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ON FUNCTION SPACES. III

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Abstract. For topological T_0-spaces $\mathbb{X}$ and $\mathbb{Y}$, we prove that the space $\mathbb{Y}$ is H-sober if and only if the function space $\mathbb{C}_{\mathcal{T}}(\mathbb{X},\mathbb{Y})$ endowed with a certain topology $\mathcal{T}$ is H-sober.

Keywords: core-compact space, function space, sober space, T_0-space.

1. Introduction

This paper continues [8]–[9]. In [8], the interplay of different topological properties for a T_0-space $\mathbb{Y}$ and its function space $\mathbb{C}(\mathbb{X},\mathbb{Y})$ (endowed with the pointwise convergence topology) was investigated. These results were extended in [9] for certain topologies on the set $C(\mathbb{X},\mathbb{Y})$ of continuous functions from a T_0-space $\mathbb{X}$ to a T_0-space $\mathbb{Y}$. Here, generalize some results from [9] and consider the general property of H-sobriety introduced in [13].

Our main result is Theorem 14. In Section 5, we give several applications of this result.

For all the notions and notation which is not defined here, we refer to the monograph of the first author [7] as well as to [8, 9].

2. Topologies on $C(\mathbb{X},\mathbb{Y})$

The proof of the following statement is straightforward.

Lemma 1. *Let $\mathbb{X}$, $\mathbb{Y}$ be T_0-spaces. If $\mathcal{T}$ is a topology on $C(\mathbb{X},\mathbb{Y})$ such that $\mathcal{P}\subseteq\mathcal{T}\subseteq\mathcal{T}_A(\leq_{\mathcal{P}})$ then $\leq_{\mathcal{P}}$ and $\leq_{\mathcal{T}}$ agree.*

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We consider the mapping

$$\xi\colon \mathbb{Y} \to C(\mathbb{X},\mathbb{Y}); \quad \xi\colon y \mapsto \xi_y, \quad \text{where } \xi_y(x) = y \text{ for all } x \in X.$$

Lemma 2. [9, Lemma 2] *Let $\mathbb{X}$, $\mathbb{Y}$ be T_0-spaces. Then $\mathcal{T}_A(\subseteq)^\sharp \subseteq \mathcal{T}_A(\leq_{\mathcal{P}})$.*

Lemma 3. [9, Lemma 3] *Let $\mathbb{X}$, $\mathbb{Y}$ be T_0-spaces. If $\mathcal{T}$ is a topology on $C(\mathbb{X},\mathbb{Y})$ such that $\mathcal{P} \subseteq \mathcal{T}_A(\subseteq)^\sharp$ then ξ is a homeomorphic embedding of $\mathbb{Y}$ into $\mathbb{C}_{\mathcal{T}}(\mathbb{X},\mathbb{Y})$.*

In what follows, we write $\xi_{\mathcal{T}}$ instead of ξ for each topology $\mathcal{T}$ on $C(\mathbb{X},\mathbb{Y})$ for which ξ is continuous.

For T_0-spaces $\mathbb{X}$ and $\mathbb{Y}$, for an element $x \in X$, and for a topology $\mathcal{T}$ on $C(\mathbb{X},\mathbb{Y})$, consider the following mapping:

$$\zeta_{x,\mathcal{T}}\colon \mathbb{C}_{\mathcal{T}}(\mathbb{X},\mathbb{Y}) \to \mathbb{Y}, \quad \zeta_{x,\mathcal{T}}\colon f \mapsto f(x).$$

Lemma 4. *Let $\mathbb{X}$, $\mathbb{Y}$ be T_0-spaces and let $x \in X$. If $\mathcal{T}$ is a topology on $C(\mathbb{X},\mathbb{Y})$ such that $\mathcal{P} \subseteq \mathcal{T}$ then $\zeta_{x,\mathcal{T}}$ is continuous.*

