A Study of Axiomatizability And Decidability of the Mathematical Structures

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Abstract—In this paper, decidability of the structures will be examined in several languages. Decidability or undecidability of mathematical structures is one of the fundamental and sometimes very difficult problems of Mathematical logic, where several examples of problems in this field are still open and unresolved even after decades. One of the goals of Mathematical Logic is the axiomatization of mathematical theoriesTarski has proved The decidability of the theory of real and complex numbers in the language of addition and multiplication, And it is proved that the theories of natural, integer, and rational numbers, in the language of addition and multiplication, are undecidable(Theorems of Gödel and Robinson).

We will proof The following problemes:

The Main Problem 1:If the structure $\langle Q; \sqsubseteq \rangle$, where $\sqsubseteq = \mid$ (divisibility relation) is decidable?

Problem 2: We will give an explicit axiomatization for $\langle Z; \times \rangle$? We will study boolean algebras. Boolean algebras are famous mathematical structures. Tarski showed the decidability of the elementary theory of Boolean algebras. N this paper, we consider the different kinds of Boolean algebras and their properties. And we present for the first-order theory of atomic Boolean algebras a quantifier elimination algorithm. The subset relation is a partial order and indeed a lattice order, And I will prove that the theory of atomic Boolean lattice orders is decidable, and furthermore admits elimination of quantifiers. So the theory of the subset relation is decidable. And we will study decidability of atomlss boolean algebra.

Keywords—Boolean algebras, Decidability, Model Theory, Quantifier-Elimination.

I. Introduction

A mathematical structure consists of a specific set (usually a set of numbers, like natural, integer, rational, real or complex numbers) in a first-order language that contains some functions, predicates or constants. The theory of a structure is the set of all first-order sentences (in the language of that structure) which are true in that structure.

Structure : $A = \langle A; \mathcal{L} \rangle$ $\qquad$ $Th(A) = \{\theta \in \mathcal{L} \mid A \models \theta\}$

For example the sentence "any number is equal to the sum of another number with itself" is false in (the domain of) integer numbers, but it is true in (the domain of) rational numbers (for e.g., 3 there are no integer n such that $n + n = 3$, but the sum of 3/2 with itself is 3).

$$\langle Z; + \rangle \not\models forall x \exists y (x = y + y) \qquad \langle Q; + \rangle \models \forall x \exists y (x = y + y)$$

The theory of a mathematical structures is decidable, when there exists an algorithm such that for each sentence given in the language of the structure, as an input, it outputs "true" or "false", depending on whether the sentence in is true or false.

In the other words, the theory of a structure is decidable, when it is possible to algorithmically determine the truth of the first-order sentences in that structure.

Decidability or undecidability of mathematical structures is one of the fundamental and sometimes very difficult problems of mathematical logic, where several examples of problems in this field are still open and unresolved even after decades.

One of the goals of Mathematical Logic is the axiomatization of mathematical theories. For example, Tarski has proved the decidability of the theory of real and complex numbers in the language of addition and multiplication, and it is proved thattheories of natural, integer, and rational numbers, in the language of addition and multiplication, are undecidable (Theorems of Gödel and Robinson).

So, the theories of the structures $\langle R; +, \times \rangle$ and $\langle C; +, \times \rangle$ are decidable, but the theories of the structures $\langle N; +, \times \rangle$ and $\langle Z; +, \times \rangle$ and $\langle Q; +, \times \rangle$ are decidable.

II. Structure-Decidable Structure-Examples of Decidable Structures With Proof-Some Examples Of Undecidable Structures

Quantifier elimination is a very powerful property, and helps in the proof of decidability.Quantifier elimination has more applications . In 1927 1927 and 1928 Alfred Tarski was in charge of the seminar on problems in logic at the University of Warsaw. He used this seminar to pursue the development of the method of quantifier elimination. A theory admites quantifier elimination if every formula is equivalent, to a formula without any quantifiers. Tarski and his students at the Warsaw seminar achieved significant results. Tarski suggested to one of his students-his names was Mojzesz Presburger-to develop an elimination-of-quantifiers procedure for the additive theory of the integer numbers. The student succeded and submitted the result as his thesis for a master's degree. The theory became known as Presburger Arithmetic. In this paper, we use quantifier elimination to show decidability of the theory of a certain structure.

Axiomatizing mathematical structure is a goal of mathematical logic. An axiom is a statement or proposition on which an abstractly defined structure is based. A theory of a structure is axiomatizable if there is a decidable set of which are all theorems of T such that every theorem of T can be proved from these axioms.

lamma1.(The Lemma of Quantifier Elimination) A theory (or a structure) admits quantifier elimination if and only if every formula of the form $\exists x(\wedge_i \alpha_i)$ is (recursively) equal with a quantifier -free formula , where each α_i is either an atomic formula or the negation of an atomic formula.

Proof:

Every formula ψ can be written (equivalency) in the prenex normal form, say

$$Q_1 x_1 Q_2 x_2 \cdots Q_n x_n \theta(x_1, x_2, \cdots, x_n)$$

where $Q_i's$ ara quantifiers and θ is quantifier-free. if $Q_n = \exists$, then let $\acute{\theta} = \theta$, and if $Q_n = \forall$, then $\acute{\theta} = \neg\theta$, (note that in the latter case $\forall x_n \equiv \neg\exists x_n \acute{\theta}$). Now, the quantifier- free formula $\acute{\theta}$ can be written in the disjunctive normal form,say $\bigvee_i \bigwedge_j \alpha_{i,j}$ where each $\alpha_{i,j}$ is a literal(i.e., an atomic or a negated atomic formula).Noting that $\exists x(\bigwedge_i \beta_i \equiv \bigvee_i \exists x \beta_i)$ we have

$$\psi \equiv Q_1 x_1 Q_2 x_2 \cdots Q_{n-1} x_{n-1} \bigcirc \bigvee \exists x_n (\bigwedge_j \alpha_{i,j})$$

where $\bigcirc$ is nothing (empty) when $Q_n = \exists$ and $\bigcirc = \neg$ when $Q_n = \forall$. Now,if $\exists x_n (\bigwedge_j \alpha_{i,j})$ is equal with a quantifier -free formula, ψ is equal with a formula with one less quantifier; counting this way one can show that ψ equal with a formula which has no quantifier.[11]

A. Decidability of Structure of Natural Numbers in Different Languages

Gödel's incompleteness theorems are two theorems of mathematical logic that demonstrate the inherent limitations of every formal axiomatic system capable of modeling basic arithmetic. These results, published by Kurt Gödel in 1931, are important both in mathematical logic and in the philosophy of mathematics. The theorems are widely, but not universally, interpreted as showing that Hilbert's program to find a complete and consistent set of axioms for all mathematics is impossible. The first incompleteness theorem states that no consistent system of axioms whose theorems can be listed by an effective procedure (i.e., an algorithm) is capable of proving all truths about the arithmetic of natural numbers. For any such consistent formal system, there will always be statements about natural numbers that are true, but that are unprovable within the system. The second incompleteness theorem, an extension of the first, shows that the system cannot demonstrate its own consistency. Employing a diagonal argument, Gödel's incompleteness theorems were the first of several closely related theorems on the limitations of formal systems. They were followed by Tarski's undefinability theorem on the formal undefinability of truth, Church's proof that Hilbert's Entscheidungsproblem is unsolvable, and Turing's theorem that there is no algorithm to solve the halting problem. We can also have decidable and complete theories. e.g. Presburger Arithmetic. and We can have undecidable and incomplete theories. e.g. Peano Arithmetic Gödel (well, actually Rosser building on Gödel) showed that any consistent, recursively axiomatizable set of sentences in the language of arithmetic which contains a certain (very small) theory is not complete (in your sense, which is what "complete" means). In particular, this means that no consistent recursively axiomatizable set of sentences in the language of arithmetic proves every true sentence of arithmetic. Decidability should not be confused with completeness. For example, the theory of algebraically closed fields is decidable but incomplete, whereas the set of all true first-order statements about nonnegative integers in the language with $+$ and $\times$ is complete but undecidable.

The original proof of the decidability of Presburger arithmetic goes by Quantifier elimination, for example,

$$\exists x(\underbrace{x + \cdots + x}_{a-times} + b = c)$$

is equivalent with $b \equiv_a c$, and such a congruence statement is easily decidable. The proof reduces every formula to a boolean combination of such basic formulas. Surely, by this reasoning, I see that,

$$Th(\mathcal{N}) = \{\phi | N \models \phi\}$$

is a decidable theory, i.e. for every formula with summation I can decide if it holds or not when interpreted in the natural numbers. But what Presburger actually is doing, is he gives a decidable set of axioms, considers the closure under deduction of these axioms. And uses the above arguments to conclude that set of all deducable formulas from the axioms is decidable The elimination of quantifier presented for addition theory.

the first complete proof of decidability of the $Th(w^+; \cdot, 1)$ of multiplication of the positive integers. He did not prove this result by a quantifier-elimination; Skolem chose five examples illustrating his decision method and declared the method completely general. the first complete proof of decidability of the $Th(w^+; \cdot, 1)$ published was given by Andrzej Mostowski1952. Mostowski considered the general problem of relating the decidability of the theory of a product of structures to decidablities of the factor structures.As an illustration of one of his results, he reduced the decidability of $Th(w^+; \cdot, 1)$ to $Th(w; +, 0, 1)$. This reduction is based on the Fundamental Theorem of Arithmetic.Let $p_0 = 2, p_1, \cdots$ be the sequence of prime numbers.every positive x can be written uniquelly in the form:

$$x = (p_0)^{n_0}(p_1)^{n_1} \cdots$$

where each $n_i \in w$ and all but finitely many n_i are 0 if, moreover

$$y = (p_0)^{m_0}(p_1)^{m_1} \cdots$$

then

$$xy = (p_0)^{n_0+m_0}(p_1)^{n_1+m_1} \cdots$$

and

$$1 = (p_0)^0 (p_1)^0 \cdots$$

In short , $(w^+; \cdot, 1)$ is isomorphic to the weak power of $(w; +, 0, 1)$

What Mostowski observed was that $(w^+; \cdot, 1)$ is weaker direct power of $(w; +, 0, 1)$, with 0 becoming 1 and + becoming $\cdot$ (and $<$ becoming the divisibility relation when $<$ added to structure). In the late 1950s, Solomon Feferman and Robert Vaught generalized Mostowski's result, but did not specifically apply for their work an elaborate quantifier -elimination to $Th(w^+; \cdot, 1)$. This was done in 1980 by Patric Cegielski, Who transformed the quantifier elimination for the theory of addition into one for the theory of multiplication.He was also to supply an axiomatization for $Th(w^+; \cdot, 1)$. Both the quantifier -elimination and the axiomatization are best to understand by thinking of $(w^+; \cdot, 1)$ as weak direct power of $(w; +, 0, 1)$.

the Peano axioms, also known as the Dedekind–Peano axioms or the Peano postulates, are axioms for the natural numbers presented by the 19th-century Italian mathematician Giuseppe Peano. These axioms have been used nearly unchanged in a number of metamathematical investigations, including research into fundamental questions of whether number theory is consistent and complete. The need to formalize arithmetic was not well appreciated until the work of Hermann Grassmann, who showed in the 1860s that many facts in arithmetics could be derived from more basic facts about the successor operation and induction. In 1881, Charles Sanders Peirce provided an axiomatization of natural number arithmetic. In 1888, Richard Dedekind proposed another axiomatization of natural number arithmetic, and in 1889, Peano published a simplified version of them as a collection of axioms in his book, The principles of arithmetic presented by a new method. The Peano axioms contain three types of statements. One of the axioms asserts the existence of at least one member of the set of natural numbers. The next are general statements about equality; in modern treatments, these are often not taken as part of the Peano axioms, The next axioms are first-order statements about natural numbers expressing the fundamental properties of the successor operation. and, a final axiom is a second-order statement of the principle of mathematical induction over the natural numbers. A weaker first-order system called Peano arithmetic is obtained by explicitly adding the addition and multiplication operation symbols and replacing the second-order induction axiom with a first-order axiom schema. It is well-known that Peano Arithmetic is undecidable.

