ON DISCRETE BOUNDARY VALUE PROBLEMS IN A QUARTER-PLANE

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ABSTRACT. We consider discrete analogue of model pseudo-differential equations in discrete plane sector using discrete variant of Sobolev–Slobodetskii spaces. Starting from the concept of wave factorization for elliptic periodic symbol we describe solvability conditions for the equations and corresponding discrete boundary value problems. We give also a comparison between discrete and continuous solutions in appropriate discrete normed spaces.

Keywords: elliptic symbol, invertibility, two-dimensional periodic Riemann problem, digital pseudo-differential operator, discrete equation, periodic wave factorization.

1. INTRODUCTION

A theory of pseudo-differential operators and equations [9, 16, 17, 6] not so long history than other mathematical subjects of analysis. Nevertheless, these operators and related boundary value problems widely arise in a lot of applied problems in physics and technique (see, for example [14] and references therein). Discrete aspects of the theory are reflected in mathematical papers more weak [11, 2] although these studies are closely related to the theory of Fourier series [5]. In our opinion the discrete theory is very important since it permits to use computer calculations to solve concrete applied problems.

We interested in studying discrete pseudo-differential equations and their solvability in appropriate discrete functional spaces. There are certain approaches to studying discrete boundary value problems for partial differential equations including finite difference method [13, 12]. But these approaches are not applicable to studying

discrete boundary value problems for elliptic pseudo-differential equations. According to this statement the first author with colleagues has started to develop discrete theory for elliptic pseudo-differential equations [21, 15]. This is main motivation, and we have started from certain canonical domains. First considerations were related to discrete m-dimensional space and half-space, and here we consider discrete quadrant.

We consider a special type of boundary conditions, namely integral conditions on a boundary. These conditions are nonlocal, and it seems, such conditions are artificial. But there are a lot of applied problems for partial differential equations with such boundary conditions [1, 4, 3], therefore it is natural way. Moreover, these conditions appear in a natural way to determine arbitrary functions in a general solution of an elliptic pseudo-differential equation.

2. Cones, periodic symbols, digital operators and equations

2.1. **Discrete spaces and transforms.** Let $\mathbb{Z}^2$ be an integer lattice in a plane. Let $K = \{x \in \mathbb{R}^2 : x = (x_1, x_2), x_1 > 0, x_2 > 0\}$ be a quadrant, $K_d = h\mathbb{Z}^2 \cap K, h > 0$. We consider functions of discrete variable $u_d(\tilde{x}), \tilde{x} = (\tilde{x}_1, \tilde{x}_2) \in h\mathbb{Z}^2$.

Let us denote $\mathbb{T}^2 = [-\pi, \pi]^2, \hbar = h^{-1}$. We consider functions defined in $\hbar\mathbb{T}^2$ as periodic functions defined in $\mathbb{R}^2$ with basic square of periods $\hbar\mathbb{T}^2$.

One can define the discrete Fourier transform for the function u_d

$$(F_d u_d)(\xi) \equiv \tilde{u}_d(\xi) = \sum_{\tilde{x} \in h\mathbb{Z}^2} e^{-i\tilde{x}\cdot\xi} u_d(\tilde{x}) h^2, \quad \xi \in \hbar\mathbb{T}^2,$$

if the latter series converges, and the function $\tilde{u}_d(\xi)$ is a periodic function in $\mathbb{R}^2$ with basic square of periods $\hbar\mathbb{T}^2$. Such discrete Fourier transform preserves all properties of integral Fourier transform, and the inverse discrete Fourier transform looks as follows

$$(F_d^{-1}\tilde{u}_d)(\tilde{x}) = \frac{1}{(2\pi)^2} \int_{\hbar\mathbb{T}^2} e^{i\tilde{x}\cdot\xi} \tilde{u}_d(\xi) d\xi, \quad \tilde{x} \in h\mathbb{Z}^2.$$

The discrete Fourier transform gives one-to-one correspondence between spaces $L_2(h\mathbb{Z}^2)$ and $L_2(\hbar\mathbb{T}^2)$ with norms

$$||u_d||_2 = \left(\sum_{\tilde{x} \in h\mathbb{Z}^2} |u_d(\tilde{x})|^2 h^2\right)^{1/2}, \quad ||\tilde{u}_d||_2 = \left(\int_{\xi \in \hbar\mathbb{T}^2} |\tilde{u}_d(\xi)|^2 d\xi\right)^{1/2}.$$

We need more general discrete functional spaces and we introduce such spaces using divided differences [13].

The divided differences of first order look as follows

$$(\Delta_1^{(1)}u_d)(\tilde{x}) = h^{-1}(u_d(\tilde{x}_1+h, \tilde{x}_2) - u_d(\tilde{x}_1, \tilde{x}_2)), \quad (\Delta_2^{(1)}u_d)(\tilde{x}) = h^{-1}(u_d(\tilde{x}_1, \tilde{x}_2+h) - u_d(\tilde{x}_1, \tilde{x}_2)),$$

and their discrete Fourier transforms are given by formulas

$$(\widetilde{\Delta_k^{(1)}u_d})(\xi) = h^{-1}(e^{-ih\cdot\xi_k} - 1)\tilde{u}_d(\xi), k = 1, 2.$$

The divided difference of second order is a divided difference of first order from divided difference of first order

$$(\Delta_1^{(2)}u_d)(\tilde{x}) = h^{-2}(u_d(\tilde{x}_1 + 2h, \tilde{x}_2) - 2u_d(\tilde{x}_1 + h, \tilde{x}_2) + u_d(\tilde{x}_1, \tilde{x}_2)),$$

$$(\Delta_2^{(2)}u_d)(\tilde{x}) = h^{-2}(u_d(\tilde{x}_1, \tilde{x}_2 + 2h) - 2u_d(\tilde{x}_1, \tilde{x}_2 + h) + u_d(\tilde{x}_1, \tilde{x}_2)),$$

with the Fourier transform

$$(\widetilde{\Delta_k^{(2)}u_d})(\xi) = h^{-2}(e^{-ih\cdot\xi_k} - 1)^2\tilde{u}_d(\xi), k = 1, 2.$$

Discrete analogue of the Laplacian is the following

$$(\Delta_d u_d)(\tilde{x}) = (\Delta_1^{(2)}u_d)(\tilde{x}) + (\Delta_2^{(2)}u_d)(\tilde{x}),$$

so that its Fourier transform is

$$(\widetilde{\Delta_d u_d})(\xi) = h^{-2}((e^{-ih\cdot\xi_1} - 1)^2 + (e^{-ih\cdot\xi_2} - 1)^2)\tilde{u}_d(\xi).$$

We use such discrete objects for constructing discrete Sobolev–Slobodetskii spaces to study wide class of discrete equations.

