambda,n)}}\right] \\ &= \frac{\mathbb{E}\left[e^{\frac{rx(n)}{n\sigma^2}S_n}\mathbf{I}(A_n)e^{\lambda \frac{x(n)}{n} S_n}\right]}{e^{\frac{rx^2(n)}{n\sigma^2}(y+\delta)}e^{A(\lambda,n)}} + \frac{\mathbb{E}\left[e^{-\frac{rx(n)}{n\sigma^2}S_n}\mathbf{I}(A_n)e^{\lambda \frac{x(n)}{n} S_n}\right]}{e^{\frac{rx^2(n)}{n\sigma^2}(\delta-y)}e^{A(\lambda,n)}} \\ &= \frac{\mathbb{E}\left[e^{\left(\frac{r}{\sigma^2}+\lambda\right)\frac{x(n)}{n} S_n}\mathbf{I}(A_n)\right]}{e^{\frac{rx^2(n)}{n\sigma^2}(y+\delta)}e^{A(\lambda,n)}} + \frac{\mathbb{E}\left[e^{\left(\lambda-\frac{r}{\sigma^2}\right)\frac{x(n)}{n} S_n}\mathbf{I}(A_n)\right]}{e^{\frac{rx^2(n)}{n\sigma^2}(\delta-y)}e^{A(\lambda,n)}} \\ &= I_1 + I_2.\end{aligned}$$

Now we find an upper bound for I_1. From (9) it follows that

$$I_1 \leqslant \frac{e^{\frac{x^2(n)}{n}\left(\frac{(r/\sigma^2+\lambda)^2}{2}\sigma^2+o(1)\right)}}{e^{\frac{x^2(n)}{n}\left(\frac{r}{\sigma^2}(y+\delta)+\frac{\lambda^2}{2}\sigma^2\right)}} = \exp\left\{\frac{x^2(n)}{n}\left(\frac{r^2}{2\sigma^2}-\frac{r\delta}{\sigma^2}+r\left(\lambda-\frac{y}{\sigma^2}\right)+o(1)\right)\right\},$$

as $n\to\infty$. If we choose $r=\delta$, then

$$I_1 \leqslant \exp\left\{\frac{x^2(n)}{n}\left(-\frac{\delta^2}{2\sigma^2}+\delta\left(\lambda-\frac{y}{\sigma^2}\right)+o(1)\right)\right\},$$

as $n\to\infty$. Similarly we get

$$I_2 \leqslant \exp\left\{\frac{x^2(n)}{n}\left(-\frac{\delta^2}{2\sigma^2}+\delta\left(\frac{y}{\sigma^2}-\lambda\right)+o(1)\right)\right\},$$

as $n\to\infty$. Thus we have

$$\text{(16)}\quad \mathbb{E}\left[\mathbf{I}(s_n\notin y^{(\delta)})\mathbf{I}(A_n)e^{\lambda\frac{x(n)}{n}S_n-A(\lambda,n)}\right] \leqslant 2\exp\left\{\frac{x^2(n)}{n}\left(-\frac{\delta^2}{2\sigma^2}+\delta\left|\frac{y}{\sigma^2}-\lambda\right|+o(1)\right)\right\},$$

as $n\to\infty$.

Let $\lambda=\frac{y}{\sigma^2}$. Thus using (9) we get

$$\text{(17)}\quad \frac{n}{x^2(n)}A\left(\frac{y}{\sigma^2},n\right)=\frac{n}{x^2(n)}\ln e^{\frac{x^2(n)}{n}\left(\frac{y^2}{2\sigma^2}+o(1)\right)}=\frac{y^2}{2\sigma^2}+o(1),$$

as $n\to\infty$. We also get

$$\text{(18)}\quad \mathbb{E}\left[\mathbf{I}(s_n\notin y^{(\delta)})\mathbf{I}(A_n)e^{\lambda\frac{x(n)}{n}S_n-A(\lambda,n)}\right]\leqslant 2\exp\left\{\frac{x^2(n)}{n}\left(-\frac{\delta^2}{2\sigma^2}+o(1)\right)\right\},$$

as $n\to\infty$. Using (15), (17) and (18), we get that

$$\begin{aligned}\liminf_{n\to\infty}\frac{n}{x^2(n)}\ln\mathbb{E}\left[\mathbf{I}(s_n\in y^{(\delta)})\mathbf{I}(A_n)\right] &\geqslant \liminf_{n\to\infty}\frac{n}{x^2(n)}\left(-\frac{x^2(n)}{n\sigma^2}y^2+A\left(\frac{y}{\sigma^2},n\right)\right.\\ &\quad\left.-\frac{|y|}{\sigma^2}\frac{x^2(n)}{n}\delta+\ln\left(1-2e^{\frac{x^2(n)}{n}\left(-\frac{\delta^2}{2\sigma^2}+o(1)\right)}\right)\right)\\ &= -\frac{y^2}{2\sigma^2}-\frac{|y|}{\sigma^2}\delta.\end{aligned}$$

That is

$$\liminf_{n\to\infty}\frac{n}{x^2(n)}\ln\mathbb{V}(s_n\in G)\geqslant -\frac{y^2}{2\sigma^2}-\frac{|y|}{\sigma^2}\delta.$$

Since the left-hand side of the above inequality does not depend on δ, by letting $\delta\to 0$, we get

$$\liminf_{n\to\infty}\frac{n}{x^2(n)}\ln\mathbb{V}(s_n\in G)\geqslant -\frac{y^2}{2\sigma^2}.$$

By using (12), we get

$$\liminf_{n\to\infty}\frac{n}{x^2(n)}\ln\mathbb{V}(s_n\in G)\geqslant -\frac{y^2}{2\sigma^2}=-I(y)\geqslant -\inf_{x\in G}I(x)-\varepsilon.$$

It remains to notice that the left-hand side of the above inequality does not depend on ε. By letting $\varepsilon\to 0$ we complete the proof. □

3. Auxiliary results

In this section we formulate auxiliary results (lemmas 1-3). Lemmas 1 and 2 are proven in works of other authors. For lemma 3 a proof will be given.