Theorem1. the theory $Th\mathcal{N}_s$ where $\mathcal{N}_s = (N, 0, s)$ admits elimination of quantifier.
Proof: [9].

Theorem2. The Theory $\mathcal{N}_L = (N, 0, S, <)$ admits elimination quantifier, and so has a decidable theory and is finitely axiomatizable.
Proof: [9].

The additive theory of natural numbers:
Presburger proof decidability of the theory $\langle N; =, + \rangle$ with quantifier elimination. One common way of quantifier elimination is to extend the language, and we add Fixed symbols 0 and 1 and an infinite set of binary relations $<_n$ for $n \geq 1$. Which is defined as follows:

$$\forall x, y \in N, x <_n y \leftrightarrow (x < y \ \& \ x \equiv y \ (mod\, n))$$

Theorem3. The following axioms at the Language $L = \{+, 0, 1, \leq, \{\equiv_m\}_{\geq 2}\}$, for the structure N allow quantifier elimination.

(A1) $x + (y + z) = (x + y) + z$

(A2) $x + y = y + x$

(A3) $x + 0 = x$

(A4) $x + z = y + z \rightarrow x = y$

(A5) $x + y = 0 \rightarrow x = y = 0$

(O1) $x \leq y \leftrightarrow \exists z (x + z = y)$

(O2) $x \leq y \lor y \leq x$

(O3) $0 \neq 1 \land \forall y (0 \leq y \leq 1 \rightarrow y = 0 \lor y = 1)$

(D1) $\forall x \exists y, z (x = n \cdot y + t \land t < \bar{n})$

Proof:
Step 1: Identify the terms

In structure $\langle N; +, 0, <, 1, \{\equiv_n\}\rangle_{n \geq 2}$, every term involving x is equal to,
$$n \cdot x + t \qquad (n \in N)$$
where x does not appear in t

Step 2: Identify Atomic Formulas and Delete $\neg$ if possible
All atomic formulas are,
$$u \leq v$$
$$u \equiv_k v$$

First, we omit the inequality behind the atoms. Because,

$x = y \leftrightarrow x \leq y \land y \leq x$
$x \neq y \leftrightarrow x + 1 \leq y \lor y + 1 \leq x$
$x \nleq y \leftrightarrow y + 1 \leq x$
$x \not\equiv_n y \leftrightarrow \bigvee_{0 < i < n} x + i \equiv_n y$

So, the following formula admits quantifier elimination.

$$\exists x(\bigwedge_i n_i \cdot x + t_i \le m_i \cdot x + s_i \wedge \bigwedge_j k_j \cdot x + u_j \equiv_{q_j} l_j \cdot x + v_j) \qquad (1)$$

Step 3: Simplify atomic formulas
So the following formula must be eliminated quantifier.

$$\exists x(\bigwedge_i r_i \le m_i \cdot x + s_i \wedge \bigwedge_j n_j \cdot x + t_j \le u_j \wedge \bigwedge_l k_l + v_l \equiv_{q_l} w_l) \qquad (2)$$

Step 4: Uniform the coefficients x
Let M is Multiply the coefficients by x

$M = \prod_i m_i \prod_j n_j \prod_l k_l$
$r_i \frac{M}{m_i} \le Mx + \frac{M}{m_i} s_i$
$Mx + \frac{M}{n_j} t_j \le \frac{M}{n_j} u_j$
$Mx + \frac{M}{k_l} v_l \equiv_{\frac{M}{k_l} q_l} \frac{M}{k_l} w_l$

So the following formula admits quantifier elimination

$$\exists x(\bigwedge_i r_i' \le Mx + s_i' \wedge \bigwedge_j Mx + t_j' \le u_j' \wedge \bigwedge_l Mx + v_l' \equiv_{q_l} w_l') \qquad (3)$$

Step 5: Remove the coefficient x
$y = Mx$. So, we have
$$\exists y(\bigwedge_i r_i' \le y + s_i' \wedge \bigwedge_j y + t_j' \le u_j' \wedge \bigwedge_l y + v_l' \equiv_{q_l} w_l' \wedge y \equiv_M 0) \qquad (4)$$

We use the following equations
$$t = s \leftrightarrow ct = cs$$
$$t < s \leftrightarrow ct < cs$$
$$t \equiv_m s \leftrightarrow ct \equiv_{cm} cs$$

so

$$\exists x(\bigwedge_i r_i \le x + s_i \wedge \bigwedge_j x + t_j \le u_j \wedge \bigwedge_l x + v_l \equiv_{q_l} w_l) \qquad (5)$$

Step 6: Identification Phrases included x

$r_i \le x + s_i \leftrightarrow r_i + t_j + v_l \le x + s_i + t_j + v_l$
$x + t_j \le u_j \leftrightarrow x + s_i + t_j + v_l \le u_j + s_i + v_l$
$x + v_l \equiv_{q_l} w_l \leftrightarrow x + s_i + t_j + v_l \equiv_{q_l} s_i + t_j + w_l$
$P = s_i + t_j + v_l$

so

$$\exists x(\bigwedge_i r_i' \le x + P \wedge \bigwedge_j x + P \le u_j' \wedge \bigwedge_l x + P \equiv_{q_l} w_l') \qquad (6)$$

we put $y = x + P$

$$\exists y(\bigwedge_i r_i' \le y \wedge \bigwedge_j y \le u_j' \wedge \bigwedge_l y \equiv_{q_l} w_l' \wedge y \ge P) \qquad (7)$$

Thus, it is enough to delete the quantifier in the following formula:
$$\exists x(\bigwedge_{i=1}^m r_i \le x \wedge \bigwedge_{j=1}^n x \le u_j \wedge \bigwedge_{l=1}^k x \equiv_{q_l} w_l) \qquad (8)$$

Step 7: Reduce Boolen Combination

A: Reduce the order
$$\exists x(r_0 \le x \wedge r_1 \le x \wedge \theta(x)) \equiv [r_0 \le r_1 \wedge \exists x(r_1 \le x \wedge \theta(x))] \vee [r_1 \le r_0 \wedge \exists x(r_0 \le x \wedge (r_0 \le x \wedge \theta(x))] \qquad (9)$$

B: $\exists x(x \le u_0 \wedge x \le u_1 \wedge \theta(x)) \equiv [u_0 \le u_1 \wedge \exists x(x \le u_0 \wedge \theta(x))] \vee [u_1 \le u_0 \wedge \exists x(x \le u_1 \wedge \theta(x))] \qquad (10)$

C: $\exists x(x \equiv_{q_0} w_0 \wedge x \equiv_{q_1} w_1 \wedge \theta(x)) \equiv \exists x(x \equiv_{q_0 \sqcup q_1} x_0 \wedge \theta(x)) \qquad (11)$

Step 8: Identify the states

$\exists x(r \le x \wedge x \le u \wedge x \equiv_q w) \equiv \bigvee_{i=0}^{q-1}(r + \bar{i} \le u \wedge r + \bar{i} \equiv_q w)$
$\exists x(r \le x \wedge x \le u) \equiv r \le u$
$\exists x(r \le x \wedge x \equiv_q w) \equiv true$
$\exists x(x \le u \wedge x \equiv_q w) \equiv \bigvee_{i=0}^{q-1}(\bar{i} \le u \wedge \bar{i} \equiv_q w)$
$\exists x(r \le x) \equiv true$
$\exists x(x \le u) \equiv true$
$\exists x(x \equiv_q w) \equiv true$
$\exists x(\quad) \equiv true.[9,3]$

■

Introduction to Decidability of the Multiplication Theory of Natural Numbers:

Mostowski deals with the notion of direct product in the theory of decision problems. This was well-known to Mostowski, who was able to prove decidability of Skolem Arithmetic through seeing it as a certain weak direct product of Presburger Arithmetic. Such that; This was well-known to Mostowski, who was able to prove decidability of Skolem Arithmetic through seeing it as a certain weak direct product of Presburger Arithmetic.

Theorem4. Theory $\langle N, \times \rangle$ admites quantifier - elimination. So has decidable theory and is axiomatizable Axiomatizing and decidability of the theory of $(\mathbf{N}; \times)$:

(A_1) $\forall x \forall y \forall z \ (x \cdot (y \cdot z)) = ((x \cdot y) \cdot z)$

(A_2) $\exists x \forall y \ x \cdot y = y \cdot x = y$

(A_3) $\forall x \forall y \ x \cdot y = y \cdot x$

(A_4) $\forall x \forall y \forall z \ (x \cdot z = y \cdot z \to x = y)$

(A_5) $\forall x \forall y \ x \cdot y = 1 \to x = y = 1$

$(A_{6,n})$ $\forall x \forall y x^n = y^n \to x = y \qquad (n \in N^*)$

$(A_{7,n})$ $\forall x \exists y \exists z \ (x = n \cdot y + z \wedge z \le n \wedge z \ne n) \wedge \forall n \in N^*$

(A_8) $\forall x \exists y \exists z(x = y^n \cdot z \wedge \forall \acute{y} \forall \acute{z}(x = \acute{y}^n \cdot \acute{z} \to z \mid \acute{z}))$

(A_9) $\forall x \exists p \ (P(p) \wedge px)$

(A_{10}) $\forall p \forall x \forall y((PR(p,x) \wedge PR(p,y)) \to x \mid y \vee y \mid x)$

(A_{11}) $\forall x \forall p \ (P(p) \to \exists y \ (y = V(p,x)))$

(A_{12}) $x = y \leftrightarrow \forall p \ (P(p) \to \exists y \ V(p,x) = V(p,y))$

(A_{13}) $\forall x \forall y \forall p(P(p) \to V(p,x \cdot y) = V(p,x) \cdot V(p,y))$

(A_{14}) $\forall x \forall y(\forall p(P(p) \to V(p,x) \mid V(p,y) \to x \mid y))$

(A_{15}) $\forall x \forall y \exists z \forall p(P(p) \to (p \mid x \to V(p,z) = V(p,y)) \wedge (px \to V(p,z) = 1)))$

(A_{16}) $\forall x \exists y \forall p \ (P(p) \to (p \not| x \to V(p,y) = 1) \wedge (p \mid x \to V(p,y) = p \cdot V(p,x))))$

(A_{17}) $\forall x \forall y \exists z \forall p((P(p) \to (p \mid x \cdot y \wedge V(p,x) \equiv_n V(p,y)) \to V(p,z) = p) \wedge (p \not| x \cdot y \vee V(p,x) \ne_n V(p,y) \to V(p,z) = 1) \ , n \in N$

Proof: in the article [3] [1] has been proven. ∎

Skolem arithmetic : The theory of the structure $(N, \times)$ is decidable.

Skolem claimed the decidablity of the theory $(\mathbf{N}; \times, =)$ by using the quantifier elimination. The first decidability proof appeared in the work of Mostowski. Cegielski axiomatized multiplication theory and proved quantifier elimination.[3]

1) Peano Arithmetic: Peano's Axiomatic System:

- 1. $\forall x \, \neg\big(S(x) = 0\big)$
- 2. $\forall x \forall y \big(S(x) = S(y) \longrightarrow x = y\big)$
- 3. $\forall x \big(x + 0 = x\big)$
- 4. $\forall x \forall y \big(x + S(y) = S(x + y)\big)$
- 5. $\forall x \big(x \cdot 0 = 0\big)$
- 6. $\forall x \forall y \big(x \cdot S(y) = x \cdot y + x\big)$
- 7. $\forall x \big[\neg(x < 0)\big]$
- 8. $\forall x \forall y \big(x < S(y) \longleftrightarrow x < y \vee x = y\big)$
- 9. $\forall x \forall y \big(x < y \vee y < x \vee x = y\big).$
- 10. $\varphi(0) \wedge \forall x \big[\varphi(x) \longrightarrow \varphi\big(S(x)\big)\big] \longrightarrow \forall x \varphi(x).$

Propostion1: Peano Arithmetic **PA** is undecidable.