Proof. Suppose that $U \in \mathcal{T}(\mathbb{Y})$ and that $f \in \zeta^{-1}_{x,\mathcal{T}}(U)$. Then $\zeta_{x,\mathcal{T}}(f) = f(x) \in U$ whence $f \in V_{x,U} \in \mathcal{P} \subseteq \mathcal{T}$. To establish that $\zeta^{-1}_{x,\mathcal{T}}(U) \in \mathcal{T}$, it suffices to prove that $V_{x,U} \subseteq \zeta^{-1}_{x,\mathcal{T}}(U)$. Indeed, if $g \in V_{x,U}$ then $\zeta_{x,\mathcal{T}}(g) = g(x) \in U$ and therefore, $g \in \zeta^{-1}_{x,\mathcal{T}}(U)$ which was to prove. □

For the following facts, we refer to Chapters 6 and 15 in [7] as well as to Sections 5.3–5.4 in [11] and to Section II-4 in [10].

Proposition 5. *Let $\mathbb{X}$, $\mathbb{Y}$ be T_0-spaces.*

(1) *If $\mathcal{T}_0$ is a proper and $\mathcal{T}_1$ is an admissible topology on $C(\mathbb{X},\mathbb{Y})$ then $\mathcal{T}_0 \subseteq \mathcal{T}_1$.*
(2) *If $\mathcal{T}_0 \subseteq \mathcal{T}_1$ and $\mathcal{T}_1$ is a proper topology on $C(\mathbb{X},\mathbb{Y})$ then $\mathcal{T}_0$ is also a proper topology on $C(\mathbb{X},\mathbb{Y})$.*
(3) *If $\mathcal{T}_0 \subseteq \mathcal{T}_1$ and $\mathcal{T}_0$ is an admissible topology on $C(\mathbb{X},\mathbb{Y})$ then $\mathcal{T}_1$ is also an admissible topology on $C(\mathbb{X},\mathbb{Y})$.*
(4) *$\mathcal{P} \subseteq \mathcal{K} \subseteq \mathcal{I} \subseteq \mathcal{T}_*$ and $\mathcal{K} \subseteq \mathcal{C}$.*
(5) *The natural topology $\mathcal{T}_*$ is the finest proper topology on $C(\mathbb{X},\mathbb{Y})$.*

For the next statement, we refer to [5] and to [7, Theorem 6.2.1], see also [11, Theorem 5.4.4].

Theorem 6. *The following conditions are equivalent for a T_0-space $\mathbb{X}$.*

(1) *$\langle \mathcal{T}(\mathbb{X}); \subseteq \rangle$ is a continuous poset.*
(2) *$\langle \mathcal{T}(\mathbb{X}), \mathcal{T}_S(\subseteq) \rangle$ is an α-space.*
(3) *For an arbitrary T_0-space $\mathbb{Y}$, there is an exponential topology on $C(\mathbb{X},\mathbb{Y})$.*
(4) *There is an exponential topology on $C(\mathbb{X},\mathbb{S})$.*

A topological space $\mathbb{X}$ which satisfies the equivalent conditions of Theorem 6 is called a *core-compact space*. A topological space $\mathbb{X}$ is *locally compact*, if for each $x \in X$ and for each $U \in \mathcal{T}(\mathbb{X})$ with $x \in U$, there is a compact set $K \subseteq X$ and an open set $V \in \mathcal{T}(\mathbb{X})$ such that $x \in V \subseteq K \subseteq U$.

For the next statement, we refer to [3], [10, Lemma II-4.2], to [7, Theorem 6.3.3], and to [11, Proposition 5.4.20].

Proposition 7. *Let $\mathbb{X}$, $\mathbb{Y}$ be T_0-spaces.*

(1) *If $\mathbb{X}$ is core-compact then $\mathcal{C} = \mathcal{I} = \mathcal{T}_*$ and this topology is exponential on $C(\mathbb{X},\mathbb{Y})$.*

(2) *If* $\mathbb{X}$ *is locally compact then* $\mathcal{K} = \mathcal{C} = \mathcal{I} = \mathcal{T}_*$ *and this topology is exponential on* $C(\mathbb{X}, \mathbb{Y})$.
(3) *If* $\mathbb{X}$ *is an* α^**-space then* $\mathcal{P} = \mathcal{K} = \mathcal{C} = \mathcal{I} = \mathcal{T}_*$ *and this topology is exponential on* $C(\mathbb{X}, \mathbb{Y})$.