Lemma 1. *Let $X, Y \in \mathcal{H}$. Then the following statements are true*

(i) *Hölder inequality: Let $p, q > 1$ and $\frac{1}{p} + \frac{1}{q} = 1$. Let $|XY| \in \mathcal{H}$, $|X|^p \in \mathcal{H}$ and $|Y|^q \in \mathcal{H}$. Then*

$$\mathbb{E}[|XY|] \leqslant (\mathbb{E}[|X|^p])^{\frac{1}{p}} \cdot (\mathbb{E}[|Y|^q])^{\frac{1}{q}}.$$

If $p = q = 2$, then we have Cauchy–Bunyakovsky–Schwarz inequality.

(ii) *Chebyshev inequality: Let $f(x)$ be a nondecreasing nonnegative function on $\mathbb{R}$ and $f(X) \in \mathcal{H}$. Then for any $x > 0$*

$$\mathbb{V}(X \geqslant x) \leqslant \frac{\mathbb{E}[f(X)]}{f(x)}.$$

The proof of lemma 1 can be found in [10].

Lemma 2. *Let $X_1, \ldots, X_n$ be random variables on $(\Omega, \mathcal{H}, \mathbb{E})$ and $\zeta_i \in [0, 1]$, $1 \leqslant i \leqslant n$ such that $\sum\limits_{i=1}^{n} \zeta_i = 1$. Then*

$$\ln \mathbb{E}\left[e^{\sum\limits_{i=1}^{n} \zeta_i X_i}\right] \leqslant \sum_{i=1}^{n} \zeta_i \ln \mathbb{E}\left[e^{X_i}\right].$$

The proof of lemma 2 can be found in [11].

Lemma 3. *For any $\lambda \in \mathbb{R}$ the following equality holds*

$$\lim_{n\to\infty} \frac{n}{x^2(n)} \ln \mathbb{E}\left[e^{\lambda \frac{x(n)}{n} S_n} \mathbf{I}(A_n)\right] = \frac{\sigma^2 \lambda^2}{2}.$$

Proof. Fix $K \geqslant m + 1$, $n \geqslant K + m$. Denote $l := \left[\frac{n}{K+m}\right]$. Define sequences

$$\xi_t := \sum_{i=1}^{K} X_{t(K+m)+i}, \quad \eta_t := \sum_{i=1}^{m} X_{t(K+m)+K+i}, \quad t = 0, 1, \ldots, l-1.$$

Since the sequence $\{X_i\}_{i=1}^{\infty}$ is m-dependent, we have that ξ_t is independent from $\xi_1, \ldots, \xi_{t-1}$, $1 \leqslant t \leqslant l-1$, and η_t is independent from $\eta_1, \ldots, \eta_{t-1}$, $1 \leqslant t \leqslant l-1$.

The sequence S_n can be represented as follows

$$S_n = \sum_{t=0}^{l-1} \xi_t + \sum_{t=0}^{l-1} \eta_t + \sum_{i=l(K+m)+1}^{n} X_i = I_1 + I_2 + I_3.$$

The rest of the proof of lemma 3 relies on the following three propositions, which we formulate and prove below.

Proposition 1. For any fixed $y \in \mathbb{R}$, $K \geqslant m+1$ the following equality holds

$$\lim_{n\to\infty} \frac{n}{x^2(n)} \ln \mathbb{E}\left[e^{y \frac{x(n)}{n} I_1} \mathbf{I}(A_n)\right] = \frac{1}{K+m} \frac{y^2 \sigma_0^2}{2},$$

where $\sigma_0^2 = \mathbb{E}\left[\xi_0^2\right]$.

Proof of proposition 1. Define

$$\Delta_t = \{i \in \mathbb{N} : 1 + t(K+m) \leqslant i \leqslant K + t(K+m)\},$$

$$\widetilde{\Delta}_n = \{i \in \mathbb{N} : 1 \leqslant i \leqslant n\} \setminus \left(\bigcup_{t=1}^{l-1} \Delta_t \right),$$

$$B_t = \left\{ \omega : \max_{i \in \Delta_t} |X_i| \leqslant x(n) \right\},$$

$$C_n = \left\{ \omega : \max_{i \in \widetilde{\Delta}_n} |X_i| \leqslant x(n) \right\}.$$

Using the fact, that ξ_t is independent from $\xi_1, \ldots, \xi_{t-1}$, $1 \leqslant t \leqslant l-1$, and the fact, that $\{X_i\}_{i=1}^{\infty}$ is strictly stationary, we get

$$\begin{aligned} \text{(19)} \quad \mathbb{E}\left[e^{y\frac{x(n)}{n}I_1}\mathbf{I}(A_n)\right] &= \mathbb{E}\left[e^{y\frac{x(n)}{n}I_1}\mathbf{I}(C_n)\prod_{t=1}^{l-1}\mathbf{I}(B_t)\right] \\ &\leqslant \mathbb{E}\left[e^{y\frac{x(n)}{n}I_1}\prod_{t=1}^{l-1}\mathbf{I}(B_t)\right] = \mathbb{E}\left[\prod_{t=1}^{l-1} e^{y\frac{x(n)}{n}\xi_t}\mathbf{I}(B_t)\right] \\ &= \prod_{t=1}^{l-1}\mathbb{E}\left[e^{y\frac{x(n)}{n}\xi_t}\mathbf{I}(B_t)\right] = \left(\mathbb{E}\left[e^{y\frac{x(n)}{n}\xi_0}\mathbf{I}(B_0)\right]\right)^l. \end{aligned}$$

Using also the remark 1, we get

$$\begin{aligned} \mathbb{E}\left[e^{y\frac{x(n)}{n}I_1}\mathbf{I}(A_n)\right] &= \mathbb{E}\left[e^{y\frac{x(n)}{n}I_1}\mathbf{I}(C_n)\prod_{t=1}^{l-1}\mathbf{I}(B_t)\right] \\ &\geqslant \mathbb{E}\left[e^{y\frac{x(n)}{n}I_1}\prod_{t=1}^{l-1}\mathbf{I}(B_t)\right] - \mathbb{E}\left[e^{y\frac{x(n)}{n}I_1}\mathbf{I}(\overline{C}_n)\prod_{t=1}^{l-1}\mathbf{I}(B_t)\right] \\ \text{(20)} \qquad &= \left(\mathbb{E}\left[e^{y\frac{x(n)}{n}\xi_0}\mathbf{I}(B_0)\right]\right)^l - \mathbb{E}\left[\mathbf{I}(\overline{C}_n)\prod_{t=1}^{l-1}e^{y\frac{x(n)}{n}\xi_t}\mathbf{I}(B_t)\right]. \end{aligned}$$

