S©MR ISSN 1813-3304

СИБИРСКИЕ ЭЛЕКТРОННЫЕ
МАТЕМАТИЧЕСКИЕ ИЗВЕСТИЯ

Siberian Electronic Mathematical Reports

http://semr.math.nsc.ru

Том 16, стр. 144–144 (2019) УДК 519.21

DOI 10.33048/semi.2019.16.xxx MSC 60J80

ON FUNCTIONAL LIMIT THEOREMS FOR BRANCHING PROCESSES WITH DEPENDENT IMMIGRATION

S.O. SHARIPOV

Abstract. In this paper we consider a triangular array of branching processes with non-stationary immigration. We prove a weak convergence of properly normalized branching processes with immigration to deterministic function under assumptions that immigration satisfies some mixing conditions, the offspring mean tends to its critical value 1 and immigration mean and variance controlled by regularly varying functions. Moreover, we obtain a fluctuation limit theorem for branching process with immigration when immigration generated by a sequence of m-dependent random variables. In this case the limiting process is a time-changed Wiener process. Our results extend the previous known results in the literature.

Keywords: Branching process, immigration, regularly varying functions, m-dependence, ρ-mixing, functional limit theorems.

1. Introduction and statement of main results

Let for each $n \geq 1$, $\left\{\xi_{k,j}^{(n)},\, k, j \geq 1\right\}$ and $\left\{\varepsilon_k^{(n)},\, k \geq 1\right\}$ be two independent families of independent identically distributed (i.i.d.) random variables with non-negative integer values which are defined on a fixed probability space $(\Omega, \mathfrak{F}, \mathbb{P})$. The sequence of branching processes with immigration $\left\{X_k^{(n)},\, k \geq 0\right\}$, $n \geq 1$, is defined by the recursion:

$$X_0^{(n)} = 0, \quad X_k^{(n)} = \sum_{j=1}^{X_{k-1}^{(n)}} \xi_{k,j}^{(n)} + \varepsilon_k^{(n)}, \quad k, n \geq 1. \tag{1}$$

Sharipov, S.O., On functional limit theorems for branching processes with dependent immigration.

Received January, 1, 2015, published March, 1, 2015.

Intuitively, for a fixed $n \geq 1$, $X_k^{(n)}$ represents the size of k-th generation of a population and $\xi_{k,j}^{(n)}$ is the offspring number of the j-th individual in the $(k-1)$-st generation and $\varepsilon_k^{(n)}$ is the number of immigrants contributing to the k-th generation. Assume that for all $n \geq 1$, offspring mean $a_n = \mathbb{E}\xi_{1,1}^{(n)} < \infty$. For each fixed $n \geq 1$, the cases when a_n is less, equal or larger than one are referred to subcritical, critical or supercritical, respectively. The family of processes (1) is called nearly critical if $a_n \to 1$ as $n \to \infty$.

From theoretical and practical points of view it is a natural problem to investigate the asymptotic behaviour of such processes. In many applications (see, for instance, [5]), it is well-known that the assumption of independence of the family of immigration individuals seems to be too strong condition which forces us to suppose that the current number of contributing immigrants depends to some degree on the previous number of immigrants. For instance, in the context of demography, this branching model can describe reasonably well the evolution of populations in which such dependence degree correspondence to the asymptotic independence of immigrants between "past" and "future".

Motivated by Rahimov's results [15], [16] on weak convergence of properly normalized scaled process (1) when immigration sequence is independent, it is a natural to ask about generalization of these results for weakly dependent immigration sequences. In this paper, we focus on weak convergence in Skorokhod topology of the properly normalized and scaled process (1) when the immigration sequence satisfies some mixing conditions.

We present the mixing conditions involved in the paper.

Definition 1. *Let $m \geq 1$ be a fixed integer. A sequence $\{\xi_k, k \geq 1\}$ of random variables is said to be m-dependent if the random vectors $(\xi_1, \ldots, \xi_j)$ and $(\xi_{j+m+1}, \ldots)$ are independent for all $j \geq 1$.*

An array $\{\xi_k^{(n)}, k, n \geq 1\}$ of random variables is said to be rowwise m-dependent if for every $n \geq 1$, $\{\xi_k^{(n)}, k \geq 1\}$ is a sequence of m-dependent random variables. We assume that m depends on the row index n and tends to infinity with an appropriate rate (see condition (C5) below).

Let $\{\xi_k, k \geq 1\}$ be a sequence of random variables defined on a fixed probability space $(\Omega, \mathfrak{F}, \mathbb{P})$. Let n and k be positive integers. Write $\mathfrak{F}_n^k = \sigma(\xi_i, n \leq i \leq k)$. Given two σ-algebras $\mathcal{B}, \mathcal{R}$ in $\mathfrak{F}$, let

$$\rho(\mathcal{B}, \mathcal{R}) = \sup_{\xi \in L^2(\mathcal{B}), \eta \in L^2(\mathcal{R})} \left| \frac{\operatorname{cov}(\xi, \eta)}{\sqrt{Var(\xi)}\sqrt{Var(\eta)}} \right|.$$

Define the ρ-mixing coefficients

$$\rho(k) = \sup_{j \geq 1} \rho\left(\mathfrak{F}_1^j, \mathfrak{F}_{j+k}^\infty\right).$$

Definition 2. *A sequence of random variables $\{\xi_k, k \geq 1\}$ is said to be ρ-mixing if $\rho(k) \to 0$ as $k \to \infty$.*

The concept of ρ-mixing dependence was introduced by Kolmogorov and Rozanov [11].

A triangular array of random variables $\left\{\xi_k^{(n)}, n, k \geq 1\right\}$ is said to be an array of rowwise ρ-mixing random variables if, for every $n \geq 1$, $\left\{\xi_k^{(n)}, k \geq 1\right\}$ is a ρ-mixing sequence of random variables. Let $\rho_n(\cdot)$ be the mixing coefficient of $\left\{\xi_k^{(n)}, k \geq 1\right\}$ for any $n \geq 1$.

Functional limit theorems (FLT for short) for (1) have a rather long history. We refer the reader to the recent survey [17] for a historical overview on this subject. We mainly focus our attention to the case when immigration process follows non-identically distributed random variables. In terms of branching process it means that immigration rate may depend on the time of immigration. Rahimov [14] considered the case when $\{\varepsilon_k, k \geq 1\}$ is a fixed sequence of independent and non-identically distributed random variables with increasing mean ($\mathbb{E}\varepsilon_k \to +\infty$, $k \to \infty$) and proved FLTs for a critical branching process with immigration. Further investigations showed that the independence assumption of immigration process can be relaxed by assuming some reasonable dependence structure: in [4] and [9], the immigration sequence is assumed to be m-dependent and ϕ-mixing, respectively, and established a FLT for (1). In [12], the authors obtained FLT for critical process (1) under assumption that the immigration satisfies ψ-mixing condition. Later, Rahimov [15], [16] considered a process given by (1) and proved diffusion and fluctuation type limit theorems for nearly critical process (1) under the assumption that the sequence $\left\{\varepsilon_k^{(n)}, k \geq 1\right\}$ is rowwise independent for each $n \geq 1$. It is shown [18] that Rahimov's result on deterministic approximation ([15], Theorem 1) still holds under the assumption that $\left\{\varepsilon_k^{(n)}, k \geq 1\right\}$ satisfies ϕ-mixing condition (see also [7]-[10], [19], [20]).

The structure of our paper is as follows. Theorem 1 shows that scaled random process (1) weakly converges to a deterministic (non-random) function when immigration satisfies ρ-mixing condition. Note that Rahimov [16] established a deterministic approximation for (1) a proof of which is based on convergence of random step processes towards a diffusion process. Since in Theorem 1 the limiting process is a deterministic, we demonstrate that it can be proven using the classical scheme based on checking the convergence of finite-dimensional distributions and tightness. The next result deals with a FLT for (1). Under assumption that immigration process is m-dependent, we prove a FLT for the fluctuation of (1) (see Theorem 2). To prove Theorem 2, we use the FLT for arrays of martingale differences from [6]. Here, m may tend to infinity with the row index at a certain rate. The use of m-dependence in Theorem 2, instead of more classical weakly dependence notions relying on the decay rate of mixing coefficients for instance, is motivated by purely technical reason. This reason is that the sequence $\left\{\eta_k^{(n)}, k \geq 1\right\}$ defined in Lemma 1 (see Section 2) is also m-dependent as soon as $\left\{\varepsilon_k^{(n)}, k \geq 1\right\}$ is m-dependent for each $n \geq 1$.

At first, we need some notations and conditions in order to introduce our results.