First, we introduce discrete analogue of the Schwartz space $S(h\mathbb{Z}^2)$ as a set of discrete functions with finite semi-norms

$$|u_d| = \sup_{\tilde{x}\in h\mathbb{Z}^2}(1 + |\tilde{x}|)^l|\Delta^{(\mathbf{k})}u_d(\tilde{x})|$$

for arbitrary $l \in \mathbb{N}, \mathbf{k} = (k_1, k_2), k_r \in \mathbb{N}, r = 1, 2,$

$$\Delta^{(\mathbf{k})}u_d(\tilde{x}) = \Delta_1^{k_1}\Delta_2^{k_2}u_d(\tilde{x}).$$

Definition 1. *A discrete distribution is called a linear continuous functional defined on the space $S(h\mathbb{Z}^2)$.*

A set of such distributions will be denoted by $S'(h\mathbb{Z}^2)$, and a value of the discrete distribution f_d on the test discrete function $u_d \in S(h\mathbb{Z}^2)$ will be denoted by (f_d, u_d).

One can introduce a concept of a support for a discrete distribution. Namely, a support of the discrete function $u_d \in S(h\mathbb{Z}^2)$ is a subset of the set $h\mathbb{Z}^2$ such that $u_d(\tilde{x}) \neq 0$ for all points $\tilde{x}$ from this subset. For an arbitrary set $M \subset \mathbb{R}^2$ we denote $M_d = M \cap h\mathbb{Z}^2$, and then one says that $f_d = 0$ in the discrete domain M_d if $(f_d, u_d) = 0, \forall u_d \in S(M_d)$, where $S(M_d) \subset S(h\mathbb{Z}^2)$ consists of discrete functions with supports in M_d. If $\widetilde{M_d}$ is a union of such M_d where $f_d = 0$ then support of the discrete distribution f_d is the set $h\mathbb{Z}^2 \setminus \widetilde{M_d}$.

Similarly [26] we can define standard operations in the space $S'(h\mathbb{Z}^2)$, but differentiation will be changed by divided difference of first order. These operations are described in [21] in details, a convergence is meant as a weak convergence in the space $S'(h\mathbb{Z}^2)$.

Example 1. If the function $f_d(\tilde{x})$ is locally summable then it generates the discrete distribution

$$(1) \qquad (f_d, u_d) = \sum_{\tilde{x}\in h\mathbb{Z}^2} f_d(\tilde{x})u_d(\tilde{x})h^2, \quad \forall u_d \in S(h\mathbb{Z}^2).$$

But there are different possibilities, for example, analogue of the Dirac mass-function

$$(\delta_d, u_d) = u_d(0),$$

which can not be represented by the formula (1).

Let $\zeta^2 = h^{-2}((e^{-ih\cdot\xi_1}-1)^2 + (e^{-ih\cdot\xi_2}-1)^2)$. We introduce the following definition.

Definition 2. *The space $H^s(h\mathbb{Z}^2)$ consists of discrete distributions and it is a closure of the space $S(h\mathbb{Z}^2)$ with respect to the norm*

$$(2) \qquad ||u_d||_s = \left(\int_{\hbar\mathbb{T}^2}(1 + |\zeta^2|)^s|\tilde{u}_d(\xi)|^2 d\xi\right)^{1/2}.$$

Let us remind that a lot of properties of such discrete spaces were studied in [7], Varying the parameter h in (2) we obtain different norms which are equivalent to the L_2-norm. But constants in this equivalence depend on h. In our constructions all constants do not depend on h.

Definition 3. The space $H^s(K_d)$ consists of discrete distributions from $H^s(h\mathbb{Z}^2)$ such that their supports belong to the set $\overline{K_d}$. A norm in the space $H^s(K_d)$ is induced by the norm of the space $H^s(h\mathbb{Z}^2)$. The space $H_0^s(K_d)$ consists of discrete distributions $f_d \in S'(h\mathbb{R}^2)$ with supports inside of K_d, and these discrete distributions must admit a continuation into the space $H^s(h\mathbb{Z}^2)$. A norm in the space $H_0^s(K_d)$ is given by the formula

$$||f_d||_s^+ = \inf ||\ell f_d||_s,$$

where infimum is taken for all continuations ℓ.

The Fourier image of the space $H^s(K_d)$ will be denoted by $\widetilde{H}^s(K_d)$.

2.2. Symbols, operators and projectors.
Let $\widetilde{A}_d(\xi)$ be a measurable periodic function in $\mathbb{R}^2$ with basic square of periods $\hbar\mathbb{T}^2$. Such functions we call symbols.

Definition 4. A digital pseudo-differential operator A_d with the symbol $A_d(\xi)$ in the discrete quadrant K_d is called an operator of the following type

$$(3) \qquad (A_d u_d)(\tilde{x}) = \sum_{\tilde{y} \in h\mathbb{Z}^2} h^2 \int\limits_{\hbar\mathbb{T}^2} \widetilde{A}_d(\xi) e^{i(\tilde{x}-\tilde{y})\cdot\xi} \tilde{u}_d(\xi) d\xi, \quad \tilde{x} \in K_d,$$

We say that the operator A_d is elliptic one if

$$ess \inf_{\xi \in \hbar\mathbb{T}^2} |A_d(\xi)| > 0.$$

A more general digital pseudo-differential operator with the symbol $\widetilde{A}_d(\tilde{x}, \xi)$ depending on a spatial variable $\tilde{x}$

$$(A_d u_d)(\tilde{x}) = \sum_{\tilde{y} \in h\mathbb{Z}^2} h^2 \int\limits_{\hbar\mathbb{T}^2} A_d(\tilde{x}, \xi) e^{i(\tilde{x}-\tilde{y})\cdot\xi} \tilde{u}_d(\xi) d\xi, \quad \tilde{x} \in {}_d,$$

can be defined in the same way, but here we consider only operators of type (3).