Now we show that for any $y \in \mathbb{R}$ the following holds

$$\text{(21)} \qquad \mathbb{E}\left[e^{y\frac{x(n)}{n}\xi_0}\mathbf{I}(B_0)\right] = \exp\left\{\frac{x^2(n)}{n^2}\left(\frac{\sigma_0^2 y^2}{2} + o(1)\right)\right\},$$

as $n \to \infty$.

Let us find an upper bound for

$$\mathbb{E}\left[e^{y\frac{x(n)}{n}\xi_0}\mathbf{I}(B_0)\right].$$

Using Taylor expansion of e^x, we get

$$\begin{aligned} \mathbb{E}\left[e^{y\frac{x(n)}{n}\xi_0}\mathbf{I}(B_0)\right] &= \mathbb{E}\left[\left(1 + y\frac{x(n)}{n}\xi_0 + y^2\frac{x^2(n)}{2n^2}\xi_0^2 + \sum_{r=3}^{+\infty} y^r\frac{x^r(n)}{r!n^r}\xi_0^r\right)\mathbf{I}(B_0)\right] \\ = \mathbb{E}\Bigg[\mathbf{I}(B_0) + y\frac{x(n)}{n}\xi_0 &- y\frac{x(n)}{n}\xi_0\mathbf{I}(\overline{B}_0) + y^2\frac{x^2(n)}{2n^2}\xi_0^2\mathbf{I}(B_0) + \sum_{r=3}^{+\infty} y^r\frac{x^r(n)}{r!n^r}\xi_0^r\mathbf{I}(B_0)\Bigg] \\ &\leqslant 1 + \mathbb{E}\left[|y|\frac{x(n)}{n}|\xi_0|\mathbf{I}(\overline{B}_0)\right] + \mathbb{E}\left[y^2\frac{x^2(n)}{2n^2}\xi_0^2\mathbf{I}(B_0)\right] \\ &\quad + \mathbb{E}\left[\sum_{r=3}^{+\infty}\frac{|y\xi_0|^r x^r(n)}{r!n^r}\mathbf{I}(B_0)\right] \end{aligned}$$

$$= 1 + \mathbb{E}_1 + \mathbb{E}\left[y^2 \frac{x^2(n)}{2n^2}\xi_0^2 \mathbf{I}(B_0)\right] + \mathbb{E}_2.$$

Now we find an upper bound for $\mathbb{E}_1$. First, by using Cauchy–Bunyakovsky–Schwarz inequality (see lemma 1, (i)), we get

$$\begin{aligned}\mathbb{E}\left[|y|\frac{x(n)}{n}|\xi_0|\mathbf{I}(\overline{B}_0)\right] &\leqslant \left(\mathbb{E}\left[y^2\frac{x^2(n)}{n^2}\xi_0^2\right]\right)^{\frac{1}{2}}\left(\mathbb{V}(\overline{B}_0)\right)^{\frac{1}{2}} \\ &= |y|\frac{x(n)}{n}(\sigma_0^2)^{\frac{1}{2}}\left(\mathbb{V}(\overline{B}_0)\right)^{\frac{1}{2}} \\ &\leqslant |y|\frac{x(n)}{n}(\sigma_0^2)^{\frac{1}{2}}\Big(K\mathbb{V}(|X_1| > x(n))\Big)^{\frac{1}{2}}.\end{aligned}$$

Then, by using Chebyshev inequality (see lemma 1, (ii)) with the function $f(x) = e^{qx^\alpha}$, property (4) and moment condition (5), we get

$$\begin{aligned}(22)\quad |y|\frac{x(n)}{n}(\sigma_0^2)^{\frac{1}{2}}\Big(K\mathbb{V}(|X_1| > x(n))\Big)^{\frac{1}{2}} &\leqslant |y|\frac{x(n)}{n}(\sigma_0^2)^{\frac{1}{2}}\frac{K^{\frac{1}{2}}\left(\mathbb{E}\left[e^{q|X_1|^\alpha}\right]\right)^{\frac{1}{2}}}{e^{\frac{q}{2}x^\alpha(n)}} \\ &\leqslant |y|\frac{x(n)K^{\frac{1}{2}}M^{\frac{1}{2}}}{ne^{\frac{q}{2}x^\alpha(n)}}\left(\sigma_0^2\right)^{\frac{1}{2}}.\end{aligned}$$

Thus we get the following bound

$$\mathbb{E}_1 \leqslant |y|\frac{x(n)K^{\frac{1}{2}}M^{\frac{1}{2}}}{ne^{\frac{q}{2}x^\alpha(n)}}\left(\sigma_0^2\right)^{\frac{1}{2}}. \tag{23}$$

Now we find an upper bound for $\mathbb{E}_2$.

$$\begin{aligned}\sum_{r=3}^{+\infty}\frac{|y\xi_0|^r x^r(n)}{r!n^r}\mathbf{I}(B_0) &\leqslant \frac{|y\xi_0|^3x^3(n)}{3!n^3}\exp\left\{\frac{|y\xi_0|x(n)}{n}\right\}\mathbf{I}(B_0) \\ &\leqslant \frac{|y|^3(|X_1| + \ldots + |X_K|)^3x^3(n)}{3!n^3}\exp\left\{\frac{x(n)|y|(|X_1| + \ldots + |X_K|)}{n}\right\}\mathbf{I}(B_0) \\ &\leqslant \frac{|y|^3K^3\max\limits_{1\leqslant r\leqslant K}|X_r|^3x^3(n)}{3!n^3}\exp\left\{\frac{x(n)|y|K\max\limits_{1\leqslant r\leqslant K}|X_r|}{n}\right\}\mathbf{I}(B_0) \\ &\leqslant \sum_{r=1}^{K}\frac{|y|^3K^3|X_r|^3x^3(n)}{3!n^3}\exp\left\{\frac{x(n)|y|K|X_r|}{n}\right\}\mathbf{I}(B_0).\end{aligned}$$