The structures $(\mathbf{N}; +, \times)$, $(\mathbf{N}; +, <)$ are undecidable. The decidablity of the structures of natural numbers in different languages is shown in the following tables so that the theories that admit QE by $\sqrt{}$ and, the theories do not admit QE by $\times$ is shown.

	N
$\{<\}$	$\langle N, < \rangle$
$\{+\}$	$\langle N, + \rangle$
$\{\times\}$	$\langle N, \times \rangle$
$\{<, +\}$	$\langle N, < + \rangle$
$\{<, \times\}$	$\langle N, <, \times \rangle$
$\{+, \times\}$	$\langle N, +, \times \rangle$
$\{+, \times, <\}$	$\langle N, +, \times, < \rangle$

Table (I): The structure Natural numbers at different Language

Structures	The decidablity of the structures
$\langle \mathbf{N}; < \rangle$	$\sqrt{}$
$\langle \mathbf{N}; + \rangle$	$\sqrt{}$
$\langle \mathbf{N}; \times \rangle$	$\sqrt{}$
$\langle \mathbf{N}; <, + \rangle$	$\sqrt{}$
$\langle \mathbf{N}; <, \times \rangle$	$\times$
$\langle \mathbf{N}; +, \times \rangle$	$\times$
$\langle \mathbf{N}; +, \times, < \rangle$	$\times$

Table (II) Decidablity of the structure of Natural numbers at different Language

Decidability of The theory of $\langle N; \sqsubseteq \rangle$:

Theorem5.The following completely axiomatizes the structure $\langle N; \sqsubseteq \rangle$ and, moreover , its theory admits quanrifier elimination, and so is decidable.[4]

- [1]$\forall x(x \sqsubseteq x)$
- [2]$\forall x, y(x \sqsubseteq y \sqsubseteq x \to x = y)$
- [3]$\forall x, y, z(x \sqsubseteq y \sqsubseteq z \to x \sqsubseteq z)$

- [4]$\forall x, y \exists z(z \sqsubseteq x, y \wedge \forall t[t \sqsubseteq x, y \to t \sqsubseteq z]), z = x \sqcap y$
- [5]$\forall x, y \exists z(x, y \sqsubseteq z \wedge \forall t[x, y \sqsubseteq t \to z \sqsubseteq t]), z = x \sqcup y$

- [6]$\forall x(1 \sqsubseteq x)$

  Definition 1. An element x of a lattice is join-irreducible iff it satisfies:

  $\forall a, b(x = a \vee b \to (x = a \, or \, x = b))$ This is denoted by $SI(x)$ (or $SI^\star(x)$ if x is not zero).

- [7]$\forall x, y[\forall z(SI(z) \to [z \sqsubseteq x \to z \sqsubseteq y]) \to x \sqsubseteq y]$

- [8]$\forall x, y, z(SI^*(x) \wedge SI^*(y) \wedge SI^*(z) \wedge [(x \sqsubseteq z \sqsubseteq z) \wedge (z \sqsubseteq x \sqsubseteq y)] \to x \sqsubseteq y \vee y \sqsubseteq x)$

- [9]$\forall x, a([SI^*(a) \wedge a \sqsubseteq x] \to \exists b SI(b) \wedge a \sqsubseteq b \sqsubseteq x \wedge \forall c(SI(c) \wedge c \sqsubseteq x, a) \to c \sqsubseteq b])$

  b is called a valuation of x

- [10]$VAL(x, a) \wedge VAL(y, b) \wedge [(a = b = 1) \vee (a = \wedge b \neq 1 \wedge \forall x[SI^*(c) \wedge b \sim c] \to c \not\sqsubseteq x) \vee (1a \sqsubseteq b)] \Rightarrow VAL(x \sqcap y, a) \& VAL(x \sqcup y, a)$

- [11]$\forall x(x \neq 0 \to \exists a(\mathbf{P(a)} \& a \sqsubseteq x))$

- [12]$\forall x(x \neq 0 \to \exists a(\mathbf{P(a)} \& a \not\sqsubseteq x)$

- [13]$\forall x \exists s \forall a(\mathbf{P(a)} \to (V(a, x) \neq 0 \to V(a, s) \neq a) \& (V(a, x) = 0 \to V(a, s) = 0)))$

  this s which is unique, is denoted by $\mathbf{SUPP}(x)$

- [14]$\forall x \forall y \exists z \forall a(\mathbf{P(a)} \to ((a \not\sqsubseteq x \to V(a, z) = V(a, y)) \& a \sqsubseteq x \to V(a, z) = 0)))$

  this z which is unique, is denoted by $\bar{\mathbf{T}}(x, y)$

- [15 − 1]$\forall a, x(SI(a, x) \to \exists y(SI(a, y) \& x \sqsubseteq y \& y \neq x \& \forall z((SI(a, z) \& xz) \to y \sqsubseteq z)))$

  this y which is unique, is denoted by $\mathbf{S}_a(x)$

- [15−2]$\forall a, x(SI(a, x) \wedge x \neq 0) \to \exists y(SI(a, y) \& \mathbf{S}_a(y) = x)).$

  this y which is unique, is denoted by $\mathbf{P}_a(x)$

- [16]$\forall x \exists y \forall a(\mathbf{P(a)} \to ((a \not\sqsubseteq x \to V(a, y) = 0) \& (a \sqsubseteq x \to V(a, y) = \mathbf{S}_a V(a, x))))$

  this y which is unique, is denoted by $\mathbf{I}(x)$

- [17]$\forall x \forall y \exists z \forall a(\mathbf{P(a)} \to (V(a, z) = 0 \, or \, a \& V(a, z) = a \leftrightarrow ((a \sqsubseteq x \, or \, a \sqsubseteq y) \& V(a, x) \sqsubseteq V(a, y)))$

proof: [6].

The Quantifier Elimination of the structure of natural numbers in different languages is shown in the following tables so that the theories that admit QE by $\sqrt{}$ and, the theories do not admit QE by $\times$ is shown.

Table (III) A Quantifier Elimination Procedure for the Natural

numbers at different Language:

Theory of	admit QE
$(\mathbf{N}, <)$	$\times$
$(\mathbf{N}, 0, <)$	$\times$
$(\mathbf{N}, 0, S, <)$	$\checkmark$
$(\mathbf{N}, 0, 1, +, \leq)$	$\times$
$(\mathbf{N}, 0, 1, +, \leq, \{\equiv_n\}_{n \geq 2})$	$\checkmark$
$(\mathbf{N}, \times)$	$\checkmark$
$(\mathbf{N}, \sqsubseteq)$	$\checkmark$

B. Decidability of Structure of Integer Numbers in Different Languages

Theorem6. The structure $(Z; 0, s)$ admits elimination of quantifier, and it has decidable theory.
Proof:
It suffices to consider a formula,

$$\exists x(\alpha_0 \wedge \cdots \wedge \alpha_q)$$

where each α_i is atomic or is the negation of an atomic formula. In the language of $\mathcal{Z}_s$ the only terms are of the from $S^k u$ where u is 0 or a variable. we may suppose that the variable x occurs in each α_i For if x does not occur in α then,

$$\exists x(\alpha \wedge \beta) \leftrightarrow \alpha \wedge \exists x\beta$$

Thus each α_i has the form $S^m x = S^n u$ or the negation of this equation, where u is 0 or a variable. We may further suppose u is different from x since $S^m x = S^n x$ could be replaced by $0 = 0$. if $m = n$ and by $0 \neq 0$ if $m \neq n$.
Case 1: Each α_i is the negation of an equation. Then the formula may be replaced by $0 = 0$.
Case 2: There is at least one α_i not negated;say α_0 is,

$$S^m x = t$$

where the term t does not contain x. Since the solution for x must be non-negative, we replace α_0 by,

$$t \neq 0 \wedge \cdots \wedge t \neq S^{m-1}0$$

Then in each other α_j we replace, say, $S^k x = u$ first by $S^{k+m} x = S^m u$ which in turn becomes $S^k t = S^m u$ We now have a formula in which x no longer occurs,so the quantifier may be omitted .[9]   ∎

Theorem7. The Theory $(Z, 0, S, <)$ admits elimination quantifier, and so has a decidable theory and is finitely axiomatizable.

S3. $\forall y(y \neq 0 \rightarrow \exists xy = Sx)$
L1. $\forall x \forall y(x < Sy \leftrightarrow x < y)$
L2. $x \not\leq 0$
L3. $\forall x \forall y(x < y \vee y < x \vee x = y)$
L4. $\forall x \forall y(x < y \rightarrow y \not< x)$
L5. $\forall x \forall y \forall z(x < y \rightarrow y < z \rightarrow x < z)$

Proof:
We consider a formula,

$$\exists x(\beta_0 \wedge \cdots \wedge \beta_n)$$

where each β_i is atomic or the negation of an atomic formula. The terms are of the form $S^k u$ Where u is 0 or a variable.There are two possibilities for atomic formula,

$$S^k u = S^l t, S^k u < S^l t$$

1. We can eliminate the negation symbol.Replace $t_1 t_2$ by $t_1 = t_2 \vee t_2 < t_1$ and replace $t_1 \neq t_2$ by $t_1 < t_2 \vee t_2 < t_1$ By regrouping the atomic formulas and noting that

$$\exists x(\phi \vee \psi) \leftrightarrow \exists x\phi \vee \exists x\psi \qquad (11)$$

we may again reach formulas of the form,

$$\exists x(\alpha_0 \wedge \cdots \wedge \alpha_q) \qquad (12)$$

where now, each α_i is atomic
2. We may suppose that the variable x occur in each α_i This is because if x does not occure in α_i then

$$\exists x(\alpha \wedge \beta) \leftrightarrow \alpha \wedge \exists x\beta \qquad (13)$$

Furthermore,we may suppose that x occurs on only one side of the equality or inequality α_i
Case1: Suppose that some α_i is an equality.Then we can proceed as in case 2 of the quantifier-elimination proof Previous theory'
Case2: Otherwise each α_i is an inequality.Then the formula can be rewritten

$$\exists x(\bigwedge_i t_i < S^{m_i} x \wedge \bigwedge_j S^{n_j} x < u_j) \qquad (14)$$

we have lower bounds on x
If the second conjuction is empty (i.e., if there are no upper bounds on x) then we can replace the formula by $0 = 0$ If the second conjunction is empty (i.e., if there are no upper bounds on x) then we an replace the formula by $\bigwedge_j S^{n_j} 0 < u_j$ which asserts that zero satisfies the upper bounds. Otherwise, we rewrite the formula successively as,

$$\exists x \bigwedge_{i,j}(t_i < S^{m_i} x \wedge S^{n_j} x < u_j) \qquad (15)$$
$$\equiv \exists x \bigwedge_{i,j}(S^{n_j} t_i < S^{m_i + n_j} x < S^{m_i} u_j) \qquad (16)$$
$$\equiv (\bigwedge_{i,j} S^{n_j+1} t_i < S^{m_i} u_j) \wedge \bigwedge_j S^{n_j} 0 < u_j \qquad (17)$$

In each case, we have arrived at a quantifier-free version of the given formula.[9]

The additive theory of Integer numbers:
Theorem8.The theory of the structure $Z = \{+, 0, 1, \leq, \{\equiv_m\}_{\geq 2}\}$, admits quantifier elimination, and this theory is decidablr theory.
$(A1)$ $x + (y + z) = (x + y) + z$
$(A2)$ $x + y = y + x$
$(A3)$ $x + 0 = x$
$(A4)$ $x + z = y + z \rightarrow x = y$
$(A5)$ $x + y = 0 \rightarrow x = y = 0$
$(O1)$ $x \leq y \leftrightarrow \exists z(x + z = y)$
$(O2)$ $x \leq y \vee y \leq x$
$(O3)$ $0 \neq 1 \wedge \forall y(0 \leq y \leq 1 \rightarrow y = 0 \vee y = 1)$
$(D1)$ $\forall x \exists y, z(x = n \cdot y + t \wedge t < \bar{n})$
Proof: [9,3].