We recall that a function $f : (0, \infty) \to (0, \infty)$ is called regularly varying at infinity if it can be represented in the form $f(x) = x^\rho l(x)$, where $\rho \in \mathbb{R}$ is called index of regular variation and $l(x)$ is a slowly varying function. If a sequence $\{f(n), n \geq 1\}$ is regularly varying with exponent ρ, we will write $\{f(n), n \geq 1\} \in R_\rho$.

Assume that for each $n \geq 1$, the variables $a_n = \mathbb{E}\xi_{1,1}^{(n)}$, $b_n := Var\left(\xi_{1,1}^{(n)}\right)$ are finite. We also assume that $\alpha(n,k) := \mathbb{E}\varepsilon_k^{(n)}$ and $\beta(n,k) := Var\left(\varepsilon_k^{(n)}\right)$ are finite for all $n, k \geq 1$. For each $n \geq 1$, let $\mathfrak{F}_k^{(n)}$ be the σ-algebra generated by $X_0^{(n)}, X_1^{(n)}, \ldots, X_k^{(n)}$.

Denote $A_n(k) = \mathbb{E}X_k^{(n)}$ and $B_n^2(k) = Var\left(X_k^{(n)}\right)$, $1 \leq k \leq n$, $n \geq 1$. By (1) and from Lemma 1 in [3], we have

$$A_n(k) = \sum_{j=1}^{k} a_n^{k-j}\alpha(n,j), \quad B_n^2(k) = \Delta_n^2(k) + \widetilde{\sigma}_n^2(k), \tag{2}$$

where

$$\Delta_n^2(k) = \frac{b_n}{1-a_n}\sum_{j=1}^{k}\alpha(n,j)\,a_n^{k-j-1}\left(1-a_n^{k-j}\right), \quad \widetilde{\sigma}_n^2(k) = \sigma_n^2(k) + 2\omega_n(k),$$

$$\sigma_n^2(k) = \sum_{j=1}^{k}\beta(n,j)\,a_n^{2(k-j)}, \quad \omega_n(k) = \sum_{j=2}^{k}\sum_{i=1}^{j-1}\operatorname{cov}\left(\varepsilon_j^{(n)}, \varepsilon_i^{(n)}\right)a_n^{2k-j-i}.$$

Note that in critical case the formula (2) coincides with the formula (2.1) in [4] and when immigration sequence is independent with formula (2.2) in [16].

We use the same time change functions as in [16] (see (2.3)):

$$\mu_\alpha(t) = \int_0^t u^\alpha e^{a(t-u)}du,$$

$$\nu_\alpha(t) = \int_0^t u^\alpha e^{a(t-u)}\left(1-e^{a(t-u)}\right)du, \quad \lambda_\beta(t) = \int_0^t u^\beta e^{2a(t-u)}du,$$

$$\varphi(t) = \begin{cases} t^{2+\alpha}, & a = 0, \\ \frac{a}{\nu_\alpha(1)}\int_0^t \mu_\alpha(u)\,e^{2a(t-u)}du, & a \neq 0. \end{cases} \tag{3}$$

Note that $\mu_\alpha(t) = \frac{t^{\alpha+1}}{\alpha+1}$ when $a = 0$ and $\lim\limits_{a\to 0}\frac{\nu_\alpha(t)}{a} = \frac{t^{\alpha+2}}{(\alpha+1)(\alpha+2)}$.

Let $\{x(n), n \geq 1\}$ and $\{y(n), n \geq 1\}$ be sequences of positive numbers. We use notation $x(n) \sim y(n)$ to denote that $\lim_{n\to\infty}\frac{x(n)}{y(n)} = 1$. The symbol $x(n) \lesssim y(n)$ means that there exist $C \in (0,\infty)$, $n_0 \in \mathbb{N}$, such that $x_n \leq Cy_n$ for all $n \geq n_0$. For the convenience of further consideration, let us accept the following conventions. Assume that all expressions of the form $x(n) \to x$, $x(n) \sim y(n)$, which appear in the sequel, are true as $n \to \infty$. For simplicity, we write $A(n) = A_n(n)$, $B^2(n) = B_n^2(n)$, $\Delta^2(n) = \Delta_n^2(n)$, $\sigma^2(n) = \sigma_n^2(n)$, $\omega(n) = \omega_n(n)$ when $k = n$. We also use the notation $\mathbb{P}$-a.s. if some relation holds almost surely; $\min(x,y) := x \wedge y$, $x, y \in \mathbb{R}$. In this paper, the symbols $\stackrel{D}{\to}$ and $\stackrel{P}{\to}$ denote the convergence of random functions in Skorokhod space $D[0,+\infty)$ with J_1-topology and convergence in probability, respectively. The indicator function of A is denoted by $I\{A\}$. In the sequel, the symbol C will denote a generic constant $(0 < C < \infty)$ which is not necessarily the same one in each appearance.

We shall make use of the following conditions.

(C1): There are sequences $\{\alpha(k), k \geq 1\} \in R_\alpha$, $\{\beta(k), k \geq 1\} \in R_\beta$ with $\alpha, \beta \geq 0$, such that, as $n \to \infty$ for each $s \geq 0$,

$$\max_{1 \leq k \leq ns} |\alpha(n,k) - \alpha(k)| = o(\alpha(n)), \quad \max_{1 \leq k \leq ns} |\beta(n,k) - \beta(k)| = o(\beta(n));$$

(C2): $a_n = 1 + an^{-1} + o(n^{-1})$ as $n \to \infty$ for some $a \in \mathbb{R}$;
(C3): $b_n = o(\alpha(n))$, $n \to \infty$;
(C4): $\alpha(n) \to \infty$, $\beta(n) = o(n\alpha^2(n))$, $n \to \infty$.
(C5): $\alpha(n) \to \infty$ and $m \to \infty$ such that $m\beta(n) = o(n\alpha(n) b_n)$; $\liminf_{n\to\infty} b_n > 0$.

Detailed discussion of conditions (C1)-(C5) are performed in [16]. Here, we replaced condition $\beta(n) = o(n\alpha(n) b_n)$ which appeared in [16] by (C5).

For $t \geq 0$, define random processes

$$X_n(t) = \frac{X^{(n)}_{[nt]}}{A(n)}, \quad Z_n(t) = \frac{X^{(n)}_{[nt]} - A_n([nt])}{B(n)}, \quad n \geq 1, \tag{4}$$

where $[x]$ denotes the integer part of nonnegative real number x.

Our first result reads as follows.

Theorem 1. *Let $\left\{\varepsilon^{(n)}_k, n,k \geq 1\right\}$ be an array of rowwise ρ-mixing random variables satisfying $\sup_{n\geq 1} \sum_{k=1}^{\infty} \rho_n(2^k) < \infty$. If conditions (C1)-(C4) hold, then*

$$\{X_n(t), t \geq 0\} \xrightarrow{D} \{\pi_\alpha(t), t \geq 0\}, \quad n \to \infty \tag{5}$$

in Skorokhod space $D[0,+\infty)$, where $\pi_\alpha(t) = \mu_\alpha(t)/\mu_\alpha(1)$, $t \geq 0$.

Remark 1. *Theorem 1 extends the corresponding results of* [16], [18] *to the case of arrays of rowwise ρ-mixing sequence. More precisely, Rahimov* [16] *obtained* (7) *when $\left\{\varepsilon^{(n)}_k, n,k \geq 1\right\}$ is a rowwise independent with conditions (C1)-(C4). In* [18], *it is considered the case when $\left\{\varepsilon^{(n)}_k, n,k \geq 1\right\}$ is ϕ-mixing with $\sum_{k=1}^{\infty} \phi^{1/2}(k) < \infty$ and established* (5).

It is natural that the examples and Corollaries 2.1-2.3 from [16] related to the maximum and the total progeny of the process remain true in the case of dependent immigration. We here provide one more example of application of Theorem 1.