There exists $C > 0$ such that for all $u > C$ the following inequality holds

$$|u|^3 \leqslant e^{\frac{q}{2}|u|^\alpha}. \tag{24}$$

Using this fact, we proceed with the bound.

$$\begin{aligned}&\sum_{r=3}^{+\infty}\frac{|y\xi_0|^r x^r(n)}{r!n^r}\mathbf{I}(B_0) \\ &\qquad\leqslant \sum_{r=1}^{K}\frac{|y|^3K^3C^3x^3(n)}{3!n^3}\exp\left\{\frac{q}{2}|X_r|^\alpha\right\}\exp\left\{\frac{x(n)|y|K|X_r|}{n}\right\}\mathbf{I}(B_0) \\ &\qquad= \sum_{r=1}^{K}\frac{|y|^3K^3C^3x^3(n)}{3!n^3}\exp\left\{|X_r|^\alpha\left(\frac{q}{2} + \frac{x^{2-\alpha}(n)|y|K}{n}\right)\right\}.\end{aligned}$$

From condition (6) it follows that for large enough n the following inequality holds

$$\frac{q}{2}+\frac{x^{2-\alpha}(n)|y|K}{n}\leqslant q.$$

By using this fact, moment condition (5) and the fact that $\{X_i\}_{i=1}^{\infty}$ is strictly stationary, we get the following bound

$$\mathbb{E}_2\leqslant\frac{|y|^3K^4C^3x^3(n)}{3!n^3}M. \tag{25}$$

By using bounds (23) and (25) we get the following bound

$$\begin{aligned}(26)\quad \mathbb{E}\left[e^{y\frac{x(n)}{n}\xi_0}\mathbf{I}(B_0)\right]&\leqslant 1+|y|\frac{x(n)K^{\frac12}M^{\frac12}}{ne^{\frac{q}{2}x^{\alpha}(n)}}\left(\sigma_0^2\right)^{\frac12}\\&+\mathbb{E}\left[y^2\frac{x^2(n)}{2n^2}\xi_0^2\mathbf{I}(B_0)\right]+\frac{|y|^3K^4C^3x^3(n)}{3!n^3}M\\&\leqslant 1+y^2\frac{x^2(n)}{2n^2}\sigma_0^2+o\left(\frac{x^2(n)}{n^2}\right),\end{aligned}$$

as $n\to\infty$.

Now we find a lower bound for

$$\mathbb{E}\left[e^{y\frac{x(n)}{n}\xi_0}\mathbf{I}(B_0)\right].$$

By using (25) and remark 1, we get the following bound

$$\begin{aligned}&\mathbb{E}\left[e^{y\frac{x(n)}{n}\xi_0}\mathbf{I}(B_0)\right]\\&\geqslant\mathbb{E}\left[\left(1+y\frac{x(n)}{n}\xi_0+y^2\frac{x^2(n)}{2n^2}\xi_0^2-\sum_{r=3}^{+\infty}\frac{|y\xi_0|^rx^r(n)}{r!n^r}\right)\mathbf{I}(B_0)\right]\\&=\mathbb{E}\left[y^2\frac{x^2(n)}{2n^2}\xi_0^2\mathbf{I}(B_0)-\left(-1+y\frac{x(n)}{n}(-\xi_0)+\sum_{r=3}^{+\infty}\frac{|y\xi_0|^rx^r(n)}{r!n^r}\right)\mathbf{I}(B_0)\right]\\&=\mathbb{E}\left[y^2\frac{x^2(n)}{2n^2}\xi_0^2\mathbf{I}(B_0)\right.\\&\left.-\left(y\frac{x(n)}{n}(-\xi_0)\mathbf{I}(B_0)-1+\mathbf{I}(\overline{B}_0)+\sum_{r=3}^{+\infty}\frac{|y\xi_0|^rx^r(n)}{r!n^r}\mathbf{I}(B_0)\right)\right]\\&\geqslant\mathbb{E}\left[y^2\frac{x^2(n)}{2n^2}\xi_0^2\mathbf{I}(B_0)\right]\\&-\mathbb{E}\left[\left(y\frac{x(n)}{n}(-\xi_0)\mathbf{I}(B_0)+\mathbf{I}(\overline{B}_0)\right)+\left(\sum_{r=3}^{+\infty}\frac{|y\xi_0|^rx^r(n)}{r!n^r}\mathbf{I}(B_0)+(-1)\right)\right]\\&\geqslant\mathbb{E}\left[y^2\frac{x^2(n)}{2n^2}\xi_0^2\mathbf{I}(B_0)\right]\\&-\mathbb{E}\left[y\frac{x(n)}{n}(-\xi_0)\mathbf{I}(B_0)+\mathbf{I}(\overline{B}_0)\right]-\mathbb{E}\left[\sum_{r=3}^{+\infty}\frac{|y\xi_0|^rx^r(n)}{r!n^r}\mathbf{I}(B_0)\right]+1\\&\geqslant\mathbb{E}_3-\mathbb{E}_4-\frac{|y|^3K^4C^3x^3(n)}{3!n^3}M+1.\end{aligned}$$