The Decidability of The multiplicative theory of Integer numbers:

Main Results Of Part B:
Theorem9.The following theory completely axiomatizes the structure $(\mathbf{Z}^{>0}; \cdot, 1)$ and,moreover , its theory admits quantifier - elimination. and so the Theory $\langle Z, \cdot \rangle$ is decidable.

(A_1) $\forall x \forall y \forall z \ (x \cdot (y \cdot z)) = ((x \cdot y) \cdot z)$
(A_2) $\exists x \forall y \ x \cdot y = y \cdot x = y$
(A_3) $\forall x \forall y \ x \cdot y = y \cdot x$
(A_4) $\forall x \forall y \forall z \ (x \cdot z = y \cdot z \rightarrow x = y)$
(A_5) $\forall x \forall y \ x \cdot y = 1 \rightarrow x = y = 1$
$(A_{6,n})$ $\forall x \forall y x^n = y^n \rightarrow x = y \qquad (n \in N^*)$
$(A_{7,n})$ $\forall x \exists y \exists z \ (x = n \cdot y + z \wedge z \leq n \wedge z \neq n) \wedge \forall n \in N^*$
(A_8) $\forall x \exists y \exists z(x = y^n \cdot z \wedge \forall \acute{y} \forall \acute{z}(x = \acute{y}^n \cdot \acute{z} \rightarrow z \mid \acute{z}))$
(A_9) $\forall x \exists p \ (P(p) \wedge px)$
(A_{10}) $\forall p \forall x \forall y((PR(p, x) \wedge PR(p, y)) \rightarrow x \mid y \vee y \mid x)$
(A_{11}) $\forall x \forall p \ (P(p) \rightarrow \exists y \ (y = V(p, x) \ )$
(A_{12}) $x = y \leftrightarrow \forall p \ (P(p) \rightarrow \exists y \ V(p, x) = V(p, y))$
(A_{13}) $\forall x \forall y \forall p(P(p) \rightarrow V(p, x \cdot y) = V(p, x) \cdot V(p, y))$
(A_{14}) $\forall x \forall y (\forall p(P(p) \rightarrow V(p, x) \mid V(p, y) \rightarrow x \mid y))$
(A_{15}) $\forall x \forall y \exists z \forall p(P(p) \rightarrow (p \mid x \rightarrow V(p, z) = V(p, y)) \wedge (px \rightarrow V(p, z) = 1)))$
(A_{16}) $\forall x \exists y \forall p \ (P(p) \rightarrow (p \,\slash x \rightarrow V(p, y) = 1) \wedge (p \mid x \rightarrow V(p, y) = p \cdot V(p, x))))$
(A_{17}) $\forall x \forall y \exists z \forall p((P(p) \rightarrow (p \mid x \cdot y \wedge V(p, x) \equiv_n V(p, y)) \rightarrow V(p, z) = p) \wedge (p \,\slash x \cdot y \vee V(p, x) \not\equiv_n V(p, y) \rightarrow V(p, z) = 1) $,$n \in N$
(A_{18}) $\forall x \exists y x + y = 0$

Proof:[3].
The structures $(\mathbf{Z}, +, \times)$ and $(\mathbf{Z}, +, <)$ are undecidabile structures.

Table (IV) :A Quantifier Elimination Procedure for the integers:

Theory of	Language	admit QE
$(\mathbf{Z}, +)$	$L = (+)$	$\times$
$(\mathbf{Z}, +, ; -; <)$	$L = (0; 1; +; -; <)$	$\times$
$(\mathbf{Z}, 0, 1, +, \leq, \{\equiv_n\}_{n \geq 2})$	$L = (0; 1, +, \leq, \{\equiv_{n_{n \geq 2}}\})$	$\checkmark$
$(\mathbf{Z}, \times)$	$L = (\cdot, v, p)$	$\checkmark$

C. Decidability of Structure of Rational Numbers in Different Languages

In this part of article, we will show decidability and axiomatization of the structure $\langle \mathbf{Q}, \sqsubseteq \rangle$. So we know decidability and axiomatization of the structure $\langle \mathbf{N}, \sqsubseteq \rangle$. we review decidability of atomlss boolean algebra, because the interval algebra of the rational numbers is atomless .

Theorem10.Theory $(Q, <)$ admits elimination of quantifier.
Proof:
1: Identify the terms
In structure $\langle Q; < \rangle$, every term involving x is equal to,
$$n \cdot x + t \qquad (n \in N)$$
where x does not appear in t
2: Identify Atomic Formulas and Delete $\neg$ if possible
omit all atomic formulas are,
$$u < v$$
$$u = v$$
First, we omit the inequality behind the atoms. Because,
$x \neq y \leftrightarrow x < y \wedge y < x$
$x \not< y \leftrightarrow x = y \vee y < x$
3: Simplify atomic formulas
So the following formula must be eliminated quantifier.
$$\exists x (\bigwedge_i r_i < m_i \cdot x + s_i \wedge \bigwedge_j n_j \cdot x + t_j < u_j \wedge \bigwedge_l k_l \cdot x + v_l = w_l) \qquad (18)$$
4:Uniform the coefficients x
Let M is Multiply the coefficients by x
$M = \prod_i m_i \prod_j n_j \prod_l k_l$
$r_i \frac{M}{m_i} < Mx + \frac{M}{m_i} s_i$
$Mx + \frac{M}{n_j} t_j < \frac{M}{n_j} u_j$
$Mx + \frac{M}{k_l} v_l = \frac{M}{k_l} w_l$
So the following formula admits quantifier elimination
$$\exists x (\bigwedge_i r_i' < Mx + s_i' \wedge \bigwedge_j Mx + t_j' < u_j' \wedge \bigwedge_l Mx + v_l' = w_l') \qquad (19)$$
5: Remove the coefficient x
$y = Mx$. So, we have
$$\exists y (\bigwedge_i r_i' < y + s_i' \wedge \bigwedge_j y + t_j' < u_j' \wedge \bigwedge_l y + v_l' = w_l') \qquad (20)$$
We use the following equations
$$t = s \leftrightarrow ct = cs$$
$$t < s \leftrightarrow ct < cs$$
so
$$\exists x (\bigwedge_i r_i < x + s_i \wedge \bigwedge_j x + t_j < u_j \wedge \bigwedge_l x + v_l = w_l) \qquad (21)$$
6: Identification Phrases included x
$r_i < x + s_i \leftrightarrow r_i + t_j + v_l < x + s_i + t_j + v_l$
$x + t_j < u_j \leftrightarrow x + s_i + t_j + v_l < u_j + s_i + v_l$
$x + v_l = w_l \leftrightarrow x + s_i + t_j + v_l = s_i + t_j + w_l$
$P = s_i + t_j + v_l$
so
$$\exists x (\bigwedge_i r_i' < x + P \wedge \bigwedge_j x + P < u_j' \wedge \bigwedge_l x + P = w_l') \qquad (22)$$
we put $y = x + P$
$$\exists y (\bigwedge_i r_i' < y \wedge \bigwedge_j y < u_j' \wedge \bigwedge_l y = w_l') \qquad (23)$$
Thus, it is enough to delete the quantifier in the following formula:
$$\exists x (\bigwedge_i r_i < x \wedge \bigwedge_j x < u_j \wedge \bigwedge_l x = w_l) \qquad (24)$$
7: Identify the states
$l \neq 0 \equiv \bigwedge_i r_i < w_0 \wedge \bigwedge_j w_0 < u_j \wedge \bigwedge_l w_0 = w_l \equiv True$

$l = 0 \equiv \exists x (\bigwedge_i r_i < x \wedge \bigwedge_j x < u_j)$
$l = j = 0 \equiv \exists x (\bigwedge_i r_i < x) \equiv True$
$l = i = 0 \equiv \exists x (\bigwedge_j x < u_j) \equiv True$
$l = 0, i, j \neq 0 \equiv \exists x (\bigwedge_i r_i < x \wedge \bigwedge_j x < u_j) \equiv \bigwedge_i \bigwedge_j r_i < u_j \equiv True$

$\blacksquare$

Decidability Mathematical Structures: Structures The Theory of Addition $(Q, +)$:
Theorem11.The Theory of Addition $(Q, +)$ admits elimination of quantifier.
Proof:
Step 1: Identify the terms
In structure $(Q, +)$, every term involving x is equal to,
$$n \cdot x + t \qquad (n \in N)$$
where x does not appear in t
Step 2: Identify Atomic Formulas
All atomic formulas are,
$$u = v$$
$$u \neq v$$
Step 3: Simplify atomic formulas
So the following formula must be eliminated quantifier.
$$\exists x (\bigwedge_i k_i \cdot x + v_i = w_i \wedge \bigwedge_j m_j \cdot x + n_j \neq s_j) \qquad (25)$$
$$\equiv \exists x (\bigwedge_i k_i \cdot x = u_i \wedge \bigwedge_j m_j \cdot x \neq t_j) \qquad (26)$$
Step 4:Uniform the coefficients x
Let M is Multiply the coefficients by x
$M = \prod_i k_i \prod_j m_j$
So the following formula admits quantifier elimination
$$\exists x (\bigwedge_i M \cdot x = u_i' \wedge \bigwedge_j M \cdot x \neq t_j') \qquad (27)$$
Step 5: Remove the coefficient x
$y = Mx$. So, we have
$$\exists y (\bigwedge_i y = u_i' \wedge \bigwedge_j y \neq t_j') \qquad (28)$$
We use the following equations
$$t = s \leftrightarrow ct = cs$$
$$t \neq s \leftrightarrow ct \neq cs$$
so
$$\exists x (\bigwedge_i x = u_i \wedge \bigwedge_j x \neq t_j) \qquad (29)$$
Step 6: Identify the states
$$i \neq 0 \equiv \bigwedge_i u_0 = u_i \wedge \bigwedge_j u_0 \neq t_j$$
$$i = 0, j \neq 0 \equiv True$$

Theorem12. the Theory $\langle Q; +, -.0, < \rangle$ admites quantifier - elimination. and so has decidable theory . Proof:
The following formula must be eliminated quantifier.
$$\exists x (\bigwedge_i n_i \cdot x = t_i \wedge \bigwedge_j 0 < m_j \cdot x + s_j) \qquad (30)$$
Likel previous proofs, admites quantifier - elimination. and so has decidable theory .

$\blacksquare$

Theorem13. the Theory $\langle Q^+; \times, 1.0^{-1}, \{R_n\}_{n \geq 2} < \rangle$ admites quantifier - elimination. and so has decidable theory .
Proof:[11]

like previous proofs, admites quantifier - elimination. and so has decidable theory .
Main Results of Part C:
We will express the axioms of rational numbers as follows:
Positive rational numbers are formed from two parts, the integer part whose denominator is one, and the Intrevel Algebra of rational numbers. The positive part of all the properties of natural numbers .So we have the axioms of $(\mathbf{N}, \sqsubseteq)$ and atomless Boolean Algebra and the axioms of $(\mathbf{Q}^+, \times)$.

A rational number is a number that can be in the form $\frac{p}{q}$ where p and q are integers and q is not equal to zero. All fractions, both positive and negative, are rational numbers. A few examples are $45, -78, 134$, and -203 Each numerator and each denominator is an integer.Are integers rational numbers? To decide if an integer is a rational number, we try to write it as a ratio of two integers. An easy way to do this is to write it as a fraction with denominator one. $3 = \frac{3}{1}, 0 = \frac{0}{1}, -8 = \frac{-8}{1}$ Since any integer can be written as the ratio of two integers, all integers are rational numbers. so $(N^+, \sqsubseteq) = (\mathbf{Q}^+, \sqsubseteq)-$ the interval algebra of the rational numbers. Therefore, we have:
We know Theory $\mathcal{N}_L = (N, 0, S, <)$ admits elimination quantifier, and so has a decidable theory and is finitely axiomatizable.