Example 1. *Let $\xi^{(n)}_{1,1}$ are Bernoulli random variables with the probability of success $1 - an^{-1}$, where $a > 0$. Assume that for each $n \geq 1$, $\left\{\varepsilon^{(n)}_k, k \geq 1\right\}$ is rowwise ρ-mixing sequence of Poisson distributed random variables with mean $\alpha(n,k) = \alpha(k)(1 + x(n))$, where $\alpha(k) \in R_\alpha$, $x(n) \to 0$, $n \to \infty$ and $\sup_{n\geq 1} \sum_{k=1}^{\infty} \rho_n(2^k) < \infty$. Then, $\beta(n,k) = \alpha(n,k)$ and condition (C1) is fulfilled with $\alpha(n) = \beta(n)$. Moreover, it is easily seen that conditions (C2)-(C4) are also satisfied. Hence, we may apply the statement of Theorem 1.*

Theorem 2. *Let for each $n \geq 1$, $\left\{\varepsilon^{(n)}_k, k \geq 1\right\}$ be a sequence of m-dependent random variables. Assume conditions (C1)-(C3) and (C5) hold and for any $\varepsilon > 0$,*

$$\mathbb{E}\left(\left(\xi^{(n)}_{1,1} - a_n\right)^2 I\left\{\left|\xi^{(n)}_{1,1} - a_n\right| > \varepsilon B(n)\right\}\right) \to 0, \quad n \to \infty. \tag{6}$$

Then

$$\{Z_n(t), t \geq 0\} \xrightarrow{D} \{W(\varphi(t)), t \geq 0\}, \quad n \to \infty \tag{7}$$

in Skorokhod space $D[0,+\infty)$, where $\{W(t), t\geq 0\}$ is the standard Brownian motion and $\varphi(t)$ is defined by (3)*.*

Remark 2. *In fact, condition* (6) *is the Lindeberg condition on offspring distributions of process* (1)*. Condition* (6) *is satisfied if $B^{-\tau}(n)\,\mathbb{E}|\xi_{1,1}^{(n)}-a_n|^{2+\tau}\to 0$ as $n\to\infty$ for some $\tau>0$ (see Remark 2.2 in* [16]*).*

Remark 3. *When $\varepsilon_k^{(n)}$ are independent and conditions (C1)-(C3) and $\alpha(n)\to\infty$, $\beta(n)=o(n\alpha(n)\,b_n)$ are fulfilled, Rahimov* [16] *obtained* (7)*. We replace condition $\beta(n)=o(n\alpha(n)\,b_n)$ by $m\beta(n)=o(n\alpha(n)\,b_n)$ which is stronger than the latter one, however, it is a natural in the context of rowwise m-dependence. Hence, Theorem 2 is an improvement for dependent immigration process.*

Example 2. *Let $\xi_{1,1}^{(n)}$ has the following distribution: $\mathbb{P}\left(\xi_{1,1}^{(n)}=n\right)=\alpha n^{-2}$, $\mathbb{P}\left(\xi_{1,1}^{(n)}=d_n\right)=d_n^{-1}$, $\mathbb{P}\left(\xi_{1,1}^{(n)}=0\right)=1-d_n^{-1}-\alpha n^{-2}$, where $\alpha\geq 0$ and $\{d_n, n\geq 1\}$ is a sequence of real numbers such that $d_n=O(n)$, $n\to\infty$. It is easily seen that $a_n=1+\alpha n^{-1}$, $b_n\sim d_n$, $n\to\infty$. Suppose that $\varepsilon_k^{(n)}$ is a rowwise m-dependent with $m=o(n)$ as $n\to\infty$ and distributed as $\mathbb{P}\left(\varepsilon_k^{(n)}=[k\ln^2 k]\right)=\frac{1}{\ln k}(1+x(n))$, $\mathbb{P}\left(\varepsilon_k^{(n)}=0\right)=1-\frac{1}{\ln k}(1+x(n))$, $n\geq 1$, $k\geq 2$. Clearly, condition (C1) holds with $\alpha(n)=n\ln n$, $\beta(n)=n^2\ln^3 n$ and conditions (C2)-(C3), (C5) are also satisfied. Condition* (6) *holds due to fact that for each fixed $\varepsilon>0$ and sufficiently large n, the set $\left\{\left|\xi_{1,1}^{(n)}-a_n\right|>\varepsilon B(n)\right\}$ is empty. Consequently, all conditions of Theorem 2 are fulfilled and it remains to apply the statement of Theorem 2.*

2. Auxiliary Results

In this section, we give some lemmas which will be used to prove our main results. We obtain from (1) that

$$\mathbb{E}\left(X_k^{(n)}\middle|\mathfrak{F}_{k-1}^{(n)}\right)=a_nX_{k-1}^{(n)}+\mathbb{E}\left(\varepsilon_k^{(n)}\middle|\mathfrak{F}_{k-1}^{(n)}\right).$$

Note that the sequence $\left\{M_k^{(n)}, k\geq 1\right\}$ for each $n\geq 1$, defined as

$$M_k^{(n)}:=X_k^{(n)}-\mathbb{E}\left(X_k^{(n)}\middle|\mathfrak{F}_{k-1}^{(n)}\right)=X_k^{(n)}-a_nX_{k-1}^{(n)}-\mathbb{E}\left(\varepsilon_k^{(n)}\middle|\mathfrak{F}_{k-1}^{(n)}\right)$$

is a martingale difference sequence with respect to the σ-algebra $\mathfrak{F}_k^{(n)}$, $k\geq 0$.

Thus,

$$M_k^{(n)}=T_k^{(n)}+N_k^{(n)},$$

where

$$T_k^{(n)}=\sum_{j=1}^{X_{k-1}^{(n)}}\left(\xi_{k,j}^{(n)}-a_n\right),\quad N_k^{(n)}=\varepsilon_k^{(n)}-\mathbb{E}\left(\varepsilon_k^{(n)}\middle|\mathfrak{F}_{k-1}^{(n)}\right). \tag{8}$$

The process $Z_n(t)$ given by (4) can be represented as

$$a_n^{-[nt]}Z_n(t)=Z_n^{(1)}(t)+Z_n^{(2)}(t), \tag{9}$$

where

$$Z_n^{(1)}(t)=\frac{1}{B(n)}\sum_{k=1}^{[nt]}a_n^{-k}M_k^{(n)},$$

$$Z_n^{(2)}(t)=\frac{1}{B(n)}\sum_{k=1}^{[nt]}a_n^{-k}\left(\mathbb{E}\left(\varepsilon_k^{(n)}\left|\mathfrak{F}_{k-1}^{(n)}\right.\right)-\alpha(n,k)\right).$$

In the following lemma we will prove that $Z_n^{(2)}(t)$ is asymptotically negligible in L^2-sense uniformly for all $t\in[0,T]$, $T>0$.

Lemma 1. *Assume for each $n\geq 1$, $\left\{\varepsilon_k^{(n)},\ k\geq 1\right\}$ be a sequence of m-dependent random variables. If conditions (C1)-(C3) and (C5) hold, then for each $T>0$,*

$$\sup_{0\leq t\leq T}\left|Z_n^{(2)}(t)\right|\xrightarrow{L^2}0.$$

Proof. If we denote $\eta_k^{(n)}=\mathbb{E}\left(\varepsilon_k^{(n)}\left|\mathfrak{F}_{k-1}^{(n)}\right.\right)-\alpha(n,k),\ k\geq 1$, then for each $n\geq 1$, the sequence $\left\{\eta_k^{(n)},k\geq 1\right\}$ also defines m-dependent random sequence. Obviously, $\mathbb{E}\left(\eta_k^{(n)}\right)^2=Var\left(\mathbb{E}\left(\varepsilon_k^{(n)}\left|\mathfrak{F}_{k-1}^{(n)}\right.\right)\right)$. By the Cauchy–Schwarz inequality and taking into account the inequality $Var\left(\mathbb{E}\left(\varepsilon_k^{(n)}\left|\mathfrak{F}_{k-1}^{(n)}\right.\right)\right)\leq\beta(n,k)$, we have

$$\mathbb{E}\left(Z_n^{(2)}(t)\right)^2\leq\frac{1}{B^2(n)}\sum_{k=1}^{[nt]}a_n^{-2k}Var\left(\mathbb{E}\left(\varepsilon_k^{(n)}\left|\mathfrak{F}_{k-1}^{(n)}\right.\right)\right)+$$

$$+\frac{2}{B^2(n)}\sum_{k=1}^{[nt]-1}\sum_{j=k+1}^{(k+m)\wedge[nt]}a_n^{-k}a_n^{-j}\sqrt{\mathbb{E}\left(\eta_k^{(n)}\right)^2}\sqrt{\mathbb{E}\left(\eta_j^{(n)}\right)^2}$$

$$\leq\frac{Cm}{B^2(n)}\sum_{k=1}^{[nt]}a_n^{-2k}\beta(n,k).\tag{10}$$

Note that the term in (10) can be rewritten as

$$\frac{Cm}{B^2(n)}\sum_{k=1}^{[nt]}a_n^{-2k}\left(\beta(n,k)-\beta(k)\right)+\frac{Cm}{B^2(n)}\sum_{k=1}^{[nt]}a_n^{-2k}\beta(k),$$

and the latter relation is dominated by

$$\frac{mn\beta(n)}{B^2(n)}\frac{1}{\beta(n)}\max_{1\leq k\leq nt}\left|\left(\beta(n,k)-\beta(k)\right)\right|\frac{1}{n}\sum_{k=1}^{[nt]}a_n^{-2k}+\frac{mn\beta(n)}{B^2(n)}\frac{1}{n}\sum_{k=1}^{[nt]}a_n^{-2k},$$

which due to conditions (C1)-(C4) vanishes to zero. Lemma 1 is proved. □

Further, for each $t\in[0,T]$, $T>0$, let $\Im_t^{(n)}$ be the σ-algebra generated by $\left\{Z_n^{(2)}(s):\ s\leq t\right\}$. Using similar arguments as in the proof of Lemma 1, it follows that

$$\mathbb{E}\left(\left|Z_n^{(2)}(t+\theta)-Z_n^{(2)}(t)\right|^2\left|\Im_t^{(n)}\right.\right)\leq\mathbb{E}\left(Q_n(t,\theta)\left|\Im_t^n\right.\right),\quad\mathbb{P}-a.s.$$

where $\theta \geq 0$ and

$$Q_n(t,\theta) = \frac{Cm}{B^2(n)} \sum_{j=[nt]+1}^{[nt+n\theta]} \left(\mathbb{E}\left(\varepsilon_j^{(n)} - \alpha(n,j) \middle| \mathfrak{F}_{j-1}^{(n)} \right) \right)^2 .$$