Now get a lower bound for $\mathbb{E}_3$. By using remark 1, moment condition (5), Cauchy–Bunyakovsky–Schwarz inequality and Chebyshev inequality with the function $f(x) = e^{qx^\alpha}$ (see lemma 1, (i) and (ii) respectively) we get the following bound

$$\begin{aligned}
\mathbb{E}\left[y^2\frac{x^2(n)}{2n^2}\xi_0^2\mathbf{I}(B_0)\right] &\geqslant \mathbb{E}\left[y^2\frac{x^2(n)}{2n^2}\xi_0^2\right] - \mathbb{E}\left[y^2\frac{x^2(n)}{2n^2}\xi_0^2\mathbf{I}(\overline{B}_0)\right]\\
&\geqslant y^2\frac{x^2(n)}{2n^2}\sigma_0^2 - y^2\frac{x^2(n)}{2n^2}(\mathbb{E}[\xi_0^4])^{\frac{1}{2}}(\mathbb{V}(\overline{B}_0))^{\frac{1}{2}}\\
&\geqslant y^2\frac{x^2(n)}{2n^2}\sigma_0^2 - y^2\frac{x^2(n)}{2n^2}(\mathbb{E}[\xi_0^4])^{\frac{1}{2}}\left(\frac{K\mathbb{E}\left[e^{q|X_1|^\alpha}\right]}{e^{qx(n)^\alpha}}\right)^{\frac{1}{2}}. \qquad (27)
\end{aligned}$$

Now we get an upper bound for $\mathbb{E}_4$. By using moment condition (5), Cauchy–Bunyakovsky–Schwarz inequality and Chebyshev inequality with the function $f(x) = e^{qx^\alpha}$ (see lemma 1, (i) and (ii) respectively) we get the following bound

$$\begin{aligned}
\mathbb{E}\left[y\frac{x(n)}{n}(-\xi_0)\mathbf{I}(B_0)+\mathbf{I}(\overline{B}_0)\right] &= \mathbb{E}\left[y\frac{x(n)}{n}(-\xi_0)+y\frac{x(n)}{n}\xi_0\mathbf{I}(\overline{B}_0)+\mathbf{I}(\overline{B}_0)\right]\\
&\leqslant \mathbb{E}\left[|y|\frac{x(n)}{n}|\xi_0|\mathbf{I}(\overline{B}_0)\right] + \mathbb{V}(\overline{B}_0)\\
&\leqslant \left(\mathbb{E}\left[y^2\frac{x^2(n)}{n^2}\xi_0^2\right]\right)^{\frac{1}{2}}\left(\frac{K\mathbb{E}[e^{q|X_1|^\alpha}]}{e^{qx^\alpha(n)}}\right)^{\frac{1}{2}}\\
&\quad + \frac{K\mathbb{E}[e^{q|X_1|^\alpha}]}{e^{qx^\alpha(n)}}\\
&= \left(y^2\frac{x^2(n)KM}{n^2e^{qx^\alpha(n)}}\sigma_0^2\right)^{\frac{1}{2}} + \frac{KM}{e^{qx^\alpha(n)}}. \qquad (28)
\end{aligned}$$

By using bounds (27) and (28) we get the following bound

$$\begin{aligned}
(29)\quad \mathbb{E}\left[e^{y\frac{x(n)}{n}\xi_0}\mathbf{I}(B_0)\right] &\geqslant y^2\frac{x^2(n)}{2n^2}\sigma_0^2 - y^2\frac{x^2(n)}{2n^2}(\mathbb{E}[\xi_0^4])^{\frac{1}{2}}\left(\frac{K\mathbb{E}\left[e^{q|X_1|^\alpha}\right]}{e^{qx(n)^\alpha}}\right)^{\frac{1}{2}}\\
&\quad - \left(y^2\frac{x^2(n)KM}{n^2e^{qx^\alpha(n)}}\sigma_0^2\right)^{\frac{1}{2}} - \frac{KM}{e^{qx^\alpha(n)}} - \frac{|y|^3K^4C^3x^3(n)}{3!n^3}M + 1\\
&= 1 + y^2\frac{x^2(n)}{2n^2}\sigma_0^2 + o\left(\frac{x^2(n)}{n^2}\right),
\end{aligned}$$

as $n \to \infty$.

Using inequalities (26) and (29) we have

$$\begin{aligned}
(30)\quad \mathbb{E}\left[e^{y\frac{x(n)}{n}\xi_0}\mathbf{I}(B_0)\right] &= 1 + y^2\frac{x^2(n)}{2n^2}\sigma_0^2 + o\left(\frac{x^2(n)}{n^2}\right)\\
&= \exp\left\{\ln\left(1 + y^2\frac{x^2(n)}{2n^2}\sigma_0^2 + o\left(\frac{x^2(n)}{n^2}\right)\right)\right\}.
\end{aligned}$$

Using the fact that $\ln(1+u) = u + o(u)$ as $u \to 0$, from (30) we get

$$\mathbb{E}\left[e^{y\frac{x(n)}{n}\xi_0}\mathbf{I}(B_0)\right] = \exp\left\{\frac{x^2(n)}{n^2}\left(\frac{\sigma_0^2y^2}{2} + o(1)\right)\right\},$$

as $n \to \infty$, as was to be shown.

Now we get an upper bound for

$$\mathbb{E}\left[\mathbf{I}(\overline{C}_n)\prod_{t=0}^{l-1} e^{y\frac{x(n)}{n}\xi_t}\mathbf{I}(B_t)\right].$$

By using Cauchy–Bunyakovsky–Schwarz inequality (see lemma 1, (i)), the fact that ξ_t is independent from $\xi_1,\dots,\xi_{t-1}$, $1\leqslant t\leqslant l-1$, the fact that $\{X_i\}_{i=1}^{\infty}$ is strictly stationary, equality (21) and the fact, that $\lim\limits_{n\to\infty}\frac{l}{n}=\frac{1}{K+m}$, we get

$$\begin{aligned}(31)\quad \mathbb{E}\left[\mathbf{I}(\overline{C}_n)\prod_{t=0}^{l-1} e^{y\frac{x(n)}{n}\xi_t}\mathbf{I}(B_t)\right] &\leqslant \left(\mathbb{E}[\mathbf{I}(\overline{C}_n)]\right)^{\frac{1}{2}}\left(\mathbb{E}\left[\prod_{t=0}^{l-1} e^{2y\frac{x(n)}{n}\xi_t}\mathbf{I}(B_t)\right]\right)^{\frac{1}{2}}\\ &\leqslant \left(\sum_{i=1}^{n}\mathbb{V}(|X_i|>x(n))\right)^{\frac{1}{2}}\left(\mathbb{E}\left[e^{2y\frac{x(n)}{n}\xi_0}\mathbf{I}(B_0)\right]\right)^{\frac{l}{2}}\\ &\leqslant \left(n\mathbb{V}(|X_1|>x(n))\right)^{\frac{1}{2}}\exp\left\{l\frac{x^2(n)}{n^2}\left(\sigma_0^2y^2+o(1)\right)\right\}\\ &\leqslant \left(n\mathbb{V}(|X_1|>x(n))\right)^{\frac{1}{2}}\exp\left\{\frac{x^2(n)}{n}\left(\frac{\sigma_0^2}{K+m}y^2+o(1)\right)\right\},\end{aligned}$$