Corollary 8. *For a core-compact space* $\mathbb{X}$ *and a* T_0*-space* $\mathbb{Y}$ *the exponential topology on* $C(\mathbb{X}, \mathbb{Y})$ *is defined by the subbasis of open sets*

$$V_{U,W} = \big\{ f \in C(\mathbb{X}, \mathbb{Y}) \mid U \prec f^{-1}(W) \big\},$$

where $\varnothing \neq U \in \mathcal{T}(\mathbb{X})$ *and* $\varnothing \neq W \in \mathcal{T}(\mathbb{Y})$.

The following statement has quite a straightforward proof and is to find as Proposition 3.2 in [4], see also [9, Proposition 15].

Proposition 9. *Let* $\mathbb{X}$ *be a core-compact space and let* $\mathcal{B} \subseteq \mathcal{T}(\mathbb{X})$ *be an additive basis of* $\mathcal{T}(\mathbb{X})$. *If open sets* $U, U', V_0, \ldots, V_n \in \mathcal{B}$, $n < \omega$, *are such that* $U \prec U' \prec V_0 \cup \ldots \cup V_n$ *then there are* $W_0, \ldots, W_n \in \mathcal{B}$ *such that*

$$W_i \prec V_i \text{ for all } i \leqslant n \text{ and } U \subseteq W_0 \cup \ldots \cup W_n \subseteq U'.$$

3. H-sober spaces

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For a topological T_0-space $\mathbb{X}$, we consider the following sets of subsets of X:

- $\mathsf{I}(\mathbb{X})$, the set of all nonempty irreducible subsets of X;
- $\mathsf{S}(\mathbb{X})$, the set of all one-element subsets of X;
- $\mathsf{D}(\mathbb{X})$, the set of all nonempty up-directed subsets of X;
- $\mathsf{WF}(\mathbb{X})$, the set of all nonempty well-filtered subsets of X;
- $\mathsf{R}(\mathbb{X})$, the set of all nonempty Rudin subsets of X;
- $\mathsf{I}^b(\mathbb{X})$, the set of all nonempty bounded irreducible subsets of X;
- $\mathsf{D}^b(\mathbb{X})$, the set of all nonempty bounded up-directed subsets of X;
- $\mathsf{WF}^b(\mathbb{X})$, the set of all nonempty bounded well-filtered subsets of X;
- $\mathsf{R}^b(\mathbb{X})$, the set of all nonempty bounded Rudin subsets of X.

Definition 1. [13] A *subset system* is a covariant functor $\mathsf{H}\colon \mathbf{Top}_0 \to \mathbf{Set}$ such that $\mathsf{S}(\mathbb{X}) \subseteq \mathsf{H}(\mathbb{X}) \subseteq 2^X$ and for each $\mathbb{Y} \in \mathbf{Top}_0$ and each continuous mapping $f\colon \mathbb{X} \to \mathbb{Y}$, we have $\mathsf{H}(f)(A) = f(A) \in \mathsf{H}(\mathbb{Y})$ for all $A \in \mathsf{H}(\mathbb{X})$.

A subset system H is an *I-system* if $\mathsf{H}(\mathbb{X}) \subseteq \mathsf{I}(\mathbb{X})$ for all $\mathbb{X} \in \mathbf{Top}_0$.

Corollary 10. *If* $\mathsf{X} \in \{\mathsf{D}, \mathsf{D}^b, \mathsf{WF}, \mathsf{WF}^b, \mathsf{R}, \mathsf{R}^b, \mathsf{I}, \mathsf{I}^b\}$ *then* X *is an I-system.*

For a subset system H, we put $\mathsf{H}_c(\mathbb{X}) = \big\{ \mathrm{cl}_{\mathbb{X}} A \mid A \in \mathsf{H}(\mathbb{X}) \big\}$.