S3.	$\forall y(y \neq 0 \rightarrow \exists xy = Sx)$
L1.	$\forall x \forall y(x < Sy \leftrightarrow x < y)$
L2.	$x \not\sqsubseteq 0$
L3.	$\forall x \forall y(x < y \vee y < x \vee x = y)$
L4.	$\forall x \forall y(x < y \rightarrow y \not< x)$
L5.	$\forall x \forall y \forall z(x < y \rightarrow y < z \rightarrow x < z)$

Proof:
We consider a formula,

$$\exists x(\beta_0 \wedge \cdots \wedge \beta_n)$$

where each β_i is atomic or the negation of an atomic formula. The terms are of the form $S^k u$ Where u is 0 or a variable.There are two possibilities for atomic formula,

$$S^k u = S^l t, S^k u < S^l t$$

1. We can eliminate the negation symbol.Replace $t_1 t_2$ by $t_1 = t_2 \vee t_2 < t_1$ and replace $t_1 \neq t_2$ by $t_1 < t_2 \vee t_2 < t_1$ By regrouping the atomic formulas and noting that

$$\exists x(\phi \vee \psi) \leftrightarrow \exists x\phi \vee \exists x\psi$$

we may again reach formulas of the form,

$$\exists x(\alpha_0 \wedge \cdots \wedge \alpha_q)$$

where now, each α_i is atomic
2. We may suppose that the variable x occur in each α_i This is because if x does not occure in α_i then

$$\exists x(\alpha \wedge \beta) \leftrightarrow \alpha \wedge \exists x\beta$$

Furthermore,we may suppose that x occurs on only one side of the equality or inequality α_i

Case1: Suppose that some α_i is an equality.Then we can proceed as in case 2 of the quantifier-elimination proof Previous theory'
Case2: Otherwise each α_i is an inequality.Then the formula can be rewritten

$$\exists x(\bigwedge_i t_i < S^{m_i} x \wedge \bigwedge_j S^{n_j} x < u_j)$$

we have lower bounds on x
If the second conjuction is empty (i.e., if there are no upper bounds on x) then we can replace the formula by $0 = 0$ If the second conjunction is empty (i.e., if there are no upper bounds on x) then we an replace the formula by $\bigwedge_j S^{n_j} 0 < u_j$ which asserts that zero satisfies the upper bounds. Otherwise, we rewrite the formula successively as,

$$\exists x \bigwedge_{i,j}(t_i < S^{m_i} x \wedge S^{n_j} x < u_j)(1)$$
$$\exists x \bigwedge_{i,j}(S^{n_j} t_i < S^{m_i + n_j} x < S^{m_i} u_j)(2)$$
$$(\bigwedge_{i,j} S^{n_j + 1} t_i < S^{m_i} u_j) \wedge \bigwedge_j S^{n_j} 0 < u_j$$

In each case, we have arrived at a quantifier-free version of the given formula.[7] ∎
And we Let $P(I)$ is denote the power set of I. We have $(P(I), \subseteq, \cap, \cup, ')$ is Boolean Algebra.
Proof:
For $P, Q, R \subseteq I, P, Q, R \in P(I)$,
we have:
$P \cap Q = Q \cap P, \qquad P \cup Q = Q \cup P$
$P \cap (Q \cap R) = (P \cap Q) \cap R, \qquad P \cup (Q \cup R) = (P \cup Q) \cup R$
$P \cap (Q \cup R) = (P \cap Q) \cup (P \cap R), \qquad P \cup (Q \cap R) = (P \cup Q) \cap (P \cup R)$
$P \cap I = P, \qquad P \cup \emptyset = P$
$P \cap P' = \emptyset, \qquad P \cup P' = I$
$\langle P(I), \subseteq \rangle$, is an atomic complemented distributive lattice and the theory of it is a finitely axiomatizable complete theory.This theory simply is the theory of $\subseteq$- based set-theoretic mereology.
And $\langle AtomicBA, \cap, \cup, ^c, \emptyset, I, = \rangle$ is an atomic Boolean Algebra, and this theory satisfies the elimination Quantifier.
   Proof:

   Quantifier-Elimination is proof next partes.
   and The following theory completely axiomatizes the theory of natural numbers with divisibility relation and , moreover , the structure $\langle N^+; \sqsubseteq \rangle = \langle Q^+; \sqsubseteq \rangle$ admits quantifier elimination ,and so the theory of the structure $\langle N^+; \sqsubseteq \rangle = (\mathbf{Q}^+, \sqsubseteq)-$ the interval algebra of the rational numbers is decidable theory. Rivew on decidableility:

   Step1: We can assume that W be a model $D = Th(\mathbf{N}^+; \sqsubseteq) = Th(\mathbf{Q}^+, \sqsubseteq)-$ the interval algebra of the rational numbers ,and a an atom of W . Such that $a \in W$ & $W \models SI(a, x)$. And $W_a = (A_a, \sqsubseteq)$ such that $A_a = \{x \in A; W \models \mathbf{SI(a, x)}\}$ and $\sqsubseteq$ is restricted to A_p . We claim W_a is a model from $(N, \leq)$.
Proof:
The following theory completely axiomatize the $(N, \leq) = (\mathbf{Q}^+, \leq)-$ the interval algebra of the rational

numbers and, moreover, the structure $(N, \leq) = (\mathbf{Q}^+, \leq)-$ the interval algebra of the rational numbers admits quantifier eimination .

L1 $\quad \forall x(x \leq x)$

L2 $\quad \forall x \forall y(x \leq y \leq x \to x = y)$

L3 $\quad \forall x, y, z(x \leq y \leq z \to x \leq z)$

L4 $\quad \forall x(0 \leq x)$

L5 $\quad \forall x \forall y(x \leq y \vee y \vee x)$

L6 $\quad \forall x \exists y(x \neq 0 \to \exists! y(x = sy))$

$W_a \models (N, \leq) = (\mathbf{Q}^+, \leq)-$ the interval algebra of the rational numbers , to prove the claim, we will examine some examples.

1) $\quad W_a \models \forall x(x \sqsubseteq x) \qquad (L1)$

2) $\quad W_a \models \forall x, y(x \sqsubseteq y \sqsubseteq x \to x = y \qquad (L2)$

3) $\quad W_a \models \forall x, y, z(x \sqsubseteq y \sqsubseteq z \to x \sqsubseteq z) \qquad (L3)$

4) $\quad W_a \models \forall x(x \neq 0 \to \exists a(\mathbf{P}(\mathbf{a}) \& a \sqsubseteq x)) \qquad (L4)$

5) $\quad W_a \models \forall x, y, z(SI^*(x) \wedge SI^*(y) \wedge SI^*(z) \wedge [(x \sqsubseteq z \sqsubseteq z) \wedge (z \sqsubseteq x \sqsubseteq y)] \to x \sqsubseteq y \vee y \sqsubseteq x) \qquad (L5)$

Step 2:
We can assume that W be a model $D = Th(\mathbf{N}^+, \sqsubseteq) = Th(\mathbf{Q}^+, \leq)-$ the interval algebra of the rational numbers ,and theory
F is the class of structures $(P_f(I), \subseteq)$. We claim W_F is a model from $((P_f(I), \subseteq, \cap, \cup, \{C_n\}_{n \geq 0},{}^c)$.
to prove the claim, we will examine some examples.
Proof:

a) $\quad W_F \models \forall x(x \sqsubseteq x) \qquad (x \subseteq x)$

b) $\quad W_F \models \forall x, y(x \sqsubseteq y \sqsubseteq x \to x = y \qquad (x \subseteq y \subseteq x \to x = y)$

c) $\quad W_F \models \forall x, y, z(x \sqsubseteq y \sqsubseteq z \to x \sqsubseteq z) \qquad (x \subseteq y \subseteq z \to x \subseteq z)$

d) $\quad W_F \models \forall x(x \neq 0 \to \exists a(\mathbf{P}(\mathbf{a}) \& a \sqsubseteq x)) \qquad (\emptyset \subseteq x)$

Proposition2: Let W be a model of DIV, p an atom of W , $a_1, \cdots, a_n$ elements of domain and $b_i = V(p, a_i)$ for $1 \leq i \leq n$ Then:

$$W \models \phi^p[a_1, \cdots, a_n] \leftrightarrow W_p \models \phi[v(p, a_1), \cdots [v(p, a_n)]$$

Proof:[4]
So, we consider that every formula $\phi(x_1, \cdots, x_n)$ is in $(\mathbf{N}, \sqsubseteq)$ equivalent to a Boolean combinaion [2] of formulas $\psi[v(p, x_1), \cdots [v(p, x_n)$ Where $\psi[y_1, \cdots, y_n)$ is a formla of $(N, \leq)$, where it is equivalent to a formula without quantifiers in the language. $W_a \equiv (N, \leq, 0, S)$ and $W_F \equiv ((P_f(I), \subseteq, \cap, \cup, \{C_n\}_{n \geq 0},{}^c)$. So, any sentence of $(\mathbf{N}, \sqsubseteq) = (\mathbf{Q}^+, \leq)-$ the interval algebra of the rational numbers is equivalent to a Boolean combination ofsentences of structures $(N, \leq, 0, S)$ and $((P_f(I), \subseteq, \cap, \cup, \{C_n\}_{n \geq 0},{}^c)$. So, $(\mathbf{N}, \sqsubseteq) = (\mathbf{Q}^+, \leq)-$ the interval algebra of the rational numbers admit Quantifier Elimination ,and it is complete. $\blacksquare$

case2:if we have the interval algebra of rational number . the intrval algebra of the rational number is aomless.Thus the proof is desidability and axiom-atizibility of atomless boolean algebra.
Proof:[Theorem 20 og part F]
so we have the following axioms for $\langle \mathbf{Q}^+, \sqsubseteq \rangle$:

$[1] \forall x(x \sqsubseteq x)$
$[2] \forall x, y(x \sqsubseteq y \sqsubseteq x \to x = y)$
$[3] \forall x, y, z(x \sqsubseteq y \sqsubseteq z \to x \sqsubseteq z)$
$\sqsubseteq$ is a strict partial order.

number theory	$\sqsubseteq$	$\sqcap$	$\sqcup$			
set theory	$\subseteq$	$\cap$	$\cup$	ϕ	X	$'$
logic	$\leq$	$\wedge$	$\vee$	0	1	$\neg$

$[4] \forall x, y \exists z(z \sqsubseteq x, y \wedge \forall t[t \sqsubseteq x, y \to t \sqsubseteq z]), z = x \sqcap y$
$[5] \forall x, y \exists z(x, y \sqsubseteq z \wedge \forall t[x, y \sqsubseteq t \to z \sqsubseteq t]), z = x \sqcup y$
Axioms 4 and 5 are equal to the following axioms in set theory:

$[4] \forall A, B \exists C(C \subseteq A, B \wedge \forall T[T \subseteq A, B \to T \subseteq C]), C = A \cap B$
$[5] \forall A, B \exists C(A, B \subseteq C \wedge \forall T[A, B \subseteq T \to C \sqsubseteq T]), C = A \cup B$

Definition: x is p-primary and denoted by $PR(p, x)$ iff we have $P(p) \wedge \forall q((P(q) \wedge p \neq q) \longrightarrow q x) x = p^n$

$[1] \forall x(x \sqsubseteq x)$
$[2] \forall x, y(x \sqsubseteq y \sqsubseteq x \to x = y)$
$[3] \forall x, y, z(x \sqsubseteq y \sqsubseteq z \to x \sqsubseteq z)$
$[4] \forall x, y \exists z(z \sqsubseteq x, y \wedge \forall t[t \sqsubseteq x, y \to t \sqsubseteq z]), z = x \sqcap y$
$[5] \forall x, y \exists z(x, y \sqsubseteq z \wedge \forall t[x, y \sqsubseteq t \to z \sqsubseteq t]), z = x \sqcup y$
We don't say that the lattice is distributive because this follows from other axioms. The lattice has a least element.
$[6] \forall x(1 \sqsubseteq x)$
Definition:An element x is join-irreducible iff it satisfies $\forall a, b(x = a \sqcup b \longrightarrow (x = a) \vee (x = b))$. This is denoted by $SI(x)$ or $SI^*(x)$ if $x \neq 1$

[2]results of Mostowski

lemma 2: x is p-primary number why?
If x is not p-primary number then we have:

$$x = \prod_i p_i^{\alpha_i} = p_i^{\alpha_i} \prod_j p_j^{\alpha_j}$$
$$= p_i^{\alpha_i} {}_j^{\alpha_j}$$

then x is not join-irreducible .