as $n\to\infty$. By using Chebyshev inequality with the function $f(x)=e^{qx^{\alpha}(n)}$ (see lemma 1, (ii)) and moment condition (5), we get

$$(32)\qquad \left(n\mathbb{V}(|X_1|>x(n))\right)^{\frac{1}{2}}\leqslant n^{\frac{1}{2}}M^{\frac{1}{2}}e^{-\frac{q}{2}x^{\alpha}(n)}.$$

By combining inequalities (31) and (32), we get the following bound

$$\begin{aligned}(33)\quad &\mathbb{E}\left[\mathbf{I}(\overline{C}_n)\prod_{t=0}^{l-1} e^{y\frac{x(n)}{n}\xi_t}\mathbf{I}(B_t)\right]\\ &\qquad\leqslant n^{\frac{1}{2}}M^{\frac{1}{2}}\exp\left\{\frac{x^2(n)}{n}\left(\frac{\sigma_0^2}{K+m}y^2+o(1)\right)-\frac{q}{2}x^{\alpha}(n)\right\}.\end{aligned}$$

Using inequalities (19), (21) and the fact that $\lim\limits_{n\to\infty}\frac{l}{n}=\frac{1}{K+m}$, we have that

$$\begin{aligned}\mathbb{E}\left[e^{y\frac{x(n)}{n}I_1}\mathbf{I}(A_n)\right] &\leqslant \left(\mathbb{E}\left[e^{y\frac{x(n)}{n}\xi_0}\mathbf{I}(B_0)\right]\right)^{l}\\ &=\exp\left\{\frac{x^2(n)}{n^2}l\left(\frac{\sigma_0^2y^2}{2}+o(1)\right)\right\}\\ &=\exp\left\{\frac{x^2(n)}{n}\frac{1}{K+m}\left(\frac{\sigma_0^2y^2}{2}+o(1)\right)\right\},\end{aligned}$$

as $n\to\infty$. From the above inequality it follows that

$$(34)\qquad \limsup_{n\to\infty}\frac{n}{x^2(n)}\ln\mathbb{E}\left[e^{y\frac{x(n)}{n}I_1}\mathbf{I}(A_n)\right]\leqslant\frac{1}{K+m}\frac{\sigma_0^2y^2}{2}.$$

By using inequalities (20), (21) and (33), we get

$$(35)\quad \mathbb{E}\left[e^{y\frac{x(n)}{n}I_1}\mathbf{I}(A_n)\right]\geqslant\exp\left\{\frac{x^2(n)}{n}\frac{1}{K+m}\left(\frac{\sigma_0^2y^2}{2}+o(1)\right)\right\}$$

$$- n^{\frac{1}{2}} M^{\frac{1}{2}} \exp\left\{ \frac{x^2(n)}{n} \left(\frac{\sigma_0^2}{K+m} y^2 + o(1) \right) - \frac{q}{2} x^{\alpha}(n) \right\}.$$

From inequality (35) it follows that

$$\frac{n}{x^2(n)} \ln \mathbb{E}\left[e^{y \frac{x(n)}{n} I_1} \mathbf{I}(A_n) \right] \geqslant$$
$$\frac{n}{x^2(n)} \ln \left(e^{\frac{x^2(n)}{n} \frac{1}{K+m} \left(\frac{\sigma_0^2 y^2}{2} + o(1) \right)} - e^{\frac{\ln nM}{2} + \frac{x^2(n)}{n} \left(\frac{\sigma_0^2}{K+m} y^2 + o(1) \right) - \frac{q}{2} x^{\alpha}(n)} \right)$$
$$= \frac{1}{K+m} \left(\frac{\sigma_0^2 y^2}{2} + o(1) \right)$$
$$+ \frac{n}{x^2(n)} \ln \left(1 - e^{\frac{\ln nM}{2} + \frac{x^2(n)}{n} \left(\frac{\sigma_0^2}{K+m} y^2 + o(1) \right) - \frac{x^2(n)}{n} \frac{1}{K+m} \left(\frac{\sigma_0^2 y^2}{2} + o(1) \right) - \frac{q}{2} x^{\alpha}(n)} \right)$$
$$= \frac{1}{K+m} \left(\frac{\sigma_0^2 y^2}{2} + o(1) \right)$$
$$+ \frac{n}{x^2(n)} \ln \left(1 - e^{-x^{\alpha}(n) \left(- \frac{\ln nM}{2x^{\alpha}(n)} - \frac{x^{2-\alpha}}{n} \left(- \frac{\sigma_0^2}{K+m} y^2 + o(1) \right) + \frac{x^{2-\alpha}(n)}{n} \frac{1}{K+m} \left(\frac{\sigma_0^2 y^2}{2} + o(1) \right) + \frac{q}{2} \right)} \right),$$

from which it follows that

$$\liminf_{n\to\infty} \frac{n}{x^2(n)} \ln \mathbb{E}\left[e^{y \frac{x(n)}{n} I_1} \mathbf{I}(A_n) \right] \geqslant \frac{1}{K+m} \frac{\sigma_0^2 y^2}{2}. \tag{36}$$

Proposition 1 follows from inequalities (34) and (36).

Proposition 2. For any fixed $y \in \mathbb{R}$, $K \geqslant m+1$ the following inequality holds

$$\lim_{n\to\infty} \frac{n}{x^2(n)} \ln \mathbb{E}\left[e^{y \frac{x(n)}{n} I_2} \mathbf{I}(A_n) \right] \leqslant \frac{1}{K+m} \frac{y^2 \widetilde{\sigma}_0^2}{2},$$

where $\widetilde{\sigma}_0^2 = \mathbb{E}\left[\eta_0^2\right]$.