We consider symbols satisfying the condition

$$(4) \qquad c_1(1 + |\zeta^2|)^{\alpha/2} \leq |A_d(\xi)| \leq c_2(1 + |\zeta^2|)^{\alpha/2}$$

with constants c_1, c_2 non-depending on h. The number $\alpha \in \mathbb{R}$ is called an order of digital pseudo-differential operator A_d.

The following simple result can be proved easily.

Lemma 1. A digital pseudo-differential operator A_d with the symbol $\widetilde{A}_d(\xi)$ is a linear bounded operator $H^s(h\mathbb{Z}^2) \to H^{s-\alpha}(h\mathbb{Z}^2)$ with a norm non-depending on h.

We study a solvability of the discrete equation

$$(5) \qquad (A_d u_d)(\tilde{x}) = v_d(\tilde{x}), \quad \tilde{x} \in K_d,$$

in the space $H^s(K_d)$ assuming that $v_d \in H_0^{s-\alpha}(K_d)$.

We will use certain special domain in two-dimensional complex space $\mathbb{C}^2$. A domain of the type $\mathcal{T}_h(K) = \hbar\mathbb{T}^2 + iK$ is called a tube domain over the quadrant K, and we will consider analytical functions $f(x+i\tau)$ in the domain $\mathcal{T}_h(K) = \hbar\mathbb{T}^2 + iK$.

Let us introduce the periodic Bochner kernel similar [26]

$$B_h(z) = \sum_{\tilde{x} \in K_d} e^{i\tilde{x} \cdot (\xi + i\tau)} h^2, \quad \xi \in \hbar\mathbb{T}^2, \quad \tau \in K,$$

and corresponding integral operator

$$(B_h \tilde{u}_d)(\xi) = \lim_{\tau \to 0, \tau \in K} \frac{1}{4\pi^2} \int_{\hbar\mathbb{T}^2} B_h(\xi + i\tau - \eta) \tilde{u}_d(\eta) d\eta.$$

Lemma 2. *For the quadrant K the operator B_h has the following form*

$$(B_h \tilde{u}_d)(\xi) = \frac{h^2}{8\pi^2} \int_{\mathbb{T}^2} \tilde{u}_d(\eta) d\eta + \lim_{\tau \to 0+} \frac{ih}{8\pi^2} \int_{\mathbb{T}^2} \cot \frac{h(\xi_1 - \eta_1 + i\tau_1)}{2} \tilde{u}_d(\eta) d\eta +$$

$$+ \lim_{\tau \to 0+} \frac{ih}{8\pi^2} \int_{\mathbb{T}^2} \cot \frac{h(\xi_2 - \eta_2 + i\tau_2)}{2} \tilde{u}_d(\eta) d\eta -$$

$$- \lim_{\tau \to 0+} \frac{h^2}{8\pi^2} \int_{\mathbb{T}^2} \cot \frac{h(\xi_1 - \eta_1 + i\tau_1)}{2} \cot \frac{h(\xi_2 - \eta_2 + i\tau_2)}{2} \tilde{u}_d(\eta) d\eta,$$

and B_h is a linear bounded operator $H^s(\hbar\mathbb{T}^2) \to H^s(\hbar\mathbb{T}^2)$ for $|s| < 1/2$. Moreover, the operator B_h is a projector $\widetilde{H}^s(h\mathbb{Z}^2) \to \widetilde{H}^s(K_d)$.

Proof. Corresponding calculations for one-dimensional discrete cone were done in [20]. We use these evaluations adapting to our two-dimensional case. Since

$$\sum_{\tilde{x}_k \in h\mathbb{Z}_+} e^{-i\tilde{x}_k z_k} h = \frac{h}{2} - \frac{ih}{2} \cot \frac{h z_k}{2}, \quad z_k = \xi_k + i\tau_k, \quad k = 1, 2.$$

then multiplying two factors and applying the Fourier property on correspondence between a product and convolution we obtain the assertion.

Boundedness of the one-dimensional operator with the kernel $h \cot \frac{hz}{2}$ for $|s| < 1/2$ was proved in [7], Theorem 6; two-dimensional case can be considered by the same method. ∎

Remark 1. The operator B_h is so called periodic bi-singular operator. Using classical results for Cauchy type integral [8, 10] one can evaluate the boundary value, but it is not important this time. Since these formulas are very huge we can do some simplifications without lost of generality. For example, we can consider the space $S_1(h\mathbb{Z}^2) \subset S_1(h\mathbb{Z}^2)$ with zeroes in coordinate axes and consider the space $H^s(h\mathbb{Z}^2)$ as closure of the set $S_1(h\mathbb{Z}^2)$ assuming that all functions of discrete variable vanish on coordinate axes. For this case the first three summands in B_h will be zero.

Lemma 3. *If $|s| < 1/2$ then the space $\widetilde{H}^s(h\mathbb{Z}^2)$ is uniquely represented as the direct sum*

$$\widetilde{H}^s(h\mathbb{Z}^2) = \widetilde{H}^s(K_d) \oplus \widetilde{H}^s(h\mathbb{Z}^2 \setminus K_d)$$

Proof. It is simple consequence of Lemma 2. Indeed, the unique representation of the function $\tilde{f} \in \widetilde{H}(h\mathbb{Z}^2)$ is the following

$$\tilde{f} = B_h \tilde{f} + (I - B_h)\tilde{f}.$$

A uniqueness of the such representation is possible only for $|s| < 1/2$. ∎

To describe a solvability picture for the discrete equation (5) we need some additional elements of multidimensional complex analysis. We give it in the next section.

3. Periodic wave factorization

This concept is a periodic analogue of the wave factorization [18]. Some first preliminary considerations and results were described in [25, 24, 23, 22].