Definition 2. [13] Let H be a subset system. A T_0-space $\mathbb{X}$ is H-*sober*, if $\mathsf{H}_c(\mathbb{X}) = \mathsf{S}_c(\mathbb{X})$.

The following statement is straightforward to prove, see [13, Proposition 4.26].

Lemma 11. *Let* H *be a subset system* [*an I-system, respectively*]. *If* $\mathbb{X}$ *is a retract of an* H*-sober space* $\mathbb{Y}$ *then* X *is also* H*-sober.*

Proof. There are a continuous onto mapping $r\colon \mathbb{Y} \to \mathbb{X}$ and an embedding $e\colon \mathbb{X} \to \mathbb{Y}$ such that $re = \mathrm{id}_{\mathbb{X}}$. Let $A \in \mathsf{H}(\mathbb{X})$; then $e(A) \in \mathsf{H}(\mathbb{Y})$. According to our assumption, there is $y \in Y$ such that $\mathrm{cl}_{\mathbb{Y}}\, e(A) = \downarrow y$. It is straightforward to check that $r(y)$ is a limit point for the set $re(A) = A$. Moreover, as $r\big(\mathrm{cl}_{\mathbb{Y}}\, e(A)\big) \subseteq \mathrm{cl}_{\mathbb{X}}\, re(A) = \mathrm{cl}_{\mathbb{X}} A$, we conclude that $\mathrm{cl}_{\mathbb{X}} A = \downarrow r(y)$. □

4. The H-sobriety of function spaces

Proposition 12. *Let* H *be a subset system [an* I*-system, respectively]. Let* $\mathbb{X}$*,* $\mathbb{Y}$ *be* T_0*-spaces and let* $\mathcal{T}$ *be a topology on* $C(\mathbb{X},\mathbb{Y})$ *such that* $\mathcal{P} \subseteq \mathcal{T} \subseteq \mathcal{T}_A(\subseteq)^\sharp$*. If* $\mathbb{C}_\mathcal{T}(\mathbb{X},\mathbb{Y})$ *is an* H*-sober space then* $\mathbb{Y}$ *is also* H*-sober.*

Proof. According to Lemma 3, $\xi_\mathcal{T}$ is a homeomorphic embedding of $\mathbb{Y}$ into $\mathbb{C}_\mathcal{T}(\mathbb{X},\mathbb{Y})$. For a fixed element $x \in X$, the mapping $\zeta_{x,\mathcal{T}}\colon \mathbb{C}_\mathcal{T}(\mathbb{X},\mathbb{Y}) \to \mathbb{Y}$ is continuous by Lemma 4. Moreover, for each $y \in Y$, we have $\zeta_{x,\mathcal{T}}\xi_\mathcal{T}(y) = \zeta_{x,\mathcal{T}}(\xi_y) = \xi_y(x) = y$, whence the space $\mathbb{Y}$ is a retract of $\mathbb{C}_\mathcal{T}(\mathbb{X},\mathbb{Y})$. The desired conclusion follows from Lemma 11. □

Proposition 13. *Let* H *be an* I*-system. For* T_0*-spaces* $\mathbb{X}$ *and* $\mathbb{Y}$*, the following statements hold.*

1. *If* $\mathbb{Y}$ *is* H*-sober, then* $\mathbb{C}(\mathbb{X},\mathbb{Y})$ *is also* H*-sober.*
2. *If* $\mathbb{X}$ *is core-compact and* $\mathbb{Y}$ *is* H*-sober, then* $\mathbb{C}_\mathcal{I}(\mathbb{X},\mathbb{Y})$ *is also* H*-sober.*

Proof. In what follows, let $\mathcal{T} \in \{\mathcal{P},\mathcal{I}\}$. Then we obtain by Lemma 2 that $\mathcal{P} \subseteq \mathcal{T} \subseteq \mathcal{T}_A(\subseteq)^\sharp \subseteq \mathcal{T}(\leq_\mathcal{P})$. This implies by Lemma 1 that $\leq_\mathcal{T}$ and $\leq_\mathcal{P}$ agree.