Proof of proposition 2. The proof is completely parallel to that of proposition 1, thus we omit it.

Proposition 3. For any fixed $y \in \mathbb{R}$, $K \geqslant m+1$ the following inequality holds

$$\limsup_{n\to\infty} \frac{n}{x^2(n)} \ln \mathbb{E}\left[e^{y \frac{x(n)}{n} I_3} \mathbf{I}(A_n) \right] \leqslant 0.$$

Proof of proposition 3. By using lemma 2 and the fact that $\{X_i\}_{i=1}^{\infty}$ is strictly stationary, we get the following bound

$$\frac{n}{x^2(n)} \ln \mathbb{E}\left[e^{y \frac{x(n)}{n} I_3} \mathbf{I}(A_n) \right]$$

$$\leqslant \frac{n}{x^2(n)} \ln \mathbb{E}\left[\exp\left\{|y|\frac{x(n)}{n}\sum_{i=l(K+m)+1}^{n}|X_i|\right\}\prod_{i=l(K+m)+1}^{n}\mathbf{I}(|X_i|\leqslant x(n))\right]$$
$$= \frac{n}{x^2(n)} \ln \mathbb{E}\left[\exp\left\{|y|\frac{x(n)}{n}\sum_{i=1}^{n-l(K+m)}|X_i|\right\}\prod_{i=1}^{n-l(K+m)}\mathbf{I}(|X_i|\leqslant x(n))\right]$$
$$\leqslant \frac{n}{x^2(n)} \ln \mathbb{E}\left[\exp\left\{|y|\frac{x(n)}{n}\sum_{i=1}^{n-l(K+m)}|X_i|\mathbf{I}(|X_i|\leqslant x(n))\right\}\right]$$
$$\leqslant \frac{n}{x^2(n)} \ln \mathbb{E}\left[\exp\left\{|y|\frac{x(n)}{n}(K+m)\sum_{i=1}^{K+m}\frac{|X_i|\mathbf{I}(|X_i|\leqslant x(n))}{K+m}\right\}\right]$$
$$\leqslant \frac{n}{x^2(n)} \sum_{i=1}^{K+m}\left(\frac{1}{K+m}\ln \mathbb{E}\left[\exp\left\{|y|\frac{x(n)}{n}(K+m)|X_i|\mathbf{I}(|X_i|\leqslant x(n))\right\}\right]\right)$$
$$= \frac{n}{x^2(n)} \ln \mathbb{E}\left[\exp\left\{|y|\frac{x(n)}{n}(K+m)|X_1|\mathbf{I}(|X_1|\leqslant x(n))\right\}\right]$$
$$= \frac{n}{x^2(n)} \ln \mathbb{E}\left[\exp\left\{|y|\frac{x(n)^{2-\alpha}}{n}(K+m)|X_1|^{\alpha}\mathbf{I}(|X_1|\leqslant x(n))\right\}\right]$$
$$\leqslant \frac{n}{x^2(n)} \ln \mathbb{E}\left[\exp\left\{|y|\frac{x(n)^{2-\alpha}}{n}(K+m)|X_1|^{\alpha}\right\}\right].$$

Using condition (6) for large enough n we have

$$|y|\frac{x(n)^{2-\alpha}}{n}(K+m)\leqslant q.$$

By using the above inequality and moment condition (5), for large enough n we get

$$\frac{n}{x^2(n)} \ln \mathbb{E}\left[e^{y\frac{x(n)}{n}I_3}\mathbf{I}(A_n)\right]\leqslant \frac{n}{x^2(n)}\ln M,$$

from which it follows that

$$\limsup_{n\to\infty}\frac{n}{x^2(n)} \ln \mathbb{E}\left[e^{y\frac{x(n)}{n}I_3}\mathbf{I}(A_n)\right]\leqslant 0. \tag{37}$$

Proposition 3 is proven.

Now we proceed with the proof of lemma 3. Fix $p_1>1, p_2>1, q_1>1, q_2>1$ such that $\frac{1}{p_1}+\frac{1}{q_1}=1$, $\frac{1}{p_2}+\frac{1}{q_2}=1$. By using Hölder inequality (see lemma 1, (i)) for any fixed $\lambda\in\mathbb{R}$ we get

$$\begin{aligned}(38)\quad &\frac{n}{x^2(n)} \ln \mathbb{E}\left[e^{\lambda\frac{x(n)}{n}S_n}\mathbf{I}(A_n)\right]\\
&\leqslant \frac{1}{q_1}\frac{n}{x^2(n)} \ln \mathbb{E}\left[e^{\lambda q_1\frac{x(n)}{n}I_3}\mathbf{I}(A_n)\right]+\frac{1}{p_1}\frac{n}{x^2(n)} \ln \mathbb{E}\left[e^{\lambda p_1\frac{x(n)}{n}(I_1+I_2)}\mathbf{I}(A_n)\right]\\
&\leqslant \frac{1}{q_1}\frac{n}{x^2(n)} \ln \mathbb{E}\left[e^{\lambda q_1\frac{x(n)}{n}I_3}\mathbf{I}(A_n)\right]+\frac{1}{p_1p_2}\frac{n}{x^2(n)} \ln \mathbb{E}\left[e^{\lambda p_1p_2\frac{x(n)}{n}I_1}\mathbf{I}(A_n)\right]\\
&\qquad+\frac{1}{p_1q_2}\frac{n}{x^2(n)} \ln \mathbb{E}\left[e^{\lambda p_1q_2\frac{x(n)}{n}I_2}\mathbf{I}(A_n)\right].\end{aligned}$$

By using propositions 1–3 and inequality (38), we get

$$\limsup_{n\to\infty} \frac{n}{x^2(n)} \ln \mathbb{E}\left[e^{\lambda \frac{x(n)}{n} S_n} \mathbf{I}(A_n)\right] \leqslant p_1 p_2 \frac{1}{K+m} \frac{\sigma_0^2 \lambda^2}{2} + \frac{p_1 q_2}{K+m} \frac{\widetilde{\sigma}_0^2 \lambda^2}{2}$$
$$= p_1 p_2 \frac{K}{K+m} \frac{\lambda^2}{2} \frac{1}{K} \mathbb{E}\left[\sum_{i=1}^{K} X_i\right]^2 + \frac{p_1 q_2}{K+m} \frac{\lambda^2}{2} \mathbb{E}\left[\sum_{i=1}^{m} X_i\right]^2 .$$