Definition 5. *A periodic wave factorization for the elliptic symbol $A_d(\xi) \in E_\alpha$ is called its representation in the form*

$$A_d(\xi) = A_{d,\neq}(\xi)A_{d,=}(\xi),$$

where the factors $A_{d,\neq}(\xi), A_{d,=}(\xi)$ admit analytical continuation into tube domains $\mathcal{T}_h(K), \mathcal{T}_h(-K)$ respectively with estimates

$$c_1(1 + |\hat{\zeta}^2|)^{\frac{\text{æ}}{2}} \leq |A_{d,\neq}(\xi + i\tau)| \leq c_1'(1 + |\hat{\zeta}^2|)^{\frac{\text{æ}}{2}},$$

$$c_2(1 + |\hat{\zeta}^2|)^{\frac{\alpha-\text{æ}}{2}} \leq |A_{d,=}(\xi - i\tau)| \leq c_2'(1 + |\hat{\zeta}^2|)^{\frac{\alpha-\text{æ}}{2}},$$

and constants c_1, c_1', c_2, c_2' non-depending on h, where

$$\hat{\zeta}^2 \equiv \hbar^2\left((e^{-ih(\xi_1+i\tau_1)} - 1)^2 + (e^{-ih(\xi_2+i\tau_2)} - 1)^2\right), \quad \xi = (\xi_1, \xi_2) \in \hbar\mathbb{T}^2, \quad \tau-(\tau_1, \tau_2) \in K.$$

The number $\text{æ} \in \mathbb{R}$ is called an index of periodic wave factorization.

Unfortunately, we have no an algorithm to construct the factorization. But the are certain examples of periodic symbols which admit such factorization. We give one of them.

If f is an arbitrary function of a discrete variable, $f \in S(h\mathbb{Z}^2)$, $supp\, f \subset K_d \cup (-K_d)$ then we have

$$f = \chi_+ f + \chi_- f,$$

where $\chi_\pm$ are indicators of $\pm K_d$. Applying the discrete Fourier transform we obtain the representation $\tilde{f} = \tilde{f}_+ + \tilde{f}_-$, and $\tilde{f}_\pm$ admit an analytical continuation into $\mathcal{T}_h(\pm K)$ according to Lemma 2. Thus, we can write $\exp\tilde{f} = \exp\tilde{f}_+\cdot\exp\tilde{f}_-$, therefore we obtain periodic wave factorization with index zero for the function $\exp\tilde{f}$.

Everywhere below we assume existence of such periodic wave factorization for the symbol $A_d(\xi)$ with index æ.

3.1. **A unique solvability.** This section is devoted to most simple case when a solution of the equation (5) exists and it is unique.

Theorem 1. *Let $|\text{æ} - s| < 1/2$. Then the equation (5) has a unique solution for arbitrary right hand side $v_d \in H_0^{s-\alpha}(K_d)$, and it is given by the formula*

$$\tilde{u}_d(\xi) - A_{d,\neq}^{-1}(\xi)B_h(A_{d,=}^{-1}(\xi)\widetilde{(\ell v_d)}(\xi)),$$

where ℓv_d is an arbitrary continuation of v_d into $H^{s-\alpha}(h\mathbb{Z}^2)$.

Proof Let ℓv_d be an arbitrary continuation of $v_d \in H_0^{s-\alpha}(K_d)$ into $H^{s-\alpha}(h\mathbb{Z}^2)$. Let us introduce the function

$$w_d(\tilde{x}) = (\ell v_d)(\tilde{x}) - (A_d u_d)(\tilde{x}),$$

so that $w(\tilde{x}) = 0$ for $\tilde{x} \notin K_d$.

Now we write the equation (5) in the form

$$(A_d u_d)(\tilde{x}) + w_d(\tilde{x}) = (\ell v_d)(\tilde{x}), \quad \tilde{x} \in h\mathbb{Z}^2,$$

and after applying the discrete Fourier transform and periodic wave factorization we obtain

$$A_{d,\neq}(\xi)\tilde{u}_d(\xi) + A_{d,=}^{-1}(\xi)\tilde{w}_d(\xi) = A_{d,=}^{-1}(\xi)\widetilde{(\ell v_d)}(\xi), \quad \xi \in \hbar\mathbb{T}^2, \tag{6}$$

We have the following inclusions according to Lemma 1 and Lemma 2

$$A_{d,\neq}(\xi)\tilde{u}_d(\xi) \in \widetilde{H}^{s-æ}(K_d), \quad A_{d,=}^{-1}(\xi)\tilde{w}_d(\xi) \in \widetilde{H}^{s-æ}(h\mathbb{Z}^2 \backslash K_d), \quad A_{d,=}^{-1}(\xi)\widetilde{(\ell v_d)}(\xi) \in \widetilde{H}^{s-æ}(h\mathbb{Z}^2),$$

and then according to Lemma 3 the right hand side of the equality (6) is uniquely represented by the sum

$$A_{d,=}^{-1}(\xi)\widetilde{(\ell v_d)}(\xi) = f_d^+(\xi) + f_d^-(\xi),$$

where

$$f_d^+(\xi) = B_h(A_{d,=}^{-1}(\xi)\widetilde{(\ell v_d)}(\xi)), \quad f_d^-(\xi) = (I - B_h)(A_{d,=}^{-1}(\xi)\widetilde{(\ell v_d)}(\xi))$$

.

Further, we rewrite the equality (6)

$$A_{d,\neq}(\xi)\tilde{u}_d(\xi) - f_d^+(\xi) = f_d^-(\xi) - A_{d,=}^{-1}(\xi)\tilde{w}_d(\xi)$$

and using the uniqueness of the representation as the direct sum $\widetilde{H}^{s-æ}(K_d) \oplus \widetilde{H}^{s-æ}(h\mathbb{Z}^2 \setminus K_d)$ we conclude that both left hand side and right hand side should be zero. Thus,

$$\tilde{u}_d(\xi) = A_{d,\neq}^{-1}(\xi)B_h(A_{d,=}^{-1}(\xi)\widetilde{(\ell v_d)}(\xi)),$$

and Theorem 1 is proved. ∎

4. Discrete boundary value problem

In this section we consider more interesting case when the equation (5) has a lot of solutions.

4.1. **Form of a discrete solution.** This section uses some results from [21] concerning a form of a discrete distribution supported at the origin.