Clearly,

$$Q_n(t,\theta) \leq Q_n(\theta) := \sup_{0 \leq t \leq T} Q_n(t,\theta), \quad \mathbb{P} - a.s.,$$

and by Lemma 1, it yields that

$$\sup_{n \geq 1} \frac{Cm \sum_{k=1}^{[nT+n\theta]} a_n^{-2k} \beta(n,k)}{B^2(n)} < \infty.$$

By setting

$$G_n(\theta) = \frac{Cm}{B^2(n)} \sum_{k=1}^{[nT+n\theta]} \left(\mathbb{E}\left(\varepsilon_k^{(n)} - \alpha(n,k) \middle| \mathfrak{F}_{k-1}^{(n)} \right) \right)^2 ,$$

we obtain

$$\mathbb{E}(G_n(\theta)) \leq \sup_{n \geq 1} \frac{Cm \sum_{k=1}^{[nT+n\theta]} a_n^{-2k} \beta(n,k)}{B^2(n)} < \infty,$$

and therefore, we get $Q_n(\theta) \leq G_n(\theta) < \infty, \mathbb{P}$-a.s.

Lemma 2. *Under assumptions of Theorem 2, it holds*

1) For each $T > 0$ and $\eta > 0$, there exists $\delta > 0$ such that

$$\sup_{n \geq 1} \mathbb{P}\left(\sup_{0 \leq t \leq T} \left| Z_n^{(2)}(t) \right| > \delta \right) \leq \eta.$$

2)

$$\lim_{\theta \to 0} \sup_{n \geq 1} \mathbb{E}(Q_n(\theta)) = 0.$$

Proof. 1) From Chebyshev's inequality and (13), we get

$$\sup_{n \geq 1} \mathbb{P}\left(\sup_{0 \leq t \leq T} \left| Z_n^{(2)}(t) \right| > \delta \right) \leq \frac{1}{\delta^2} \sup_{n \geq 1} \frac{Cm \sum_{k=1}^{[nT]} a_n^{-2k} \beta(n,k)}{B^2(n)}. \tag{11}$$

If we denote the right hand side of (11) by M, then, by Lemma 1, it follows that $M < \infty$. Now, for each fixed η and t, it remains to choose δ such that $\delta \geq (M/\eta)^{1/2}$. Thus, we have proved claim 1).

2) Since $\sup_{0<\theta<1} \mathbb{E}(Q_n(\theta)) \leq \mathbb{E}(G_n(1)) \to 0$, then, for each $\varepsilon > 0$, there exists $N > 0$ such that for all $\theta \in (0,1)$, $n > N$, $\mathbb{E}(Q_n(\theta)) < \varepsilon$. While for all $n \leq N$ observing that $\mathbb{E}(Q_n(\theta)) \to 0$ when $\theta \to 0$, one may choose $\theta \in (0,1)$ such that for $\theta \leq \theta_0$ and all $k = 1, 2, \ldots, N$, it yields $\mathbb{E}(Q_k(\theta)) < \varepsilon$. Thus, we have $\sup_n \mathbb{E}(Q_n(\theta)) < \varepsilon$ for $\theta \leq \theta_0$ which proves claim 2).

Lemma 2 is proved. □

Lemma 3. *The set of probability distributions of the process $\left\{ Z_n^{(2)}(t), t \geq 0 \right\}$ is relatively compact in $D[0,+\infty)$.*

Proof. From Lemma 2 we see that conditions (a) and (b) in [2] (see Theorem 8.6) are fulfilled. Lemma 3 is proved. □

Now we provide some lemmas which are taken from [16].

Lemma 4. *Assume* $\{x(n), n \geq 1\} \in R_\rho$ *and condition (C2) holds. Then for each fixed* $T > 0$ *and for all* $\rho \geq 0$, $\theta \in \mathbb{R}$,

$$\sup_{0 \leq s \leq T} \left| \frac{1}{nx(n)} \sum_{k=1}^{[ns]} a_n^{k\theta} x(k) - \int_0^s t^\rho e^{t\theta a} dt \right| \to 0.$$

Lemma 5. *If conditions (C1) and (C2) hold, then uniformly in* $s \in [0, T]$ *for each fixed* $T > 0$,

1) $\lim\limits_{n\to\infty} \frac{A_n([ns])}{n\alpha(n)} = \mu_\alpha(s)$, $\lim\limits_{n\to\infty} \frac{\sigma_n^2([ns])}{n\beta(n)} = \mu_\beta(s)$,

2) $\lim\limits_{n\to\infty} \frac{\Delta_n^2([ns])}{n^2\alpha(n)b_n} = \begin{cases} \nu_\alpha(s)/a, & if\, a \neq 0, \\ s^{\alpha+2}\big/(\alpha+1)(\alpha+2), & if\, a = 0. \end{cases}$

Lemma 6. *If conditions (C1) and (C2) hold, then uniformly in* $s \in [0, T]$ *for any* $\theta \in \mathbb{R}$, $s \geq 0$,

1) $\lim\limits_{n\to\infty} \frac{1}{n^3\alpha(n)b_n} \sum_{i=1}^{[ns]} a_n^{\theta i} \Delta_n^2(i) = \begin{cases} (1/a)\int_0^s e^{u\theta a}\nu_\alpha(u)\,du, & if\, a \neq 0, \\ s^{\alpha+3}\big/(\alpha+1)(\alpha+2)(\alpha+3), & if\, a = 0, \end{cases}$

2) $\lim\limits_{n\to\infty} \frac{1}{n^2\beta(n)} \sum_{i=1}^{[ns]} a_n^{\theta i} \sigma_n^2(i) = \int_0^s e^{u\theta a}\mu_\beta(u)\,du$,

3) $\lim\limits_{n\to\infty} \frac{1}{n^2\alpha(n)} \sum_{i=1}^{[ns]} a_n^{\theta i} A_n(i) = \int_0^s e^{u\theta a}\mu_\alpha(u)\,du$.

Lemma 7. *Let* $\left\{\xi_{k,i}^{(n)}, k, i, n \geq 1\right\}$ *be a triangular array of random variables defined in* (1) *and* $T_k^{(n)}$ *is defined by* (8). *Then,*

1) $\mathbb{E}\left(\left(T_k^{(n)}\right)^2 \middle| \mathfrak{F}_{k-1}^{(n)}\right) = b_n X_{k-1}^{(n)}$,

2) $\mathbb{E}\left(\left(\sum_{1\leq i\neq j\leq X_{k-1}^{(n)}} \left(\xi_{k,i}^{(n)} - a_n\right)\left(\xi_{k,j}^{(n)} - a_n\right)\right)^2 \middle| \mathfrak{F}_{k-1}^{(n)}\right) = 2b_n^2 X_{k-1}^{(n)} \left(X_{k-1}^{(n)} - 1\right)$.

Lemma 8. *For the variable* $T_n(k)$ *defined in* (10) *and for all* $\varepsilon > 0$,

$$\mathbb{E}\left(T_n^2(k)\, I(|T_n(k)| > \varepsilon) \middle| \mathfrak{F}_{k-1}^{(n)}\right) \leq I_1 + I_2, \tag{12}$$

where

$$I_1 = X_{k-1}^{(n)} \mathbb{E}\left(\left(\xi_{k,i}^{(n)} - a_n\right)^2 I\left(\left|\xi_{k,i}^{(n)} - a_n\right| > \varepsilon/2\right)\right),$$

$$I_2 = 4b_n\varepsilon^{-2}\left(X_{k-1}^{(n)}\right)^2 + \sqrt{2b_n}\,\varepsilon^{-1}\left(X_{k-1}^{(n)}\right)^{3/2}.$$

The next lemma provides the moment inequality for sums of ρ-mixing random variables which comes from [21].