$[7] \forall x, y [\forall z (SI(z)[z \sqsubseteq x \to z \sqsubseteq y]) \to x \sqsubseteq y]$
Propostion 1: $\forall x, y \Leftrightarrow \forall z (SI(z)[z \sqsubseteq x \leftrightarrow z \sqsubseteq y])$

$[8] \forall x, y, z (SI^*(x) \wedge SI^*(y) \wedge SI^*(z) \wedge [(x, y \sqsubseteq z \vee z \sqsubseteq x, y)] \to x \sqsubseteq y \vee y \sqsubseteq x)$
Propostion 3: $xy \Leftrightarrow x \sqsubseteq y \vee y \sqsubseteq x$

$[9] \forall x, a([SI^*(a) \wedge a \sqsubseteq x] \to \exists b (SI(b)^* \wedge a \sqsubseteq b \sqsubseteq x \wedge \forall c (SI(c) \wedge c \sqsubseteq x, a) \to c \sqsubseteq b])$
Propostion 4: $(1) \forall x, y (x \sqsubseteq y \leftrightarrow \forall a VAL(x, a) \to a \sqsubseteq y)$
$(2) \forall x, y (x = y \leftrightarrow \forall a VAL(x, a) \to VAL(y, a))$
Proof:
$(1) x \sqsubseteq y \leftrightarrow \forall a VAL(x, a) \to a \sqsubseteq y$
For only if $\to$
direction is trivial.Because:
$$x \sqsubseteq y \to \forall a (VAL(x, a) \to a \sqsubseteq x) \to a \sqsubseteq y$$
For other direction, We consider that $\forall a (VAL(x, a))$
then by axiom (7) we show: $\forall z (SI(z)[z \sqsubseteq x \to z \sqsubseteq y])$
so,
for the z arbitrary, we suppose that $SI(z), z \sqsubseteq x$ by (9),
$\exists b, SI(b), z \sqsubseteq b \wedge \forall c (SI(c) \wedge z \sqsubseteq c \sqsubseteq x \to c \sqsubseteq b)$
then we have, $VAL(x, b)$. so, $b \sqsubseteq y$ then $z \sqsubseteq b \Rightarrow z \sqsubseteq y$
so, $x \sqsubseteq y$.
$[10] VAL(x, a) \wedge VAL(y, b) \wedge [(a = b = 1) \vee (a = \wedge b \neq 1 \wedge \forall x [SI^*(c) \wedge b \sim c] \to c \not\sqsubseteq x) \vee (1 \sqsubseteq a \sqsubseteq b)] \Rightarrow VAL(x \sqcap y, a) \& VAL(x \sqcup y, a)$
Proposition 4:
$$\forall x, y, z (x \sqcap (y \sqcup z) = (x \sqcap y) \sqcup (x \sqcap z))$$
Atom: $a \neq 0 \forall x (x \leq a \to (x = 0 \vee x = a))$, we denotes by $A(a)$.

$[11] \forall x (x \neq 0 \to \exists a (\mathbf{P(a)} \& a \sqsubseteq x))$
$[12] \forall x (x \neq 0 \to \exists a (\mathbf{P(a)} \& a \not\sqsubseteq x))$
Propostion 5: $\forall x (SI^*(x) \to \exists! a (P(a) a \sqsubseteq x))$

lemma3. $\exists! a (P(a) a \sqsubseteq x) \to SI^*(a)$
Proof:
$a = b \sqcup c \Rightarrow b \sqsubseteq a \Rightarrow b = a \sqrt{} \vee b = 1$
$b = 1 \Rightarrow a = 1 = c \sqrt{}$

so we have the following axioms for $\langle \mathbf{Q}^+, \sqsubseteq \rangle$:

$[1] \forall x (x \sqsubseteq x)$
$[2] \forall x, y (x \sqsubseteq y \sqsubseteq x \to x = y)$
$[3] \forall x, y, z (x \sqsubseteq y \sqsubseteq z \to x \sqsubseteq z)$
$[4] \forall x, y \exists z (z \sqsubseteq x, y \wedge \forall t [t \sqsubseteq x, y \to t \sqsubseteq z]), z = x \sqcap y$
$[5] \forall x, y \exists z (x, y \sqsubseteq z \wedge \forall t [x, y \sqsubseteq t \to z \sqsubseteq t]), z = x \sqcup y$
$[6] \forall x (1 \sqsubseteq x)$
$[7] \forall x, y [\forall z (SI(z)[z \sqsubseteq x \to z \sqsubseteq y]) \to x \sqsubseteq y]$
$[8] \forall x, y, z (SI^*(x) \wedge SI^*(y) \wedge SI^*(z) \wedge [(x, y \sqsubseteq z \vee z \sqsubseteq x, y)] \to x \sqsubseteq y \vee y \sqsubseteq x)$
$[9] \forall x, a([SI^*(a) \wedge a \sqsubseteq x] \to \exists b (SI(b)^* \wedge a \sqsubseteq b \sqsubseteq x \wedge \forall c (SI(c) \wedge c \sqsubseteq x, a) \to c \sqsubseteq b])$

$[10] VAL(x, a) \wedge VAL(y, b) \wedge [(a = b = 1) \vee (a = \wedge b \neq 1 \wedge \forall x [SI^*(c) \wedge b \sim c] \to c \not\sqsubseteq x) \vee (1 \sqsubseteq a \sqsubseteq b)] \Rightarrow VAL(x \sqcap y, a) \& VAL(x \sqcup y, a)$
$[11] \forall x (x \neq 0 \to \exists a (\mathbf{P(a)} \& a \sqsubseteq x))$
$[12] \forall x (x \neq 0 \to \exists a (\mathbf{P(a)} \& a \not\sqsubseteq x))$
$[13] \forall x \exists s \forall a (\mathbf{P(a)} \to (V(a, x) \neq 0 \to V(a, s) \neq a) \& (V(a, x) = 0 \to V(a, s) = 0)))$
$s = \mathbf{SUPP}(x)$
$[14] \forall x \forall y \exists z \forall a (\mathbf{P(a)} \to ((a \not\sqsubseteq x \to V(a, z) = V(a, y)) \& a \sqsubseteq x \to V(a, z) = 0)))$
$z = \bar{\mathbf{T}}(x, y)$
$[15 - 1] \forall a, x (SI(a, x) \to \exists y (SI(a, y) \& x \sqsubseteq y \& y \neq x \& \forall z ((SI(a, z) \& x \sqsubseteq z) \to y \sqsubseteq z)))$
$y = \mathbf{S}_a(x)$
$[15 - 2] \forall a, x (SI(a, x) \wedge x \neq 0) \to \exists y (SI(a, y) \& \mathbf{S}_a(y) = x))$.
$y = \mathbf{P}_a(x)$
$[16] \forall x \exists y \forall a (\mathbf{P(a)} \to ((a \not\sqsubseteq x \to V(a, y) = 0) \& (a \sqsubseteq x \to V(a, y) = \mathbf{S}_a V(a, x))))$
$y = \mathbf{I}(x)$
$[17] \forall x \forall y \exists z \forall a (\mathbf{P(a)} \to (V(a, z) = 0 ora \& V(a, z) = a \leftrightarrow ((a \sqsubseteq x ora \sqsubseteq y) \& V(a, x) \sqsubseteq V(a, y)))$
$[18] x \sqcap y = y \sqcap x \qquad x \sqcup y = y \sqcup x$
$[19] x \sqcap (y \sqcap z) = (x \sqcap y) \sqcap z \qquad x \sqcup (y \sqcup z) = (x \sqcup y) \sqcup z$
$[20] (x \sqcap y) \sqcup y = y \qquad (x \sqcup y) \sqcap y = y$
$[21] x \sqcap (y \sqcup z) = (x \sqcap y) \sqcup (x \sqcap z) \qquad x \sqcup (y) = (x \sqcup y) \sqcap (x \sqcup z)$
$[22] x \sqcap x^{-1} = 1 \qquad x \sqcup x^{-1} = x$
$[23] \neg P(a)$
$[24] \forall x, y, z (x \cdot (y \cdot z)(x \cdot y) \cdot z)$
$[25] \forall x, y (0 < x < y \to \exists z (x^{2n} < z < y^{2n})), n \geq 1$
$[26] \forall v_1, \cdots, v_l \exists x \forall z \bigwedge_{k=1}^{l} (x^n \cdot v_k \neq z^{m_k})$

The axiomatization is a modeling in the first order language L, $\langle \mathbf{Q}^+, \sqsubseteq \rangle$, will be a model of this language.

Theorem14: The theory of the rational numbers $(\mathbf{Q}, \sqsubseteq)$ is decidable, and moreover axiomatizable.
Proof:
Decidablity for the theory of the rational numbers $(\mathbf{Q}^+, \sqsubseteq)$:

$$p \sqsubseteq q \leftrightarrow \exists m \in \mathbf{N}^+ (p \cdot m = q)$$

Structure $(\mathbf{Q}^+, \sqsubseteq)$ is equal to structure $(\mathbf{Q}^+, \times)$ First,We conclude decidablity $(\mathbf{Q}^+, \times)$ of paper [11] so, the structure $(\mathbf{Q}^+, \sqsubseteq)$ Based on the article [11] is decidable . The decidability of the structure of rational numbers in different languages is shown in the following tables so that the theories of decidable by $\sqrt{}$ and, undecidable theories by $\times$ is shown.

The structure of rational numbers in different languages $TableV$

L	Q
$\{<\}$	$\langle Q, < \rangle$
$\{+\}$	$\langle Q, + \rangle$
$\{\times\}$	$\langle Q, \times \rangle$
$\{<, +, -, 0\}$	$\langle Q, <, + \rangle$
$\{\sqsubseteq\}$	$\langle Q, \sqsubseteq \rangle$

Table VI: The decidablity of the structure of rational numbers in different languages

Structures	The decidablity of the structures
$\langle \mathbf{Q}; < \rangle$	$\checkmark$
$\langle \mathbf{Q}; + \rangle$	$\checkmark$
$\langle \mathbf{Q}; \times \rangle$	$\checkmark$
$\langle \mathbf{Q}; <, + \rangle$	$\checkmark$
$\langle \mathbf{Q}; \sqsubseteq \rangle$	$\checkmark$

D. Deciability of structures of real numbers at different Language

Theorem14. The structure $(\mathbf{R}; <)$ admites quantifier elimination , so has a decidable theory.

Proof:

Quantifier Elimination Procedure for $(\mathbf{R}; <)$

$$\underbrace{\exists x [\bigwedge_{i=1}^{m} x = x_i \wedge \bigwedge_{j=1}^{n} z_j < x \wedge \bigwedge_{k=1}^{p} x < u_k]}_{\varphi}$$

If $m > 0$:

$$R \models \varphi \Longleftrightarrow \bigwedge_{i=2}^{m} x_1 = x_i \wedge \bigwedge_{j=1}^{n} z_j < x_1 \wedge \bigwedge_{k=1}^{p} x_1 < u_k$$

If $m = 0$ then we distinguish 3 subcases:

If $n = 0$, then $R \models \varphi \Longleftrightarrow$ true, because R has no minimum. If $p = 0$, then $R \models \varphi \Longleftrightarrow$ true, because R has no maximum. If $n > 0$ and $p > 0$, then
$$R \models \bigwedge_{j=1}^{n} \bigwedge_{k=1}^{p} z_j < u_k$$

Proof:

$'' \rightarrow'' <$ is transitive.