Theorem 2. *Let $æ - s = n + \delta, n \in \mathbb{N}, |\delta| < 1/2$. Then a general solution of the equation* (5) *has the following form*

$$\tilde{u}_d(\xi) = A_{d,\neq}^{-1}(\xi)Q_n(\xi)B_h(Q_n^{-1}(\xi)A_{d,=}^{-1}(\xi)\widetilde{(\ell v_d)}(\xi)) + A_{d,\neq}^{-1}(\xi)\left(\sum_{k=0}^{n-1} \tilde{c}_k(\xi_1)\hat{\zeta}_2^k + \tilde{d}_k(\xi_2)\hat{\zeta}_1^k\right),$$

where $Q_n(\xi)$ is an arbitrary polynomial of order n of variables $\zeta_k = \hbar(e^{-ih\xi_k} - 1), k = 1, 2$, satisfying the condition (4) *for $\alpha = n$, $\tilde{c}_k(\xi_1), \tilde{d}_k(\xi_2), k = 0, 1, \cdots, n-1$, — are arbitrary functions from $H^{s_k}(h\mathbb{T}), s_k = s - æ + k - 1/2$.*

The a priori estimate

$$||u_d||_s \leq const\left(||f||_{s-\alpha}^+ + \sum_{k=0}^{n-1}([c_k]_{s_k} + [d_k]_{s_k})\right),$$

holds, where $[\cdot]_{s_k}$ denotes a norm in $H^{s_k}(h\mathbb{T})$, and const does not depend on h.

Proof. We start from the equality (6). Let $Q_n(\xi)$ be an arbitrary polynomial of order n of variables $\zeta_k = \hbar(e^{-ih\xi_k} - 1), k = 1, 2$, satisfying the condition (4) for $\alpha = n$. We multiply the equality (6) by $Q_n^{-1}(\xi)$

$$Q_n^{-1}(\xi)A_{d,\neq}(\xi)\tilde{u}_d(\xi) + Q_n^{-1}(\xi)A_{d,=}^{-1}(\xi)\tilde{w}_d(\xi) = Q_n^{-1}(\xi)A_{d,=}^{-1}(\xi)\widetilde{(\ell v_d)}(\xi), \quad \xi \in \hbar\mathbb{T}^2, \tag{7}$$

We have in view of Lemma 1

$$Q_n^{-1}(\xi)A_{d,=}^{-1}(\xi)\widetilde{(\ell v_d)}(\xi) \in \widetilde{H}^{s-\text{æ}+n}(h\mathbb{Z}^2),$$

and since $s - \text{æ} + n = -\delta$ then according to Lemma 3 we write the unique decomposition

$$Q_n^{-1}(\xi)A_{d,=}^{-1}(\xi)\widetilde{(\ell v_d)}(\xi) = F_d^+(\xi) + F_d^-(\xi),$$

where

$$F_d^+(\xi) = B_h(Q_n^{-1}(\xi)A_{d,=}^{-1}(\xi)\widetilde{(\ell v_d)}(\xi)), \quad F_d^-(\xi) = (I - B_h)(Q_n^{-1}(\xi)A_{d,=}^{-1}(\xi)\widetilde{(\ell v_d)}(\xi)).$$

Taking into account this fact we rewrite the equality (7) in the form

$$A_{d,\neq}(\xi)\tilde{u}_d(\xi) + A_{d,=}^{-1}(\xi)\tilde{w}_d(\xi) = Q_n(\xi)F_d^+(\xi) + Q_n(\xi)F_d^-(\xi),$$

and further,

$$A_{d,\neq}(\xi)\tilde{u}_d(\xi) - Q_n(\xi)F_d^+(\xi) = Q_n(\xi)F_d^-(\xi) - A_{d,=}^{-1}(\xi)\tilde{w}_d(\xi),$$

Since $F_d^+(\xi) \in \widetilde{H}^{s-\text{æ}+n}(K_d), F_d^-(\xi) \in \widetilde{H}^{s-\text{æ}+n}(h\mathbb{Z}^2 \setminus K_d)$ then according to Lemma 1 we conclude $Q_n(\xi)F_d^+(\xi) \in \widetilde{H}^{s-\text{æ}}(K_d), Q_n(\xi)F_d^-(\xi) \in \widetilde{H}^{s-\text{æ}}(h\mathbb{Z}^2 \setminus K_d)$. Applying the inverse discrete Fourier transform we obtain an equality for two discrete distributions. The left hand side vanishes at least under one condition $\tilde{x}_1 < 0$ or $\tilde{x}_2 < 0$, and the right hand side vanishes under the condition $.\tilde{x}_1 > 0, \tilde{x}_2 > 0$, Thus, it should be a discrete distribution supported on sides of the discrete quadrant $\{(\tilde{x}_1, \tilde{x}_2) \in h\mathbb{Z}^2 : \{\tilde{x}_1 > 0, \tilde{x}_2 = 0\} \cup \{\tilde{x}_1 = 0, \tilde{x}_2 > 0\}\}$. Using corresponding result from [21] we obtain the following form for this discrete distribution

$$\sum_{k=0}^{n-1} \left(c_k(\tilde{x}_1)(\Delta_2^{(k)}\delta_d)(\tilde{x}_2) + d_k(\tilde{x}_2)(\Delta_1^{(k)}\delta_d)(\tilde{x}_1) \right),$$

where all summands should be elements of the space $H^{s-\text{æ}}(h\mathbb{Z}^2)$.

The left question is how much summands we need in the right-hand side. Counting principle is a very simple because every summand should belong to the space $\widetilde{H}^s(\hbar\mathbb{T}^2)$.

Let us consider the summand $c_k(\xi_1)\zeta_2^k$. Taking into account that order of $A_{d,+}^{-1}(\xi)$ is $-\text{æ}$ we need to verify the finiteness of the $H^{s-\text{æ}}$-norm for $c_k(\xi_1)\zeta_2^k$. We have

$$||c_k(\Delta_2^{(k)}\delta_d)||_{s-\text{æ}}^2 = \int\limits_{\hbar\mathbb{T}^2} (1 + |\zeta^2|)^{s-\text{æ}}||c_k(\xi_1)\zeta_2^k|^2 d\xi$$

$$= \int\limits_{\hbar\mathbb{T}^2} (1 + |\zeta^2|)^{s-\text{æ}}||c_k(\xi_1)|^2|\zeta_2^k|^2 d\xi \leq a_1\hbar^{2(s-\text{æ}+k+1/2)} \int\limits_{\hbar\mathbb{T}} |c_k(\xi_1)|^2 d\xi_1$$

$$\leq a_2 \int\limits_{\hbar\mathbb{T}} (1 + |\zeta_1{}^2|)^{s-\text{æ}+k+1/2}|c_k(\xi_1)|^2 d\xi_1,$$

and the constants a_1, a_2 do not depend on h.