Let $F \in \mathsf{H}\big(\mathbb{C}_\mathcal{T}(\mathbb{X},\mathbb{Y})\big)$; then for each $x \in X$, we have $F(x) = \{f(x) \mid f \in F\} = \zeta_{x,\mathcal{T}}(F) \in \mathsf{H}(\mathbb{Y})$. The H-sobriety of $\mathbb{Y}$ implies that for each $x \in X$, there is $g(x) \in Y$ such that $\downarrow g(x) = \mathrm{cl}_\mathbb{Y} F(x)$. We show that the mapping $g\colon X \to Y$ is continuous. Indeed, let $x \in g^{-1}(U)$ for some $U \in \mathcal{T}(\mathbb{X})$. Then $g(x) \in U$, whence $f(x) \in U$ for some $f \in F$. Suppose that $x' \in f^{-1}(U) \in \mathcal{T}(\mathbb{X})$; that is, $f(x') \in U$. Since $f(x') \leq g(x')$, we conclude that $g(x') \in U$. Therefore, $x \in f^{-1}(U) \subseteq g^{-1}(U)$ whence $g^{-1}(U) \in \mathcal{T}(\mathbb{X})$. This proves that g is continuous.

As $\leq_\mathcal{T}$ and $\leq_\mathcal{P}$ agree, we conclude that $F \subseteq \downarrow g$ whence $\mathrm{cl}_\mathcal{T} F \subseteq \downarrow g$. In order to establish the H-sobriety of $\mathbb{C}_\mathcal{T}(\mathbb{X},\mathbb{Y})$, it suffices to show that g is a limit point for F in $\mathcal{T}$. Indeed, let $g \in V \in \mathcal{T}$.

Consider first the case when $\mathcal{T} = \mathcal{P}$. This means by the definition of the pointwise convergence topology that there are $x_0,\dots,x_n \in X$ and $U_0,\dots,U_n \in \mathcal{T}(\mathbb{Y})$ such that $g \in V_{x_0,U_0} \cap \ldots \cap V_{x_n,U_n} \subseteq V$. Thus, $g(x_i) \in U_i$ for all $i \leqslant n$. Since $g(x_i)$ is a limit point of the set $F(x_i)$, we conclude that $F \cap V_{x_i,U_i} \neq \varnothing$ for all $i \leqslant n$. Therefore $F \cap V_{x_0,U_0} \cap \ldots \cap V_{x_n,U_n} \neq \varnothing$ as F is an irreducible set, whence $F \cap V \neq \varnothing$, which is our desired conclusion.

Consider now the case when $\mathcal{T} = \mathcal{I}$. We prove an auxiliary claim first.

Claim 1. If $U \in \mathcal{T}(\mathbb{X})$ and $W \in \mathcal{T}(\mathbb{Y})$ are such that $g \in V_{U,W}$ then there is $f \in F$ such that $f \in V_{U,W}$.

Proof of Claim. First of all, we claim that

$$g^{-1}(W) = \bigcup\{f^{-1}(W) \mid f \in F\}.$$

Indeed, if $x \in f^{-1}(W)$ for some $f \in F$ then $g(x) \geq f(x) \in W$ whence $g(x) \in W$. Conversely, if $g(x) \in W$ then $f(x) \in W$ for some $f \in F$ as $g(x)$ is a limit point in $\mathbb{Y}$ for the set $F(x)$.