$'' \leftarrow''$ there exists $x \in R$ with

$max_j z_j < x < min_k u_k$.

Theorem15. The structure $\langle \mathbf{R}; <, + \rangle$ admits quantifier elimination, and so is decidable. [1]

Proof:

It suffices to prove that the following formula is equivalent to a formula without quantifier.

$$\exists x (\bigwedge_{i<l} t_i p_i \cdot x \wedge \bigwedge_{j<m} q_j \cdot x < s_j \wedge \bigwedge_{k<n} r_k \cdot x = u_k) \qquad (31)$$

Consider the coefficients p_i, q_j, r_k
are equal. As a result, we have the following equivalence

$$\exists y (\bigwedge_{i<l} t_i < y \wedge \bigwedge_{j<m} y < s_j \wedge \bigwedge_{k<n} y = u_k) \qquad (32)$$

Now the quantifier of this formula is easily removed.

Theorem16. The structure $\langle R; + \rangle$ admits quantifier elimination, and so is decidable. [11]

Proof:

Each term included x is equivalent to $k \cdot x + t$. So each atomic formula contains x equal to $k \cdot x = t$. For term t

without x and positive integers k It is enough to delete the suras of the following formula:

$$\exists x (\bigwedge_{i<l} n_i \cdot x = t_i \wedge \bigwedge_{j<k} m_j \cdot x \neq s_j) \qquad (33)$$

We can assume all of $n_i s$ and $m_j s$ are equal to each other. As a result, we show the following equivalence

$$\exists x (\bigwedge_{i<l} q \cdot x = t_i \wedge \bigwedge_{j<k} q \cdot x \neq s_j) \qquad (34)$$

We consider $y = q \cdot x$ so, we have

$$\exists y (\bigwedge_{i<l} y = t_i \wedge \bigwedge_{j<k} y \neq s_j) \qquad (35)$$

If $l > 0$ so we have

$$\bigwedge_{i<l} t_0 = t_i \wedge \bigwedge_{j<k} t_0 \neq s_j \qquad (36)$$

If $l = 0$ so we have $\bigwedge_{j<k} y \neq s_j$ and so is equivalent with the quantifier-free formula $0 = 0$

Theorem17. The structure $(\mathbf{R}; \times)$ admits quantifier elimination, and so is decidable. [11]

Theorem18. The structure $(\mathbf{R}; \times, 0^{-1}, 0, -1, P^3)$ admits quantifier elimination, and so is decidable. [11]

Table (VII) : A Quantifier Elimination Procedure for the Reals Numbers at different Language:

Theory of	Language	admit QE
$(\mathbf{R}, <)$	$L = (<)$	$\checkmark$
$(\mathbf{R}, 0, +, -)$	$L = (0; +; -)$	$\checkmark$
$(\mathbf{R}, 0, +, -, <)$	$L = (0, +, -, <)$	$\checkmark$
$(\mathbf{R}, \times)$	$L = (\times, 0^{-1}, 0, 1, -1, P)$	$\checkmark$

E. Deciability of structures of complex numbers at different Language

The additive of theory of the complex number is similiar to the additive theory of real and rational number, and so has a decidable theory. It is interesting , we know that the proof desidability of thr theory of $(\mathbf{C}; +)$ and $(\mathbf{R}; +)$ and $(\mathbf{Q}; +)$ is easire than $(\mathbf{Z}; +), (\mathbf{N}; +)$.

Theorem19. The theoty of $(\mathbf{C}; \times)$ admits quantifier elimination , and so has a decidable theory.[11]

Table $(VIII)$: A Quantifier Elimination Procedure for the complex Numbers at different Language:

Theory of	admit QE
$(\mathbf{C}, +)$	$\checkmark$
$(\mathbf{C}, \times)$	$\checkmark$

F. Deciding Boolean Algebras:

Boolean algebras was to begin with intoduced by Boole in an effort to automate reasoning. Since that they have been extensively studied, and have proved fundemental

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[3]postivity property

in numerous application areas. At the consider of Boolean algebras, we show decidability and undecidability questions for the class of Boolean algebras,And We describe an algorithm for deciding the Boolean algebras. A basic result of Tarski is that the elementary theory of Boolean algebras is decidable. Even the theory of Boolean algebras with a distinguished ideal is decidable. On the other hand, the theory of a Boolean algebra with a distinguished subalgebra is undecidable. Both the decidability results and undecidablity results extend in various ways to Boolean algebras in extensions of first-order logic.

- Definition:Atoms are exactly the minimal nonzero elements, i.e. a is an atom iff $0 \leq a$ and $0 < x \leq a \Longrightarrow x = a$.

An algorithm for deciding the theory Atomic Boolean algebras: We present an algorithm and show how decide.We have some definitions:

- $L = \{\subseteq, \cap, \cup, A \setminus B, =, \emptyset, C_n, E_n, n \in \mathbf{N}^+\}$
- $\mathbf{A}(a) \leftrightarrow \forall x[x \subseteq a \to x = \emptyset \lor x = a] \land a \neq \emptyset$
- $C_n(x) \equiv \exists a_1 \cdots a_n (\bigwedge_{i<j} a_i \neq a_j \land \bigwedge_{i=1}^n \mathbf{A}(a_i) \land \bigwedge_{i=1}^n a_i \subseteq X)$
- $E_{n(x)} \equiv C_n(x) \land \neg C_{n+1}(x)$
- The next step of the algorithm is eliminate $=$:
  Because: $a = b \Longleftrightarrow a \subseteq b \land b \subseteq a$
- eliminate $\subseteq$
  Because: $a \subseteq b \Longleftrightarrow a \setminus b = \emptyset \leftrightarrow E_0(a - b)$
- And eliminat: $\neg$:
  Because: $\neg C_n(x) \Longleftrightarrow \bigvee_i E_i(x)$
  $\neg E_n(x) \Longleftrightarrow C_{n+1}(x) \lor \bigvee_i E_i(x)$

Quantifier-Elimination for Boolean formulas is as follows:

- $L = \{\cap, \cup, \setminus, =, \{C_n\}, \{E_n\}, n \in \mathbf{N}^+\}$
  We have the following
- $R = \{= |\{C_n\}_{n \geq 0}| \subseteq |\{E_n\}_{n \geq 0}\}$
  $F = \{A|F_1 \land F_2|F_1 \lor F_2|\neg F|\exists F|\forall F\}$
  $A = \{B_1 = B_2|B_1 \subseteq B_2|C_n(B), E_n(B)\}$
  $B = \{x|\emptyset|I|B_1 \cap B_2|B_1 \cup B_2|B^c\}$
  $n = \{0|1|2|\cdots\}$

So it is enough to consider only the following formulas: $C_n(x) = |x| \geq n, E_n(x) = |x| = n$. Contradictions of liters are eliminated according to the above definitions.

$\neg |x| = n \leftrightarrow |x| = 0 \lor \cdots \lor |x| = n - 1 \lor |x| \geq n + 1$
$\neg |x| \geq n \leftrightarrow |x| = 0 \lor \cdots \lor |x| = n - 1$

So at this step we've removed some of the relationships as follow :
1.Eliminate equality

$a = b \leftrightarrow a \subseteq b \land b \subseteq a$

2. Delete inclusion

$a \subseteq b \leftrightarrow |a \cap b^c| = 0$

3.Eliminate contradictions

$\neg C_n(x) \leftrightarrow \bigvee_{i<n} E_i(x)$
$\neg E_n(x) \leftrightarrow C_{n+1}(x) \lor \bigvee_{i<n} E_i(x)$

Language to Quantifier-Elimination

$$\cap, \cup, {}^c, \emptyset, \{C_n\}_{n \geq 0}, \{E_n\}_{n \geq 0}$$

term:

$$x, \emptyset, \cap, \cup, {}^c$$

Quantifier Elimination:
In the resulting formula, each set variable x occurs in some term $|t(x)|$. each set expression $|t(x)|$ as a union of cubes (regions in the Venn diagram). The cubes have the form $\bigcap_{i=1}^n x_i^{\alpha_i}$ where $x_i^{\alpha_i}$ is either x_i or x_i^c ; there are $m = 2^n$ cubes . The resulting formula is then equivalent to

$$\exists x(\land_i C_{n_i}(t_i(x)) \land (\land_j E_{n_j}(t_j'(x))))$$

for example:

$$\exists x(|x \cap c| \geq 3 \land |x \cap c| \geq 7 \land |c - x| = 2)$$
$$\exists x(C_3(x \cap c) \land C_7(x \cap c) \land E_2(c - x)) \equiv C_9(c)$$
$$\exists x(C_5(x \cap c) \land C_7(x \cap d) \land E_6(c - x)) \equiv C_{11}(c) \land C_7(d)$$

TABLE(IX):More explained of quantifier elimination in atomic boolean algebras

The main formula	Deleted form
$\exists z \cdots |x \cap z| \geq k \land |x \cap z^c| \geq l \cdots$	$|x| \geq k + l$
$\exists z \cdots |x \cap z| = k \land |x \cap z^c| \geq l \cdots$	$|x| \geq k + l$
$\exists z \cdots |x \cap z| \geq k \land |x \cap z^c| = l \cdots$	$|x| \geq k + l$
$\exists z \cdots |x \cap z| = k \land |x \cap z^c| = l \cdots$	$|x| = k + l$

A Boolean Algebra is atomless if it has no atoms.Every atomless Boolean algebras with more than one element must be infinite.Indeed,the unit 1 is different from zero, so there is a non-zero element p_1 strictly below 1;otherwise, 1 would be an atom. Because p_1 is not zero,there must be a non-zero element p_2 strictly below p_1; otherwise, p_1 decreasing sequence of elements $1 > p_1 > p_2 > \cdots$.

atomless boolean algebra: we have the interval algebra of rational number . the intrval algebra of the rational number is aomless.

lemma4. We have in every Boolean algebra:

$$p \subset P, q \subset Q, P \cap Q = \emptyset, P \cdot Q = 0 \to p + q + Q$$

Proof: $p + q \subseteq P + Q$.
To show:
$p + q \neq P + Q$ We assume
$p + q = P + Q$ so.
$(P + Q) \cdot \bar{p} = P \cdot \bar{p} + Q \cdot \bar{p}$
Because $Q \cap p = \emptyset$ we have $= P \cdot \bar{p} + Q = (p + q) \cdot \bar{p}$
$p \cdot \bar{p} + q \cdot \bar{p}$
$= q \qquad (q \cap p = \emptyset)$
$= q$
$(P + Q) \cdot \bar{p} \cdot \bar{q} = q \cdot \bar{q} = \emptyset$
$P \cdot \bar{p} \cdot \bar{q} + Q \cdot \bar{p} \cdot \bar{q} = 0 \Longleftrightarrow P \cdot \bar{q} + Q \cdot \bar{p} = 0$
$\Longleftrightarrow P \cdot \bar{q} = Q \cdot \bar{p} = 0$
Which contradicts with the assumption
lemma5. The following formulas are equivalent:

$\exists x(rx = 0 \land s\bar{x} = 0 \land \bigwedge_{i=1}^m u_i x \neq 0 \land \bigwedge_{j=}^n v_j \bar{x} \neq 0)$ \qquad (37)
$rs = 0 \land \exists y(\bigwedge_{i=1}^m u_i \bar{r} y \neq 0 \land \bigwedge_{j=}^n v_j \bar{s} \bar{y} \neq 0)$ \qquad (38)

Proof:
for only direction,
If there is x such that,

$$rx = 0 \land s\bar{x} = 0 \land \bigwedge_{i=1}^{m} u_i x \neq 0 \land \bigwedge_{j=}^{n} v_j \bar{x} \neq 0)$$
$$rx = 0 \land rs\bar{x} = 0 \Rightarrow rs(x + \bar{x}) = 0 \Rightarrow rs(1) = 0 \Rightarrow rs = 0$$
$$u_i x \neq 0 \Rightarrow u_i x(r + \bar{r}) \neq 0$$
$$\Rightarrow u_i xr + u_i x\bar{r} \neq 0$$
$$u_i x\bar{r} \neq 0$$
$$\Rightarrow \exists x(\bigwedge_{i=1}^{m} u_i \bar{r} x \neq 0 \land \bigwedge_{j=}^{n} v_j \bar{s}\bar{x} \neq 0)$$

for other direction,we Suppose $rs = 0$, there is y such that ,

$$\bigwedge_{i=1}^{m} u_i \bar{r} y \neq 0 \land \bigwedge_{j=}^{n} v_j \bar{s}\bar{y} \neq 0$$

We put,

$$x = \bar{r} \cdot (s + y)$$
$$\bar{x} = r + \bar{s}\bar{y} = (r + \bar{s})(r + \bar{y})$$
$$\bar{s} \cdot (r + \bar{y})$$

We show,

$$\bigwedge_{i=1}^{m} u_i x \neq 0 \land \bigwedge_{j=}^{n} v_j \bar{x} \neq 0$$
$$u_i x = u_i \bar{r}(s + y) = u_i \bar{r}s + u_i \bar{r}y \supseteq u_i \bar{r}y \neq 0$$
$$v_j \bar{x} = v_j \bar{s}(r + \bar{y}) = v_j \bar{s}r + v_j \bar{s}\bar{y} \supseteq v_j \bar{s}\bar{y} \neq 0$$

Theorem20. The theory of atomless Boolean algebra in the language $L = \langle 0, 1, \land, \lor, \neg, = \rangle$ is accepts the quantifier elimination , so has a decidable theory. And is axiomatizable.