The last summand should be $(n-1)$th because for nth summand we obtain a positive growth: for $k = n$ we have $s_n = s - \text{æ} - n + 1/2 = -n - \delta + n + 1/2 = -\delta + 1/2 > 0$.

A priori estimates can be obtained in the same way described in [21]. ∎

4.2. **The Dirichlet discrete boundary condition.** We consider here first simple case with discrete Dirichlet boundary conditions. We suppose in this section that $æ - s = 1 + \delta, |\delta| < 1/2, v_d \equiv 0$. It follows from Theorem 2 that we have the following general solution of the equation (5)

$$(8) \qquad \tilde{u}_d(\xi) = A_{d,\neq}^{-1}(\xi)(\tilde{c}_0(\xi_1) + \tilde{d}_0(\xi_2)),$$

where $c_0, d_0 \in H^{s-æ-1/2}(\hbar\mathbb{Z})$ are arbitrary functions. To determine uniquely these functions we add the discrete Dirichlet conditions on angle sides

$$(9) \qquad u_{d|_{\tilde{x}_1=0}} = f_d(\tilde{x}_2), \qquad\qquad u_{d|_{\tilde{x}_2=0}} = g_d(\tilde{x}_1).$$

Thus, we have the discrete Dirichlet problem (5),(9).

First, we apply the discrete Fourier transform to discrete conditions (9) and obtain the following form

$$(10) \qquad \int\limits_{-\hbar\pi}^{\hbar\pi} \tilde{u}_d(\xi_1, \xi_2)d\xi_1 = \tilde{f}_d(\xi_2), \quad \int\limits_{-\hbar\pi}^{\hbar\pi} \tilde{u}_d(\xi_1, \xi_2)d\xi_2 = \tilde{g}_d(\xi_1).$$

Substituting (10) into (8) we obtain the following relations

$$\int\limits_{-\hbar\pi}^{\hbar\pi} \tilde{u}_d(\xi)d\xi_1 = \int\limits_{-\hbar\pi}^{\hbar\pi} A_{d,\neq}^{-1}(\xi)\tilde{c}_0(\xi_1)d\xi_1 + \tilde{d}_0(\xi_2)\int\limits_{-\hbar\pi}^{\hbar\pi} A_{d,\neq}^{-1}(\xi)d\xi_1$$

$$\int\limits_{-\hbar\pi}^{\hbar\pi} \tilde{u}_d(\xi)d\xi_2 = \tilde{c}_0(\xi_1)\int\limits_{-\hbar\pi}^{\hbar\pi} A_{d,\neq}^{-1}(\xi)d\xi_2 + \int\limits_{-\hbar\pi}^{\hbar\pi} A_{d,\neq}^{-1}(\xi)\tilde{d}_0(\xi_2)d\xi_2$$

Let us denote

$$\int\limits_{-\hbar\pi}^{\hbar\pi} A_{d,\neq}^{-1}(\xi)d\xi_1 \equiv \tilde{a}_0(\xi_2), \quad \int\limits_{-\hbar\pi}^{\hbar\pi} A_{d,\neq}^{-1}(\xi)d\xi_2 \equiv \tilde{b}_0(\xi_1)$$

and suppose that $\tilde{a}_0(\xi_2), \tilde{b}_0(\xi_1) \neq 0, \forall \xi_1 \neq 0, \xi_2 \neq 0$.

Therefore, we have the following system of two linear integral equations with respect to two unknown functions $\tilde{c}_0(\xi_1), \tilde{d}_0(\xi_2)$

$$(11) \qquad \begin{cases} \int\limits_{-\hbar\pi}^{\hbar\pi} K_1(\xi)\tilde{c}_0(\xi_1)d\xi_1 + \tilde{d}_0(\xi_2) = \tilde{F}_d(\xi_2) \\ \tilde{c}_0(\xi_1) + \int\limits_{-\hbar\pi}^{\hbar\pi} K_2(\xi)\tilde{d}_0(\xi_2)d\xi_2 = \tilde{G}_d(\xi_1), \end{cases}$$

where we have used the following notations

$$\tilde{F}_d(\xi_2) = \tilde{f}_d(\xi_2)\tilde{a}_0^{-1}(\xi_2), \quad \tilde{G}_d(\xi_1) = \tilde{g}_d(\xi_1)\tilde{b}_0^{-1}(\xi_1),$$

$$K_1(\xi) = A_{d,\neq}^{-1}(\xi)\tilde{b}_0^{-1}(\xi_1), = \quad K_2(\xi) = A_{d,\neq}^{-1}(\xi)\tilde{a}_0^{-1}(\xi_2).$$

Unique solvability conditions for the system (11) will be equivalent to unique solvability for the discrete Dirichlet problem (5),(9).

Thus, we obtain the following result.

Proposition 1. *Let $f_d, g_d \in H^{s-1/2}(\mathbb{R}_+), s > 1/2, v_d \equiv 0$. Then the discrete Dirichlet problem (5),(9) is reduced to the equivalent system of linear integral equations (11).*

4.3. Non-local discrete boundary condition. We consider here the $æ - s = 1 + \delta, |\delta| < 1/2$ for the equation (5) with different boundary conditions, namely
$$(12)$$
$$\sum_{\tilde{x}_1 \in h\mathbb{Z}_+} u_d(\tilde{x}_1, \tilde{x}_2)h = f_d(\tilde{x}_2), \quad \sum_{\tilde{x}_2 \in h\mathbb{Z}_+} u_d(\tilde{x}_1, \tilde{x}_2)h = g_d(\tilde{x}_1), \quad \sum_{\tilde{x} \in h\mathbb{Z}_{++}} u_d(\tilde{x}_1, \tilde{x}_2)h^2 = 0.$$

These additional conditions will help us to determine uniquely the unknown functions c_0, d_0 in the solution (8).