Furthermore, the inclusion $g \in V_{U,W}$ means that $U \prec g^{-1}(W) = \big\{f^{-1}(W) \mid f \in F\big\}$. Therefore, there are $f_0,\dots,f_k \in F$ such that $U \prec f_0^{-1}(W) \cup \ldots \cup f_k^{-1}(W)$. As $\mathbb{X}$ is core-compact, $\mathcal{T}(\mathbb{X})$ is an α-space in the Scott topology by Theorem 6. Thus, there is $U' \in \mathcal{T}(\mathbb{X})$ such that

$$U \prec U' \prec f_0^{-1}(W) \cup \ldots \cup f_k^{-1}(W).$$

Taking $\mathcal{B} = \mathcal{T}(\mathbb{X})$ and applying Proposition 9, we obtain that there are open sets $S_0, \ldots, S_k \in \mathcal{T}(\mathbb{X})$ such that

$$S_i \prec f_i^{-1}(W) \text{ for all } i \leqslant k \text{ and } U \subseteq S_0 \cup \ldots \cup S_n \subseteq U'.$$

For each $i \leqslant k$, we have $S_i \prec f_i^{-1}(W)$ whence $f_i \in V_{S_i,W} \in \mathcal{J}$ for all $i \leqslant k$. This means that $f_i \in F \cap V_{S_i,W} \neq \varnothing$ for all $i \leqslant k$. Therefore, $F \cap V_{S_0,W} \cap \ldots \cap V_{S_n,W} \neq \varnothing$ as F is an irreducible set.

To establish that $F \cap V_{U,W} \neq \varnothing$, it suffices to prove that $V_{S_0,W} \cap \ldots \cap V_{S_n,W} \subseteq V_{U,W}$. Indeed, if $h \in V_{S_0,W} \cap \ldots \cap V_{S_n,W}$ then $h \in V_{S_i,W}$ for each $i \leqslant k$. This implies that $S_i \prec h^{-1}(W)$ for all $i \leqslant k$. Therefore,

$$\begin{aligned} h^{-1}(W) \in \operatorname{int}_{\mathcal{T}(\mathbb{X})} \uparrow S_0 \cap \ldots \cap \operatorname{int}_{\mathcal{T}(\mathbb{X})} \uparrow S_k \subseteq \operatorname{int}_{\mathcal{T}(\mathbb{X})} \big(\uparrow S_0 \cap \ldots \cap \uparrow S_k \big) = \\ = \operatorname{int}_{\mathcal{T}(\mathbb{X})} \uparrow (S_0 \cup \ldots \cup \uparrow S_k). \end{aligned}$$

Hence, $U \subseteq S_0 \cup \ldots \cup S_k \prec h^{-1}(W)$ yielding $h \in V_{U,W}$, which implies the desired inclusion. □

Consider now the general case. By Corollary 8, there are open sets $U_0, \ldots, U_n \in \mathcal{T}(\mathbb{X})$ and $W_0, \ldots, W_n \in \mathcal{T}(\mathbb{Y})$ such that $g \in V_{U_0,W_0} \cap \ldots \cap V_{U_n,W_n} \subseteq V$. By Claim 1, there are $f_0, \ldots, f_n \in F$ such that $f_i \in F \cap V_{U_i,W_i} \neq \varnothing$ for all $i \leqslant n$. Therefore, $F \cap V_{U_0,W_0} \cap \ldots \cap V_{U_n,W_n} \neq \varnothing$ as F is an irreducible set, whence $F \cap V \neq \varnothing$, which is again our desired conclusion. □

Remark 1. The statement of Proposition 13(1) was established in [13, Theorem 4.28]. A stronger statement than the statement of Proposition 13(2) was announced in [2, Theorem 4.24]. However, the proof of this statement in [2] contains a gap which is in line 10 counting from the bottom of the proof of Theorem 4.24 there.

Theorem 14. *Let H be an I-system. For a T_0-space $\mathbb{Y}$, the following conditions are equivalent.*

1. *$\mathbb{Y}$ is H-sober.*
2. *$\mathbb{C}_{\mathcal{J}}(\mathbb{X}, \mathbb{Y})$ is H-sober for each core-compact space $\mathbb{X}$.*
3. *$\mathbb{C}_{\mathcal{J}}(\mathbb{X}, \mathbb{Y})$ is H-sober for some [core-compact] space $\mathbb{X}$.*
4. *$\mathbb{C}(\mathbb{X}, \mathbb{Y})$ is H-sober for each T_0-space $\mathbb{X}$.*
5. *$\mathbb{C}(\mathbb{X}, \mathbb{Y})$ is H-sober for some T_0-space $\mathbb{X}$.*
6. *$\mathbb{C}_{\mathcal{T}}(\mathbb{X}, \mathbb{Y})$ is H-sober for some T_0-space $\mathbb{X}$ and some topology $\mathcal{T}$ on $C(\mathbb{X}, \mathbb{Y})$ such that $\mathcal{P} \subseteq \mathcal{T} \subseteq \mathcal{T}_A(\subseteq)^\sharp$.*