Lemma 9. *Let* $\{\xi_i, i \geq 1\}$ *be a sequence of* ρ*-mixing random variables with* $\mathbb{E}\xi_i = 0$, $\mathbb{E}\xi_i^2 < \infty$, *and* $\sum_{i=1}^\infty \rho(2^i) < \infty$. *Then there exists an absolute constant* K *such that*

$$\mathbb{E}\left|\sum_{k=1}^n \xi_k\right|^2 \leq \exp\left\{K\left(1 + \sum_{i=1}^\infty \rho\left(2^i\right)\right)\right\} \sum_{k=1}^n \mathbf{E}\xi_k^2. \tag{13}$$

In the proofs we need the following FLT from [6] (see Theorem VIII.3.33).
Theorem A. Let for each $n \geq 1$, $\{U_k^n, k \geq 1\}$ be a sequence of martingale difference

with respect to some filtration $\{\Re_k^n, k \geq 1\}$, such that the conditional Lindeberg condition:

$$\sum_{k=1}^{[nt]} \mathbb{E}\left((U_k^n)^2 I\{|U_k^n| > \varepsilon\} \middle| \Re_{k-1}^{(n)}\right) \xrightarrow{P} 0, \quad n \to \infty \tag{14}$$

holds for all $\varepsilon > 0$ and $t \geq 0$. Then

$$\sum_{k=1}^{[nt]} U_k^n \xrightarrow{D} U(t), \quad n \to \infty$$

in Skorokhod space $D[0, +\infty)$, where $U(t)$ is a continuous Gaussian martingale with mean zero and covariance function $C(t)$, $t \geq 0$, if and only if

$$\sum_{k=1}^{[nt]} \mathbb{E}\left((U_k^n)^2 \middle| \Re_{k-1}^n\right) \xrightarrow{P} C(t), \quad n \to \infty \tag{15}$$

for all $t \geq 0$.

3. Proofs of the main theorems

In this section, we provide the proofs of our main results.

Proof of Theorem 1. We denote by $\left\{Y_{n,i}^j(k), k \geq i\right\}$, $1 \leq j \leq \varepsilon_i^{(n)}$-the Galton-Watson branching process generated by j-th particle arriving at the moment i in the n-th series with $Y_{n,i}^j(i) = 1$. From our assumptions it follows that for each $n \geq 1$ processes $\left\{Y_{n,i}^j(k), k \geqslant i\right\}$, $i, j \geqslant 1$ are independent; $Y_{n,i}^j(k+i)$, $k \geq 0$ has the same distribution as $Y_{n,1}^1(k)$, $k \geq 1$. Denote

$$f_{n,k}(z) = \mathbb{E}e^{izY_{n,1}^1(k)}, \quad \Psi_n(z,t) = \mathbb{E}e^{izX_{[nt]}^{(n)}}, \quad z \in \mathbb{R}, \quad t \geq 0.$$

The form of $\Psi_n(z,t)$ is given by relation (3) in [13].

It is known that in order to prove (7), it suffices to verify it for each finite interval. Therefore, let us fix some $T > 0$ and denote by $\{X_n(t), t \geq 0\} \overset{D(T)}{\to} \{\pi_\alpha(t), t \geq 0\}$ the weak convergence of distribution $X_n(t)$ to distribution of $\pi_\alpha(t)$ in $D[0,T]$.

We only consider the case $a \neq 0$. Then, it is well-known that

$$\mathbb{E}Y_{n,1}^1(k) = a_n^k, \quad \mathbb{E}\big(Y_{n,1}^1(k)\big)^2 = \frac{a_n^{k-1}\left(a_n^k - 1\right)}{a_n - 1} b_n + a_n^{2k}. \tag{16}$$

Since the assumption $b_n < \infty$, $n \geq 1$ is equivalent to $\mathbb{E}\big(Y_{n,1}^1(k)\big)^2 < \infty$, we see that the Taylor series expansion is valid for the characteristic function $f_{n,k}(z)$:

$$f_{n,k}(z) = 1 + iz\mathbb{E}Y_{n,1}^1(k) - \frac{z^2}{2}\mathbb{E}\big(Y_{n,1}^1(k)\big)^2 + \frac{z^2}{2}\tau_{n,k}(z), \quad k, n \geq 1, \tag{17}$$

where $\tau_{n,k}(z)$ is the remainder term and $|\tau_{n,k}(z)| \leq 3\mathbb{E}\big(Y_{n,1}^1(k)\big)^2$, $\tau_{n,k}(z) \to 0$ as $z \to 0$. Now using decomposition $\ln x = x - 1 + O\left((x-1)^2\right)$, $x \to 1$ and from relations (16), (17), we obtain $\mathbb{P}$-a.s.

$$\ln \prod_{k=1}^{[nt]} f_{n,[nt]-k}^{\varepsilon_k^{(n)}}\left(\frac{z}{A(n)}\right) = izI_n^{(1)}(t) - \frac{z^2}{2}\left(I_n^{(2)}(t) + I_n^{(3)}(t) + I_n^{(4)}(t)\right), \tag{18}$$

where

$$I_n^{(1)}(t) = \frac{1}{A(n)} \sum_{k=1}^{[nt]} a_n^{[nt]-k} \varepsilon_k^{(n)},$$

$$I_n^{(2)}(t) = \frac{1}{A^2(n)} \sum_{k=1}^{[nt]} \varepsilon_k^{(n)} \mathbb{E}\big(Y_{n,1}^1([nt]-k)\big)^2,$$

$$I_n^{(3)}(t) = \frac{1}{A^2(n)} \sum_{k=1}^{[nt]} \varepsilon_k^{(n)} \tau_{n,[nt]-k}\left(\frac{z}{A(n)}\right), \quad I_n^{(4)}(t) = \frac{1}{A^2(n)} \sum_{k=1}^{[nt]} a_n^{2([nt]-k)} \varepsilon_k^{(n)}.$$

We treat each term of relation (18) separately. Let us start with $I_n^{(1)}(t)$.

$$\mathbb{E}I_n^{(1)}(t) = \frac{1}{A(n)} \sum_{k=1}^{[nt]} a_n^{[nt]-k} (\alpha(n,k) - \alpha(k)) + \frac{1}{A(n)} \sum_{k=1}^{[nt]} a_n^{[nt]-k} \alpha(k). \tag{19}$$

From [16], we know that the first term in (19) tends to zero and the second term converges to $\pi_\alpha(t)$ uniformly in $t \in [0,T]$ for each fixed $T > 0$. Thus, we derive that $\mathbb{E}I_n^{(1)}(t) \to \pi_\alpha(t)$ as $n \to \infty$ uniformly in $t \geq 0$.

Now consider the variance of $I_n^{(1)}(t)$. Let us consider a sequence $\left\{\zeta_k^{(n)}, k, n \geq 1\right\}$ where $\zeta_k^{(n)} = a_n^{[nt]-k} \varepsilon_k^{(n)}$, $k, n \geq 1$. We see that both sequences $\left\{\varepsilon_k^{(n)}, k \geq 1\right\}$ and $\left\{\zeta_k^{(n)}, k, n \geq 1\right\}$ generate the same σ-algebras. Thus, $\left\{\zeta_k^{(n)}, k, n \geq 1\right\}$ is also ρ-mixing with $\mathbb{E}\zeta_k^{(n)} = a_n^{[nt]-k} \alpha(n,k)$. Consequently, by (13), we have

$$\frac{1}{A^2(n)} \mathbb{E}\left(\sum_{k=1}^{[nt]} \zeta_k^{(n)}\right)^2 \leq \frac{C}{A^2(n)} \sum_{j=1}^{[nt]} \mathbb{E}\left(\zeta_j^{(n)}\right)^2$$

$$\leq \frac{C}{A^2(n)} \sum_{j=1}^{[nt]} a_n^{2([nt]-j)} \beta(n,j) + \frac{C}{A^2(n)} \sum_{j=1}^{[nt]} a_n^{2([nt]-j)} \alpha^2(n,j). \tag{20}$$

It was shown in [16] (see pp. 362-363) that under conditions $n\beta(n) = o\big(A^2(n)\big)$ and (C1) both terms of (20) converge to zero as n tends to infinity.