Indeed, using the discrete Fourier transform we rewrite the conditions (12) as follows

$$(13) \qquad \tilde{u}_d(0, \xi_2) = \tilde{f}_d(\xi_2), \quad \tilde{u}_d(\xi_1, 0) = \tilde{g}_d(\xi_1), \quad \tilde{u}_d(0, 0) = 0.$$

Now we substitute the formulas (13) into (8). The first two equality are

$$\tilde{u}_d(0, \xi_2) = A_{d,\neq}^{-1}(0, \xi_2)(\tilde{c}_0(0) + \tilde{d}_0(\xi_2)) = \tilde{f}_d(\xi_2),$$

$$\tilde{u}_d(\xi_1, 0) = A_{d,\neq}^{-1}(\xi_1, 0)(\tilde{c}_0(\xi_1) + \tilde{d}_0(0)) = \tilde{g}_d(\xi_1).$$

It implies the following relations according to the third condition

$$\tilde{f}_d(0) = \tilde{g}_d(0), \quad \text{and from which} \quad \tilde{c}_0(0) + \tilde{d}_0(0) = 0, \quad \text{and} \quad \tilde{c}_0(0) = \tilde{d}_0(0) = 0.$$

Then we have at least formally

$$(14) \qquad \tilde{u}_d(\xi) = A_{d,\neq}^{-1}(\xi) \left(A_{d,\neq}(\xi_1, 0)\tilde{g}_d(\xi_1) + A_{d,\neq}(0, \xi_2)\tilde{f}_d(\xi_2) \right)$$

It is left to formulate and to prove exactly the obtained result.

Theorem 3. *Let $f_d, g_d \in H^{s+1/2}(h\mathbb{Z}), v_d \equiv 0$. Then the discrete problem (5),(12) has unique solution which is given by the formula (14).*

The a priori estimate

$$||u_d||_s \leq const(||f_d||_{s+1/2} + ||g_d||_{s+1/2})$$

holds with a const non-depending on h

Proof. We need to prove the a priori estimate only. Let us consider the first summand

$$||A_{d,\neq}^{-1}(\xi)A_{d,\neq}(\xi_1, 0)\tilde{g}_d(\xi_1)||_s^2 = \int_{\hbar\mathbb{T}^2} |A_{d,\neq}^{-1}(\xi_1, \xi_2)A_{d,\neq}(\xi_1, 0)\tilde{g}_d(\xi_1)|^2(1+|\zeta^2|)^s d\xi_1 d\xi_2 \leq$$

$$\leq C\hbar^{2s} \int_{\hbar\mathbb{T}^2} |g_d(\xi_1)|^2 d\xi \leq C_1\hbar^{2s+1} \int_{-\hbar\pi}^{\hbar\pi} |g_d(\xi_1)|^2 d\xi_1 \leq$$

$$\leq C_2 \int_{-\hbar\pi}^{\hbar\pi} |g_d(\xi_1)|^2(1+|\zeta_1^2|)^{s+1/2} d\xi_1 = ||g_d||_{s+1/2}^2.$$

The second summand has the same estimate. ∎

5. A COMPARISON BETWEEN DISCRETE AND CONTINUOUS SOLUTIONS

The continuous analogue of the discrete boundary value problem is the following [19].

Let A be a pseudo-differential operator with the symbol $A(\xi), \xi = (\xi_1, \xi_2)$ satisfying the condition

$$c_1(1 + |\xi|)^\alpha \leq |A(\xi)| \leq c_2(1 + |\xi|)^\alpha.$$

and admitting the wave factorization with respect to the quadrant K with index æ.

We consider the equation

(15) $$(Au)(x) = 0, \quad x \in K,$$

with the following additional conditions

(16) $$\int\limits_0^{+\infty} u(x_1, x_2)dx_1 = f(x_2), \quad \int\limits_0^{+\infty} u(x_1, x_2)dx_2 = g(x_1), \quad \int\limits_{-K} u(x)dx = 0.$$

A solution of the problem (15),(16) is sought in the space $H^s(K)$ [18] and boundary functions are taken from the space $H^{s+1/2}(\mathbb{R}_+)$. Such problem was considered in [19] and it has the solution

(17) $$\tilde{u}(\xi) = A_{\neq}^{-1}(\xi)\left(A_{\neq}(\xi_1, 0)\tilde{g}(\xi_1) + A_{\neq}(0, \xi_2)\tilde{f}(\xi_2)\right)$$

under condition that the symbol $A(\xi)$ admits the wave factorization with respect to the quadrant K

$$A(\xi) = A_{\neq}(\xi)A_{=}(\xi)$$

with index æ such that $æ - s = 1 + \delta, |\delta| < 1/2$.

To construct a discrete boundary value problem which is good approximation to (15),(16) we need to choose $A_d(\xi)$ and f_d, g_d in a special way. First, we introduce the operator l_h which acts as follows. For a function u defined in $\mathbb{R}$ we take its Fourier transform $\tilde{f}$ then we take its restriction on $\hbar T$ and periodically extend it to $\mathbb{R}$. Finally, we take its inverse discrete Fourier transform and obtain the function of discrete variable $(l_h u)(\tilde{x}), \tilde{x} \in h\mathbb{R}$. Thus, we put

$$f_d = l_h f, \quad g_d = l_h g.$$

Second, the symbol of digital operator A_d we construct in the same way. If we have the wave factorization for the symbol $A(\xi)$ then we take restrictions of factors on $\hbar\mathbb{T}^2$ and the periodic symbol $A_d(\xi)$ is a product of these restrictions. For such f_d, g_d and the symbol $A_d(\xi)$ we obtain the following result.

Theorem 4. Let $f, g \in S(\mathbb{R}), æ > 1$. Then we have the following estimate for solutions u and u_d of the continuous problem (15),(16) and the discrete one (5),(12)

$$|u(\tilde{x}) - u_d(\tilde{x})| \leq C(f, g)h^\beta,$$

where the const $C(f, g)$ depends on functions f, g, $\beta > 0$ can be an arbitrary number.

Proof. We need to compare two functions (14) and (17), more exactly their inverse discrete Fourier transform and inverse Fourier transform at points $\tilde{x} \in K_d$.