Since the left-hand side of the above inequality does not depend on K, by letting $K \to \infty$, we get

$$\limsup_{n\to\infty} \frac{n}{x^2(n)} \ln \mathbb{E}\left[e^{\lambda \frac{x(n)}{n} S_n} \mathbf{I}(A_n)\right] \leqslant p_1 p_2 \frac{\sigma^2 \lambda^2}{2}.$$

Since the left-hand side of the above inequality does not depend on p_1, p_1, q_1 and q_2, by letting $p_1 \to 1$ and $p_2 \to 1$, we get

$$\limsup_{n\to\infty} \frac{n}{x^2(n)} \ln \mathbb{E}\left[e^{\lambda \frac{x(n)}{n} S_n} \mathbf{I}(A_n)\right] \leqslant \frac{\sigma^2 \lambda^2}{2}. \tag{39}$$

Now we find a lower bound for

$$\liminf_{n\to\infty} \frac{n}{x^2(n)} \ln \mathbb{E}\left[e^{\lambda \frac{x(n)}{n} S_n} \mathbf{I}(A_n)\right].$$

Using Hölder inequality (see lemma 1, (i)), we have that

$$\ln \mathbb{E}\left[e^{\lambda \frac{x(n)}{p_1 p_2 n} I_1} \mathbf{I}(A_n)\right] = \ln \mathbb{E}\left[e^{\lambda \frac{x(n)}{p_1 p_2 n} (I_1 + I_2 - I_2)} \mathbf{I}(A_n)\right]$$
$$\leqslant \frac{1}{p_1} \ln \mathbb{E}\left[e^{\lambda \frac{x(n)}{p_2 n} (I_1 + I_2)} \mathbf{I}(A_n)\right] + \frac{1}{q_1} \ln \mathbb{E}\left[e^{-\lambda \frac{q_1 x(n)}{p_1 p_2 n} I_2} \mathbf{I}(A_n)\right]$$
$$\leqslant \frac{1}{p_1} \ln \mathbb{E}\left[e^{\lambda \frac{x(n)}{p_2 n} (I_1 + I_2 + I_3 - I_3)} \mathbf{I}(A_n)\right] + \frac{1}{q_1} \ln \mathbb{E}\left[e^{-\lambda \frac{q_1 x(n)}{p_1 p_2 n} I_2} \mathbf{I}(A_n)\right]$$
$$\leqslant \frac{1}{p_1 p_2} \ln \mathbb{E}\left[e^{\lambda \frac{x(n)}{n} (I_1 + I_2 + I_3)} \mathbf{I}(A_n)\right] + \frac{1}{p_1 q_2} \ln \mathbb{E}\left[e^{-\lambda \frac{q_2 x(n)}{p_2 n} I_3} \mathbf{I}(A_n)\right]$$
$$+ \frac{1}{q_1} \ln \mathbb{E}\left[e^{-\lambda \frac{q_1 x(n)}{p_1 p_2 n} I_2} \mathbf{I}(A_n)\right].$$

By using the above we get

$$\frac{n}{x^2(n)} \ln \mathbb{E}\left[e^{\lambda \frac{x(n)}{n} S_n} \mathbf{I}(A_n)\right] \geqslant -\frac{p_2}{q_2} \frac{n}{x^2(n)} \ln \mathbb{E}\left[e^{-\lambda \frac{q_2 x(n)}{p_2 n} I_3} \mathbf{I}(A_n)\right] \tag{40}$$
$$+ p_1 p_2 \frac{n}{x^2(n)} \ln \mathbb{E}\left[e^{\lambda \frac{x(n)}{p_1 p_2 n} I_1} \mathbf{I}(A_n)\right] - \frac{p_1 p_2}{q_1} \frac{n}{x^2(n)} \ln \mathbb{E}\left[e^{-\lambda \frac{q_1 x(n)}{p_1 p_2 n} I_2} \mathbf{I}(A_n)\right].$$

By using propositions 1-3 and inequality (40), we get

$$\liminf_{n\to\infty} \frac{n}{x^2(n)} \ln \mathbb{E}\left[e^{\lambda \frac{x(n)}{n} S_n} \mathbf{I}(A_n)\right]$$
$$\geqslant \frac{1}{p_1 p_2} \frac{K}{K+m} \frac{\lambda^2}{2} \frac{1}{K} \mathbb{E}\left[\sum_{i=1}^{K} X_i\right]^2 - \frac{q_1}{p_1 p_2} \frac{1}{K+m} \frac{\lambda^2}{2} \mathbb{E}\left[\sum_{i=1}^{m} X_i\right]^2 .$$

Since the left-hand side of the above inequality does not depend on K, by letting $K \to \infty$, we get

$$\liminf_{n\to\infty} \frac{n}{x^2(n)} \ln \mathbb{E}\left[e^{\lambda \frac{x(n)}{n} S_n} \mathbf{I}(A_n)\right] \geqslant \frac{1}{p_1 p_2} \frac{\sigma^2\lambda^2}{2}.$$

Since the left-hand side of the above inequality does not depend on p_1, p_1, q_1 and q_2, by letting $p_1 \to 1$ and $p_2 \to 1$, we get

$$\liminf_{n\to\infty} \frac{n}{x^2(n)} \ln \mathbb{E}\left[e^{\lambda \frac{x(n)}{n} S_n} \mathbf{I}(A_n)\right] \geqslant \frac{\sigma^2\lambda^2}{2}. \tag{41}$$

By combining (39) and (41), we get

$$\lim_{n\to\infty} \frac{n}{x^2(n)} \ln \mathbb{E}\left[e^{\lambda \frac{x(n)}{n} S_n} \mathbf{I}(A_n)\right] = \frac{\sigma^2\lambda^2}{2}.$$

□

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