Combining together the above bounds, we have uniformly in $t \geq 0$,

$$I_n^{(1)}(t) \xrightarrow{P} \pi_\alpha(t). \tag{21}$$

Consider $I_n^{(2)}(t)$. Applying Lemmas 4-5 and using the fact that $A(n) \sim Cn\alpha(n)$, we derive that

$$\mathbb{E}I_n^{(2)}(t) = \frac{\Delta_n^2([nt])}{A^2(n)} + \frac{1}{A^2(n)} \sum_{j=1}^{[nt]} a_n^{2([nt]-j)} \alpha(n,j) \to 0.$$

Since $I_n^{(2)}(t) \geq 0$ $\mathbb{P}$-a.s., we have

$$I_n^{(2)}(t) \xrightarrow{P} 0. \tag{22}$$

Using similar arguments as in the proof of $I_n^{(2)}(t)$ and from the inequality $|\tau_{n,k}(z)| \le 3\mathbb{E}\big(Y_{n,1}^1(k)\big)^2$, one establishes that

$$\mathbb{E}\left|I_n^{(3)}(t)\right| \le \sum_{k=1}^{[nt]} \alpha(n,k)\left|\tau_{n,[nt]-k}\left(\frac{z}{A(n)}\right)\right|$$

$$\lesssim \frac{1}{A^2(n)}\Delta_n^2([nt]) + \frac{1}{A^2(n)}\sum_{k=1}^{[nt]} a_n^{2([nt]-k)}\alpha(n,k) \to 0.$$

Thus by Chebyshev's inequality, we deduce that

$$I_n^{(3)}(t) \xrightarrow{P} 0. \tag{23}$$

Similarly to the proof of (22), we find that $I_n^{(4)}(t)$ converges to zero in probability.

From the above and (18), (21)-(23), it follows

$$\ln\prod_{k=1}^{[nt]} f_{n,[nt]-k}^{\varepsilon_k^{(n)}}\left(\frac{z}{A(n)}\right) \xrightarrow{P} iz\pi_\alpha(t), \quad t \ge 0.$$

Consequently, the application of Lebesgue dominated convergence theorem gives us

$$\Psi_n\left(\frac{z}{A(n)}, t\right) \to e^{iz\pi_\alpha(t)}, \quad t \ge 0.$$

Hence, uniformly with respect to $t \ge 0$,

$$X_n(t) \xrightarrow{P} \pi_\alpha(t), \quad n \to \infty. \tag{24}$$

Due to the fact that the limiting distribution in (24) is a degenerate, the convergence of the finite-dimensional distributions follows from (24). Hence, the finite-dimensional distributions of random process $\{X_n(t), t \in [0,T]\}$ converge in probability to finite-dimensional distributions of $\{\pi_\alpha(t), t \ge 0\}$.

It remains to prove the tightness of $\{X_n(t), t \in [0,T]\}$ in $D[0,T]$. We will prove that for sufficiently large n and $t_1 \le t \le t_2$,

$$\mathbb{E}\left(|X_n(t) - X_n(t_1)|\,|X_n(t_2) - X_n(t)|\right) \le C(t_2 - t_1)^2, \tag{25}$$

which is sufficient for tightness (see Theorem 13.5 in [1]). We use the following inequality

$$\mathbb{E}(X_n(t) - X_n(s))^2 \le K_n^{(1)}(t,s) + K_n^{(2)}(t,s),$$

where $0 < s < t \le T$ and

$$K_n^{(1)}(t,s) = \frac{3}{A^2(n)}\left(B_n^2([nt]) + B_n^2([ns])\right),$$

$$K_n^{(2)}(t,s) = \frac{3}{A^2(n)}\left((A_n([nt]) - A_n([ns]))^2\right).$$

From conditions of Theorem 1, for sufficiently large n and $0 < s < t \le T$, we obtain

$$K_n^{(1)}(t,s) + K_n^{(2)}(t,s) \le \frac{Cb_n}{\alpha(n)}(\nu_\alpha(t) + \nu_\alpha(s))$$

$$+\frac{C}{n^2\alpha^2(n)}\sup_{n\ge 1}\sum_{i=1}^{\infty}\rho_n(2^i)\sum_{k=1}^{[nt]} a_n^{2(n-k)}\beta(n,k)$$

$$+\frac{C}{n^2\alpha^2(n)}\sup_{n\geq 1}\sum_{i=1}^{\infty}\rho_n\left(2^i\right)\sum_{k=1}^{[ns]}a_n^{2(n-k)}\beta(n,k)+T^{2\alpha}(t-s)^2,$$

which implies (25). This completes the proof of Theorem 1.

In the proof of next theorem we use (9) and divide the proof of Theorem 2 into two propositions, which together will imply our result.

Proposition 1. *Under conditions of Theorem 2, we have*

$$Z_n^{(2)}(t)\xrightarrow{D}W(\varphi(t)),\quad n\to\infty$$

in Skorokhod space $D[0,+\infty)$, where $\{W(t), t\geq 0\}$ is the standard Brownian motion.

Proof. First note that since $M_k^{(n)}$ is a martingale difference then the sequence $\{U_k^n, k\geq 1\}$ where $U_k^n = a_n^{-k}M_k^{(n)}/B(n)$ for each $n\geq 1$, defines a sequence of martingale differences with respect to the filtration $\mathfrak{F}_k^{(n)}$, $k\geq 0$. Hence, we need to show that all conditions of Theorem A are fulfilled. First, we will prove that (15) is satisfied. Observe that

$$\mathbb{E}\left(\left(M_k^{(n)}\right)^2\middle|\mathfrak{F}_{k-1}^{(n)}\right)=b_nX_{k-1}^{(n)}+\mathbb{E}\left(\left(\varepsilon_k^{(n)}\right)^2\middle|\mathfrak{F}_{k-1}^{(n)}\right)-\left(\mathbb{E}\left(\varepsilon_k^{(n)}\middle|\mathfrak{F}_{k-1}^{(n)}\right)\right)^2$$

which yields

$$\sum_{k=1}^{[nt]}\mathbb{E}\left(\left(U_k^n\right)^2\middle|\mathfrak{F}_{k-1}^{(n)}\right)=J_n^{(1)}(t)+J_n^{(2)}(t),\tag{26}$$

where

$$J_n^{(1)}(t)=\frac{b_n}{B^2(n)}\sum_{k=1}^{[nt]}a_n^{-2k}X_{k-1}^{(n)},$$

$$J_n^{(2)}(t)=\frac{1}{B^2(n)}\sum_{k=1}^{[nt]}a_n^{-2k}\left(\mathbb{E}\left(\left(\varepsilon_k^{(n)}\right)^2\middle|\mathfrak{F}_{k-1}^{(n)}\right)-\left(\mathbb{E}\left(\varepsilon_k^{(n)}\middle|\mathfrak{F}_{k-1}^{(n)}\right)\right)^2\right).$$

Consider $J_n^{(1)}(t)$. Since $B^2(n)\sim\Delta^2(n)$ and by using Lemmas 5-6, we get uniformly for each $t\geq 0$,

$$\mathbb{E}J_n^{(1)}(t)=\frac{b_n}{B^2(n)}\sum_{k=1}^{[nt]}a_n^{-2k}A_n(k-1)\to\varphi^*(t),\tag{27}$$

where $\varphi^*(t)=(a/\nu_\alpha(1))\int_0^t\mu_\alpha(u)e^{-2au}du$ if $a\neq 0$ and $\varphi^*(t)=t^{2+\alpha}$ if $a=0$.

Now consider the variance of $J_n^{(1)}(t)$. It is easy to check that

$$Var\left(J_n^{(1)}(t)\right)=R_n^{(1)}(t)+R_n^{(2)}(t),\tag{28}$$

where

$$R_n^{(1)}(t)=\frac{b_n^2}{B^4(n)}\sum_{k=1}^{[nt]}a_n^{-4k}B_n^2(k-1),$$

$$R_n^{(2)}(t)=\frac{2b_n^2}{B^4(n)}\sum_{i=1}^{[nt]-2}\sum_{j=i+1}^{[nt]-1}a_n^{-2(i+j)}\operatorname{cov}\left(X_i^{(n)},X_j^{(n)}\right).$$

We will show that $R_n^{(1)}(t) \to 0$. With this aim, we first consider $\widetilde{\sigma}^2(n)$. By the moment inequality for m-dependent random variables,

$$\widetilde{\sigma}^2(n) = \mathbb{E}\left(\sum_{i=1}^{n} a_n^{n-i}\left(\varepsilon_i^{(n)} - \alpha(n,i)\right)\right)^2 \leq m\sum_{i=1}^{n} a_n^{2(n-i)}\beta(n,i) = m\sigma^2(n).$$

From Lemma 5 and condition (C5), it follows that $B^2(n) \sim \Delta^2(n)$, $\widetilde{\sigma}^2(n) = o\left(\Delta^2(n)\right)$. Thus from above and by Lemma 6, one can have that

$$R_n^{(1)}(t) = \frac{b_n^2}{B^4(n)}\sum_{k=1}^{[nt]} a_n^{-4k}B_n^2(k-1) + \frac{b_n^2}{B^4(n)}\sum_{k=1}^{[nt]} a_n^{-4k}\widetilde{\sigma}_n^2(k-1)$$

$$\sim \frac{Cn^3\alpha(n)b_n^3}{n^4\alpha^2(n)b_n^2} + \frac{Cmn^2\beta(n)b_n^2}{n^4\alpha^2(n)b_n^2} \to 0. \tag{29}$$