- $[1]x \land y = y \land x \qquad x \lor y = y \lor x$
- $[2]x \land (y \land z) = (x \land y) \land z \qquad x \lor (y \lor z) = (x \lor y) \lor z$
- $[3](x \land y) \lor y = y \qquad (x \lor y) \land y = y$
- $[4]x \land (y \lor z) = (x \land y) \lor (x \land z) \qquad x \lor (y \land z) = (x \lor y) \land (x \lor z)$
- $[5]x \land x^c = 0 \qquad x \lor x^c = 1$
- $[6]\neg At(x) \qquad 0 \neq 1$

Proof:

$F = \{A|F_1 \land F_2|F_1 \lor F_2|\neg F|\exists F|\forall F\}$
$A = \{t_1 = t_2\}$
$T = \{x|0|1|t_1 \lor t_2|t_1 \land t_2|\neg t\}$

we have:
$t = s$ and $t \neq s$ so

$$t = \bigcup_{i \in I}(\bigcap_{j \in J} i, j)$$

such that i, j Is variable or complement variable.
. Terms included x:

$$x \cdot r + \bar{x} \cdot s$$

Atomic formulas:

$t = s \leftrightarrow t \cdot \bar{s} + \bar{t} \cdot s = 0$
$t \neq s \leftrightarrow t \cdot \bar{s} + \bar{t} \cdot s \neq 00$

Atomic formulas include x:

$r \cdot x + s \cdot \bar{x} = 0$
$\leftrightarrow r \cdot x = 0 \land s \cdot \bar{x} = 0$

$\leftrightarrow x \subseteq \bar{r} \land s \subseteq x \leftrightarrow sr = \emptyset$

Contradiction of atomic formulas include x:

$r \cdot x + s \cdot \bar{x} \neq 0 \leftrightarrow r \cdot x \neq 0 \lor s \cdot \bar{x} \neq 0$

so it is enough to eliminate quantifiers of the folloeing formulas:

$\exists x(rx = 0 \land s\bar{x} = 0 \land \bigwedge_{i=1}^{m} u_i x \neq 0 \land \bigwedge_{j=}^{n} v_j \bar{x} \neq 0)$ \qquad (39)
$\equiv rs = 0 \land \exists y(\bigwedge_{i=1}^{m} u_i \bar{r} y \neq 0 \land \bigwedge_{j=}^{n} v_j \bar{s}\bar{y} \neq 0)$ \qquad (40)

becuase:

if the left-side holds then we have:

if there is x such that

$$rs = 0 \land \exists y(\bigwedge_{i=1}^{m} u_i \bar{r} y \neq 0 \land \bigwedge_{j=}^{n} v_j \bar{s}\bar{y} \neq 0)$$

so

$rsx = 0, rs\bar{x} = 0 \to rs(x + \bar{x}) = 0 \to rs = 0$
$u_i x \neq 0 \Rightarrow u_i x(r + \bar{r}) \neq 0$
$\Rightarrow u_i xr + u_i x\bar{r} \neq 0$
$\Rightarrow u_i x\bar{r} \neq 0$
$v_j \bar{x} \neq 0 \to v_j \bar{x}(s + \bar{s}) \neq 0$
$\to v_j \bar{x}s + v_j \bar{x}\bar{s} \neq 0$
$\to v_j \bar{x}\bar{s} \neq 0$
$\to \exists x(\bigwedge_{i=1}^{m} u_i \bar{r} x \neq 0 \land \bigwedge_{j=}^{n} v_j \bar{s}\bar{x} \neq 0)$

if the right-side holds. so,

we assume there was $rs \neq 0$ and y such that

$$\bigwedge_{i=1}^{m} u_i \bar{r} y \neq 0 \land \bigwedge_{j=}^{n} v_j \bar{s}\bar{y} \neq 0$$

we put

$x = \bar{r}(s + y)$
$\bar{x} = r + \bar{s}\bar{y} = (r + \bar{s}) \cdot (r + \bar{y})$
$= \bar{s} \cdot (r + \bar{y})$

so $rx = 0, s\bar{x} = 0$ it is enough to show

$\bigwedge_{i=1}^{m} u_i x \neq 0 \land \bigwedge_{j=}^{n} v_j \bar{x} \neq 0$
$u_i x = u_i \bar{r}(s + y) = u_i \bar{r}s + u_i \bar{r}y \supseteq u_i \bar{r}y$
$v_j \bar{x} = v_j \bar{s}(r + \bar{y}) = v_j \bar{s}r + v_j \bar{s}\bar{y} \supseteq v_j \bar{s}\bar{y} \neq 0$

.

so it suffices to eliminate the quantifier of the formula

$\exists y(\bigwedge_{i=1}^{m} a_i y \neq 0 \land \bigwedge_{j=}^{n} b_j \bar{y} \neq 0)$ \qquad (41)
$\equiv \bigwedge_{i=1}^{m} a_i \neq 0 \land \bigwedge_{j=}^{n} b_j \neq 0$ \qquad (42)

The only if $\Longrightarrow$ direction is trivial.
For the other directon , we consider all cells C_α includin a_i, b_j
both cells are distinctly distinct $C_\alpha \cap C_\beta = \emptyset, C_\alpha \cdot C_\beta = 0$
each set is equall tocommunity of cells contained in it
$Z = \sum_{C_\alpha \subseteq Z} C_\alpha$ for all Z and any cell C we have $C \subseteq Z$ with $C \subseteq Z$.

$$Z \neq 0 \leftrightarrow \exists \alpha(C_\alpha \subseteq Z \land C_\alpha \neq 0)$$

for any cells is not equall zero C_α from being atomless d_α there is such that

$$0 \neq d_\alpha C_\alpha \neq 0$$

if $C_\alpha = 0$ put $d_\alpha = 0$ we put it now $y = \sum_{C_\alpha \neq 0} d_\alpha$.

$a_i y \neq 0 : a_i y = a_i \sum_{C_\alpha \neq 0} d_\alpha = \sum_{C_\alpha \neq 0} a_i d_\alpha$
$\supseteq a_i d_\beta \supseteq c_\beta d_\beta = d_\beta \neq 0$
$0 \neq a_i = \sum_{C_\beta a_i} C_\beta$
$\exists \beta : C_\beta a_i \wedge \bar{C}_\beta \neq 0$
$0 \neq d_\beta C_\beta \neq 0$
$b_j \bar{y} \neq 0 : b_j \bar{y} = b_j \prod_{C_\alpha \neq 0} \bar{d}_\alpha = \prod_{C_\alpha \neq 0} b_j \bar{d}_\alpha$
forall C_α we have $C_\alpha \subseteq b_j$ with $C_\alpha \subseteq \bar{b}_j \to \bar{C}_\alpha \supseteq b_j$
if $C_\alpha \neq 0$ then $d_\alpha C_\alpha \to \bar{d}_\alpha \supseteq \bar{C}_\alpha \supseteq b_j$ so $b_j \bar{d}_\alpha = b_j$
$b_j \bar{y} = \prod_{C_\alpha \subseteq b_j} b_j \bar{d}_\alpha = \prod_{0 \neq C_\alpha \subseteq b_j} b_j \bar{d}_\alpha$
$C_\alpha = 0 \to \bar{d}_\alpha = 1$
$C_\alpha \neq 0 \to 0 \neq d_\alpha C_\alpha \subseteq b_j$
$\sum_{0 \neq C_\alpha \subseteq b_j} d_\alpha \sum_{0 \neq C_\alpha \subseteq b_j} C_\alpha = b_j$
$b_j \sum \bar{d}_{apha} \neq 0$
$b_j \prod_{0 \neq C_\alpha \subseteq b_j} d_{alpha} \neq 0$
$b_j \bar{y} \neq 0$

The above proof we proved theory of Boolean algebras
by the quantifier-elimination is decidable.
The first-order theory of Boolean algebras, established
by Alfred Tarski in 1940 (found in 1940 but announced
in 1949).

Theory of	Proved by	A method of proof
Boolean algebras	by Tarski in 1949	by model completeness

Table (X) :Decidability of Boolean algebras

III. Conclusion

We could completey axiomatize the theoty $\langle Q^+; \sqsubseteq \rangle$
:Indeed , the theory of the structure $\langle Q^+; \sqsubseteq \rangle$ is
decidable. But we result decidability theorical, we leave
open the problem of finding a $\langle Q^+; \sqsubseteq \rangle$ such that admits
quantifier elimnation.The theory $\langle N; \times \rangle$ is decidable and
axiomatizable.So the theory $\langle Z; \times \rangle$ we proved by
methods of decidability of $\langle N; \times \rangle$ is decidable and
axiomatizable, and this paper we present an explicit
axioomatization for the theory $\langle Z; \times \rangle$. In this paper,
decidability (i.e., there exists an algorithm that decides
whether a given sentence is derivable from the theory) of
the structures study in different languages and introduce
ways that it allows quantifier elimination (for the
theory) and review some classical theorems and give for
some of old results ,new proofs.

The Quantifier Elimination of the structure and
decidability of them in different languages is shown in
the following tables so that the theories that admit QE
by $\sqrt{}$ and, the theories do not admit QE by $\times$ is shown.

Table (XI) :The Quantifier Elimination of the structure
and decidability of them in different languages

	N	Z	Q	R	C
$\{+\}$	$\sqrt{}$	$\sqrt{}$	$\sqrt{}$	$\sqrt{}$	$\sqrt{}$
$\{\times\}$	$\sqrt{}$	$\sqrt{}$	$\sqrt{}$	$\sqrt{}$	$\sqrt{}$
$\{+, \times\}$	$\times$	$\times$	$\times$	$\sqrt{}$	$\sqrt{}$
$\{\sqsubseteq\}$	$\sqrt{}$	?	$\sqrt{}$	?	?

IV. Declarations

Ethical Approval: All procedures performed in studies involving human participants wereby the ethical standards of the institutional and national research committee and with the 1964 Helsinki declaration and its later amendments or comparable ethical standards. This article does not contain any studies with animals performed by any of the authors. The present study is part of a Ph.D. thesis. Competing interests: The author declare no competing interests. Authors' contributions: Not applicable Funding: This research received no specific grant from any funding agency in the public, commercial, or not-forprofit sectors. Availability of data and materials: The present study is part of a Ph.D. thesis.

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