We have

$$u_d(\tilde{x}) - u(\tilde{x}) = \frac{1}{4\pi^2} \left(\int\limits_{\hbar\mathbb{T}^2} e^{i\tilde{x}\cdot\xi}\tilde{u}_d(\xi)d\xi - \int\limits_{\mathbb{R}^2} e^{i\tilde{x}\cdot\xi}\tilde{u}(\xi)d\xi \right) =$$

$$= \frac{1}{4\pi^2} \int\limits_{\mathbb{R}^2\setminus\hbar\mathbb{T}^2} e^{i\tilde{x}\cdot\xi}A_{\neq}^{-1}(\xi) \left(A_{\neq}(\xi_1,0)\tilde{g}(\xi_1) + A_{\neq}(0,\xi_2)\tilde{f}(\xi_2) \right) d\xi,$$

since according to our choice for A_d, f_d, g_d the functions $\tilde{u}_d$ and $\tilde{u}$ coincide in points $\xi \in \hbar\mathbb{T}^2$.

We will estimate one summand.

$$\left| \frac{1}{4\pi^2} \int\limits_{\mathbb{R}^2\setminus\hbar\mathbb{T}^2} e^{i\tilde{x}\cdot\xi}A_{\neq}^{-1}(\xi)A_{\neq}(\xi_1,0)\tilde{g}(\xi_1)d\xi \right| \leq C \int\limits_{\hbar\pi}^{+\infty} \frac{d\xi_2}{(1+|\xi_1|+|\xi_2|)^{\text{æ}}} \int\limits_{\hbar\pi}^{+\infty} |\xi_1|^{-\gamma}d\xi_1,$$

since $\tilde{g} \in S(\mathbb{R})$. It implies the required estimate. ∎

CONCLUSION

In this paper we have considered two-dimensional cone only, but the authors continue to work in multidimensional situations and we hope to obtain results similar to a discrete half-space [21, 15].

As first practical applications the authors plan to study discrete variant of a quarter-plane problem in diffraction theory [14] and elasticity theory [18]. We hope it will useful application of the developed theory.

REFERENCES

1. B Alexeev, *Nonlocal astrophysics*, Elsevier, Amsterdam, 2017.
2. L. Botchway, R. Kibiti, and M Ruzhansky, *Difference equations and pseudo-differential equations on $\mathbf{z}^n$*, arXiv:1705.07564v1 [math.FA] (2017), 1–29.
3. F Camilli, R.D. Maio, and A Tosin, *Measure-valued solutions to nonlocal transport equations on networks*, J. Differ. Equations **264** (2018), no. 12, 7213–7241.
4. W. Chen and S Kim, *Bubble-tower solutions to asymptotically critical nonlocal elliptic equations on conformal infinities*, J. Differ. Equations **265** (2018), no. 19, 5115–5155.
5. R Edwards, *Fourier series. a modern introduction*, Springer-Verlag, Heidelberg, 1982.
6. G Eskin, *Boundary value problems for elliptic pseudodifferential equations*, AMS, Providence, 1981.
7. L Frank, *Spaces of network functions*, Math. USSR Sb. **15** (1971), no. 2, 182–226.
8. F.D Gakhov, *Boundary value problems*, Dover Publications, Mineola, 1981.
9. L Hoermander, *Analysis of partial differential operators, vol. i–iv*, Springer-Verlag, Berlin, 1983.
10. N.I Muskhelishvili, *Singular integral equations*, North Holland, Amsterdam, 1976.
11. V Rabinovich, *Wiener algebra of operators on the lattice $\mu\mathbf{z}^n$ depending on the small parameter $\mu > 0$*, Complex Var. Ell. Equ. **58** (2013), no. 6, 751–766.
12. V Ryaben'kii, *Method of difference potentials and its applications*, Springer-Verlag, Berlin–Heidelberg, 2002.
13. A Samarskii, *The theory of difference schemes*, CRC Press, Boca Raton, 2001.
14. F.-O Speck, *From sommerfeld diffraction problems to operator factorisation*, Constr. Math. Anal.. **2** (2019), no. 4, 183–216.
15. O. A. Tarasova and V. B Vasilyev, *To the theory of discrete boundary value problems*, 4open **2** (2019), no. 17, 1–7.
16. M Taylor, *Pseudodifferential operators*, Princeton University Press, Princeton, 1981.
17. F Treves, *Introduction to pseudodifferential operators and fourier integral operators*, Springer, New York, 1980.

18. V. B Vasil'ev, *Wave factorization of elliptic symbols: Theory and applications*, Kluwer Academic Publishers, Dordrecht–Boston–London, 2000.

19. _____, *On some new boundary-value problems in nonsmooth domains*, J. Math. Sci. **173** (2011), no. 2, 225–230.

20. A. Vasilyev and V Vasilyev, *Discrete singular operators and equations in a half-space*, Azerb. J. Math. **3** (2013), no. 1, 84–93.

21. A. V. Vasilyev and V. B Vasilyev, *Pseudo-differential operators and equations in a discrete half-space*, Math. Model. Anal. **23** (2018), no. 3, 492–506.

22. V Vasilyev, *Discrete equations and periodic wave factorization*, Proceedings of the International Conference on Analysis and Applied Mathematics (ICAAM-2016) (Melville) (A. Ashyralyev and A. Lukashov, eds.), AIP Conf. Proc., vol. 1759, AIP Publishing, 2016, p. 020126.

23. _____, *Discreteness, periodicity, holomorphy, and factorization*, Integral Methods in Science and Engineering (New York) (C. Constanda, M. Dalla Riva, P.D. Lamberti, and P. Musolino, eds.), Theoretical Technique, vol. 1, Birkhauser, 2017, pp. 315–324.

24. _____, *On discrete boundary value problems*, Proceedings of the International Conference "Functional Analysis in Interdisciplinary Applications" (FAIA2017) (Melville) (T. Kal'menov and M. Sadybekov, eds.), AIP Conf. Proc., vol. 1880, AIP Publishing, 2017, p. 050010.

25. _____, *The periodic cauchy kernel, the periodic bochner kernel, discrete pseudo-differential operators*, Proceedings of the International Conference on Numerical Analysis and Applications (ICNAAM-2016) (Melville) (T. Simos and C. Tsitouras, eds.), AIP Conf. Proc., vol. 1863, AIP Publishing, 2017, p. 140014.

26. V.S Vladimirov, *Generalized functions in mathematical physics*, Mir, Moscow, 1979.

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