Regarding the term $R_n^{(2)}(t)$, we use the equality

$$\operatorname{cov}\left(X_i^{(n)}, X_j^{(n)}\right) = a_n^{j-i}B_n^2(i) + \sum_{k=i+1}^{j}\sum_{l=1}^{i} a_n^{j-k-i-l}\operatorname{cov}\left(\varepsilon_k^{(n)}, \varepsilon_l^{(n)}\right).$$

Unlike the equality (4.8) in [16], the latter formula contains a new term because of dependence of the immigration sequence, so that

$$R_n^{(2)}(t) = \frac{2b_n^2}{B^4(n)}\sum_{i=1}^{[nt]-2}\sum_{j=i+1}^{[nt]-1} a_n^{-2(i+j)}a_n^{j-i}B_n^2(i)$$

$$+\frac{2b_n^2}{B^4(n)}\sum_{i=1}^{[nt]-2}\sum_{j=i+1}^{[nt]-1} a_n^{-2(i+j)}\sum_{k=i+1}^{j}\sum_{l=1}^{i} a_n^{j-k-i-l}\operatorname{cov}\left(\varepsilon_k^{(n)}, \varepsilon_l^{(n)}\right). \tag{30}$$

The first term on the right hand-side of (30) can be estimated as

$$\frac{2b_n^2}{B^4(n)}\sum_{i=1}^{[nt]-2} a_n^{-3i}B_n^2(i)\sum_{j=1}^{[nt]-1} a_n^{-j} \lesssim \frac{Cn^3\alpha(n)b_n^3}{n^4\alpha^2(n)b_n^2} + \frac{Cmn^3\beta(n)b_n^2}{n^4\alpha^2(n)b_n^2} \to 0. \tag{31}$$

For the second term of (30), we have

$$\frac{2b_n^2}{B^4(n)}\sum_{i=1}^{[nt]-2}\sum_{j=i+1}^{[nt]-1}\sum_{k=i+1}^{j}\sum_{l=1}^{i} a_n^{-3i-j-k-l}\operatorname{cov}\left(\varepsilon_l^{(n)}, \varepsilon_k^{(n)}\right)$$

$$\lesssim \frac{b_n^2}{B^4(n)}\sum_{i=1}^{[nt]-2}\sum_{j=i+1}^{[nt]-1} a_n^{-3i-j}\sum_{k=2}^{[nt]}\sum_{l=1}^{k-1} a_n^{-k-l}\operatorname{cov}\left(\varepsilon_l^{(n)}, \varepsilon_k^{(n)}\right)$$

$$\lesssim \frac{mb_n^2}{B^4(n)}\sum_{i=1}^{[nt]-2}\sum_{j=i+1}^{[nt]-1} a_n^{-3i-j}\sum_{k=2}^{[nt]} a_n^{-2k}\beta(n,k) \to 0. \tag{32}$$

From (30)-(32) we deduce that $R_n^{(2)}(t) \to 0$. Thus, by (28), (29), we infer that

$$Var\left(J_n^{(1)}(t)\right) \to 0.$$

From above and by relations (27)-(32), we get

$$J_n^{(1)}(t) \xrightarrow{P} \varphi^*(t). \tag{33}$$

We now prove that $J_n^{(2)}(t) \xrightarrow{P} 0$. Indeed, note that

$$\mathbb{E}J_n^{(2)}(t) = \frac{1}{B^2(n)} \sum_{k=1}^{[nt]} a_n^{-2k} \left(\mathbb{E}\left(\varepsilon_k^{(n)}\right)^2 - (\alpha(n,k))^2 \right)$$

$$+\frac{1}{B^2(n)} \sum_{k=1}^{[nt]} a_n^{-2k} \left((\alpha(n,k))^2 - \mathbb{E}\left(\mathbb{E}\left(\varepsilon_k^{(n)} \middle| \mathfrak{F}_{k-1}^{(n)}\right)\right)^2 \right)$$

$$= \frac{1}{B^2(n)} \sum_{k=1}^{[nt]} a_n^{-2k} \left(Var\left(\varepsilon_k^{(n)}\right) - Var\left(\mathbb{E}\left(\varepsilon_k^{(n)} \middle| \mathfrak{F}_{k-1}^{(n)}\right)\right) \right)$$

$$\leq \frac{1}{B^2(n)} \sum_{k=1}^{[nt]} a_n^{-2k} \beta(n,k). \tag{34}$$

In order to bound the term in (34), we first rewrite it as

$$\frac{1}{B^2(n)} \sum_{k=1}^{[nt]} a_n^{-2k} (\beta(n,k) - \beta(k)) + \frac{1}{B^2(n)} \sum_{k=1}^{[nt]} a_n^{-2k} \beta(k),$$

and due to conditions of Theorem 2 and Lemma 4,

$$\frac{1}{\beta(n)} \max_{1 \leq k \leq nt} |\beta(n,k) - \beta(k)| \frac{n\beta(n)}{B^2(n)} \frac{1}{n} \sum_{k=1}^{[nt]} a_n^{-2k} + \frac{n\beta(n)}{B^2(n)} \frac{1}{n} \sum_{k=1}^{[nt]} a_n^{-2k} \to 0,$$

which proves

$$J_n^{(2)}(t) \xrightarrow{P} 0. \tag{35}$$

Consequently, collecting (33) with (35), and by (26) we obtain

$$\sum_{k=1}^{[nt]} \mathbb{E}\left((U_k^n)^2 \middle| \mathfrak{F}_{k-1}^n \right) \xrightarrow{P} \varphi^*(t). \tag{36}$$

We turn now to the proof of (14). Using a simple inequality $(x+y)^2 \leq 2\left(x^2+y^2\right)$, where $x, y \in \mathbb{R}$, we find that

$$L(n) := \frac{1}{B^2(n)} \sum_{k=1}^{[nt]} a_n^{-2k} \mathbb{E}\left(\left(M_k^{(n)}\right)^2 I\left\{ \left| a_n^{-k} M_k^{(n)} \right| > \varepsilon B(n) \right\} \middle| \mathfrak{F}_{k-1}^{(n)} \right)$$

$$\leq 2\left(L_1(n) + L_2(n)\right),$$

where

$$L_1(n) = \frac{1}{B^2(n)} \sum_{k=1}^{[nt]} a_n^{-2k} \mathbb{E}\left(\left(T_k^{(n)}\right)^2 I\left\{ \left| a_n^{-k} M_k^{(n)} \right| > \varepsilon B(n) \right\} \middle| \mathfrak{F}_{k-1}^{(n)} \right),$$

$$L_2(n) = \frac{1}{B^2(n)} \sum_{k=1}^{[nt]} a_n^{-2k} \mathbb{E}\left(\left(N_k^{(n)}\right)^2 I\left\{ \left| a_n^{-k} M_k^{(n)} \right| > \varepsilon B(n) \right\} \middle| \mathfrak{F}_{k-1}^{(n)} \right).$$

First we estimate $L_1(n)$. Note that for any random variables X and Y, and for all $\varepsilon > 0$, one has

$$I\{|X+Y| > \varepsilon\} \le I\{|X| > \varepsilon/2\} + I\{|Y| > \varepsilon/2\}. \tag{37}$$

The application of (37) gives us the bound

$$L_1(n) \le L_{1,1}(n) + L_{1,2}(n),$$

where

$$L_{1,1}(n) = \frac{1}{B^2(n)} \sum_{k=1}^{[nt]} a_n^{-2k} \mathbb{E}\left(\left(T_k^{(n)}\right)^2 I\left\{ \left| a_n^{-k} T_k^{(n)} \right| > \varepsilon B(n)/2 \right\} \middle| \mathfrak{F}_{k-1}^{(n)} \right),$$

$$L_{1,2}(n) = \frac{1}{B^2(n)} \sum_{k=1}^{[nt]} a_n^{-2k} \mathbb{E}\left(\left(T_k^{(n)}\right)^2 I\left\{ \left| a_n^{-k} N_k^{(n)} \right| > \varepsilon B(n)/2 \right\} \middle| \mathfrak{F}_{k-1}^{(n)} \right).$$

Consider $L_{1,1}(n)$. Using (12) from Lemma 8, it is proved [16] that $L_{1,1}(n)$ is bounded by relations $S_1(n,t)$, $S_2(n,t)$ and $S_3(n,t)$. Then, by conditions (C1)-(C3), (C5) and (6), it is also shown that each of the terms $S_i(n,t)$, $i = 1,2,3$, converges in probability to zero. Hence,

$$L_{1,1}(n) \xrightarrow{P} 0. \tag{38}$$

The detailed proof of (38) is omitted since the proof also remains true in our context.

Further, by using Lemma 7 and taking into account the independence between the offspring and immigration sequences, we get

$$L_{1,2}(n) = \frac{b_n}{B^2(n)} \sum_{k=1}^{[nt]} a_n^{-2k} X_{k-1}^{(n)} \mathbb{E}\left(I\left\{ \left| a_n^{-k} N_k^{(n)} \right| > \varepsilon B(n)/2 \right\} \middle| \mathfrak{F}_{k-1}^{(n)} \right).$$

Thus, from Markov inequality, for any $\gamma > 0$, it yields

$$\mathbb{P}\left(L_{1,2}(n) > \gamma\right) \le \frac{b_n}{\gamma B^2(n)} \sum_{k=1}^{[nt]} a_n^{-2k} \mathbb{E} X_{k-1}^{(n)} \mathbb{E}\left(I\left\{ \left| a_n^{-k} N_k^{(n)} \right| > \varepsilon B(n)/2 \right\} \right). \tag{39}$$

In order to bound the right-hand side of (39), we first apply Cauchy-Schwarz inequality, the inequality $\sqrt{(x^2+y^2)} \le \sqrt{2}\,(|x|+|y|)$, $x, y \in \mathbb{R}$ and then Chebyshev inequality, overall, we obtain

$$\begin{aligned}
&\frac{b_n}{B^2(n)} \sum_{k=1}^{[nt]} a_n^{-2k} \left(\mathbb{E}\left(X_{k-1}^{(n)} \right)^2 \right)^{1/2} \left(\mathbb{E}\left(I\left\{ \left| a_n^{-k} N_k^{(n)} \right| > \varepsilon B(n)/2 \right\} \right) \right)^{1/2} \\
&\le \frac{b_n}{B^2(n)} \sum_{k=1}^{[nt]} a_n^{-2k} \left(B_n^2(k-1) + A_n^2(k-1) \right)^{1/2} \left(\mathbb{P}\left\{ \left| a_n^{-k} N_k^{(n)} \right| > \varepsilon B(n)/2 \right\} \right)^{1/2} \\
&\le \frac{C b_n}{B^3(n)} \sum_{k=1}^{[nt]} a_n^{-2k} \left(B_n(k-1) + A_n(k-1) \right) \left(\mathbb{E}\left(a_n^{-k} N_k^{(n)} \right)^2 \right)^{1/2} \\
&\le \frac{C b_n}{B^3(n)} \sum_{k=1}^{[nt]} a_n^{-3k} \beta^{1/2}(n,k) \left(B_n(k-1) + A_n(k-1) \right) \\
&\lesssim \frac{b_n \left(B_n([nt]) + A_n([nt]) \right)}{B^3(n)} \sum_{k=1}^{[nt]} a_n^{-3k} \beta^{1/2}(n,k) \to 0.
\end{aligned}$$

where in last step we used the properties of regularly varying functions and $B^2(n) \sim Cn^2\alpha(n)\, b_n,\ A(n) \sim Cn\alpha(n)$. Therefore, we deduce that

$$L_{1,2}(n) \xrightarrow{P} 0. \tag{40}$$

Recalling (38) and (40), we have

$$L_1(n) \xrightarrow{P} 0. \tag{41}$$

In order to estimate $L_2(n)$, it suffices to note that

$$\mathbb{E}L_2(n) \le \frac{1}{B^2(n)} \sum_{k=1}^{[nt]} a_n^{-2k} \mathbb{E}\left(N_k^{(n)}\right)^2$$

$$\le \frac{2}{B^2(n)} \sum_{k=1}^{[nt]} a_n^{-2k} \left(\beta(n,k) + \mathbb{E}\left(\mathbb{E}\left(\varepsilon_k^{(n)} \middle| \mathfrak{F}_{k-1}^{(n)}\right) - \alpha(n,k)\right)^2\right)$$

$$\le \frac{4}{B^2(n)} \sum_{k=1}^{[nt]} a_n^{-2k} \beta(n,k) \to 0.$$

Therefore,

$$L_2(n) \xrightarrow{P} 0. \tag{42}$$

From (41) and (42), we conclude

$$L(n) \xrightarrow{P} 0. \tag{43}$$

Collecting (36) and (43), we have proved that $Z_n^{(1)}(t) \xrightarrow{D} W(\varphi^*(t))$ in $D[0,+\infty)$. Now, it remains to note that $CW\left(t/C^2\right)$ is a standard Wiener process for any $C \neq 0$. This ends the proof of Proposition 1. □

Proposition 2. *Assume for each $n \ge 1$, $\left\{\varepsilon_k^{(n)},\ k \ge 1\right\}$ be a sequence of m-dependent random variables. If conditions (C1)-(C3) and (C5) hold, then*

$$Z_n^{(2)}(t) \xrightarrow{D} 0,$$

in Skorokhod space $D[0,+\infty)$.

Proof. The assertion of Proposition 2 follows from Lemmas 2-3 immediately. □

Proof of Theorem 2. The proof of Theorem 2 is a straightforward consequence of Propositions 1-2.

Acknowledgements. The author would like to thank Vladimir Vatutin and Ibrahim Rahimov for fruitful discussions and useful comments and an anonymous reviewer for indicating inaccuracies in the previous version of the paper.

References

[1] P. Billingsley, *Convergence of Probability Measures. 2nd edn.*, Wiley, New York, 1999.

[2] S.N. Ethier, T.G. Kurtz, *Markov Processes. Characterization and Convergence*, Wiley, New York, 1986.

[3] V.V Golomoziy, S.O. Sharipov, *On central limit theorems for branching processes with dependent immigration*, Bulletin of Taras Shevchenko National University of Kyiv Series: Physics and Mathematics, **37** (2020), 7–17.

[4] H. Guo, M. Zhang, *A fluctuation limit theorem for a critical branching process with dependent immigration*, Statistics and Probability Letters, **94** (2014), 29–38.

[5] P. Haccou, P. Jagers, V.A. Vatutin, *Branching Processes: Variation, Growth and Extinction of Populations*, Cambridge University Press, Cambridge, 2005.

[6] J. Jacod, A.N. Shiryaev, *Limit Theorems for Stochastic Processes*, 2nd ed. Springer, Berlin, 2003.

[7] Y.M. Khusanbaev, *Nearly critical branching processes and limit theorems*, Ukr. Math. J., **61** (2009), 127–133.

[8] Y.M. Khusanbaev, *On asymptotics of branching processes with immigration*, Discrete Math. Appl., **27**:2 (2017), 73–80.

[9] Y.M. Khusanbaev, S.O. Sharipov, *Functional limit theorem for critical branching processes with dependent immigration*, Uzbek Mathematical Journal, **3** (2017), 149–158.

[10] Y.M. Khusanbaev, S.O. Sharipov, *On the asymptotic behavior of branching processes with stationary immigration*, Bulletin of National University of Uzbekistan: Mathematics and Natural Sciences, **3** (2020), 59–73.

[11] A. N. Kolmogorov, Y.A. Rozanov, *On strong mixing conditions for stationary Gaussian processes*, Theory Probab. Appl., **5** (1960), 204–208.

[12] D. Li, M. Zhang, *Limit Theorem and Parameter Estimation for a Critical Branching Process with Mixing Immigration*, Chinese Journal of Applied Probability and Statistics, **36**:4 (2020), 331–341.

[13] S.V. Nagaev, *A limit theorem for branching processes with immigration*, Theory Probab. Appl., **20**:3 (1975), 178–180.

[14] I. Rahimov, *Functional limit theorems for critical processes with immigration*, The Advances in Applied Probability, **39**:4 (2007), 1054–1069.

[15] I. Rahimov, *Deterministic approximation of a sequence of nearly critical branching processes*, Stochastic Analysis and Applications, **26**:5 (2008), 1013–1024.

[16] I.Rahimov, *Approximation of fluctuations in a sequence of nearly critical branching processes*, Stochastic Models, **25**:2 (2009), 348–373.

[17] I.Rahimov, *Homogeneous branching processes with non-homogeneous immigration*, Stochastics and Quality Control, **36** (2021), 165–183.

[18] S.O. Sharipov, *Deterministic approximation for nearly critical branching processes with dependent immigration*, Uzbek Mathematical Journal, **3** (2018), 156–165.

[19] S.O. Sharipov, *Central limit theorem for branching process with immigration*, AIP Conference Proceedings, **2365** (2021), 060020.

[20] S.O. Sharipov, *On limit theorems for branching processes with dependent immigration*, Global and Stochastic Analysis, **8** (2021), 151–160.

[21] S.A. Utev, *Sums of weakly dependent random variables*, Siberian Math. J., **32**:4 (1991), 675–690.

Sadillo Olimjonovich Sharipov
V.I.Romanovskiy Institute of Mathematics,
4b, University street,
100174, Tashkent, Uzbekistan
E-mail address: sadi.sharipov@yahoo.com