Proof. Statement (1) implies (2) and (4) by Proposition 13. Furthermore, statement (2) implies (3), statement (4) implies (5), and each of statements (3) and (5) implies (6) in a trivial way. Finally, (6) implies (1) by Proposition 12. □

5. Applications

Corollary 15. *Let H be any of I-systems from the set*

$$\mathsf{X} \in \{\mathsf{D}, \mathsf{D}^b, \mathsf{WF}, \mathsf{WF}^b, \mathsf{R}, \mathsf{R}^b, \mathsf{I}, \mathsf{I}^b\}.$$

For a T_0-space $\mathbb{Y}$, the following conditions are equivalent.

1. *$\mathbb{Y}$ is H-sober.*
2. *$\mathbb{C}_{\mathcal{J}}(\mathbb{X}, \mathbb{Y})$ is H-sober for each core-compact space $\mathbb{X}$.*
3. *$\mathbb{C}_{\mathcal{J}}(\mathbb{X}, \mathbb{Y})$ is H-sober for some [core-compact] space $\mathbb{X}$.*
4. *$\mathbb{C}(\mathbb{X}, \mathbb{Y})$ is H-sober for each T_0-space $\mathbb{X}$.*

(5) $\mathbb{C}(\mathbb{X},\mathbb{Y})$ *is* H*-sober for some* T_0*-space* $\mathbb{X}$*.*
(6) $\mathbb{C}_{\mathcal{T}}(\mathbb{X},\mathbb{Y})$ *is* H*-sober for some* T_0*-space* $\mathbb{X}$ *and some topology* $\mathcal{T}$ *on* $C(\mathbb{X},\mathbb{Y})$ *such that* $\mathcal{P}\subseteq\mathcal{T}\subseteq\mathcal{T}_A(\subseteq)^\sharp$*.*

Remark 2. Corollary 15 generalizes Theorems 9 and 12 from [9], see also [12].

In [2], the following two problems were posed.

Problem 1. [2, Question 4.19] For a T_0-space $\mathbb{X}$ and an H-sober space $\mathbb{Y}$, is the function space $\mathbb{C}_{\mathcal{K}}(\mathbb{X},\mathbb{Y})$ H-sober?

Problem 2. [2, Question 4.20] For a T_0-space $\mathbb{X}$ and a sober [well-filtered] space $\mathbb{Y}$, is the function space $\mathbb{C}_{\mathcal{K}}(\mathbb{X},\mathbb{Y})$ sober [well-filtered, respectively]?

It follows from Proposition 7(2) and Theorem 14 that the following holds.

Corollary 16. *Let* H *be an I-system. For a* T_0*-space* $\mathbb{Y}$*, the following conditions are equivalent.*

(1) $\mathbb{Y}$ *is* H*-sober.*
(2) $\mathbb{C}_{\mathcal{K}}(\mathbb{X},\mathbb{Y})$ *is* H*-sober for each locally compact space* $\mathbb{X}$*.*
(3) $\mathbb{C}_{\mathcal{K}}(\mathbb{X},\mathbb{Y})$ *is* H*-sober for some [locally compact] space* $\mathbb{X}$*.*
(4) $\mathbb{C}_{\mathcal{T}}(\mathbb{X},\mathbb{Y})$ *is* H*-sober for some* T_0*-space* $\mathbb{X}$ *and some topology* $\mathcal{T}$ *on* $C(\mathbb{X},\mathbb{Y})$ *such that* $\mathcal{P}\subseteq\mathcal{T}\subseteq\mathcal{T}_A(\subseteq)^\sharp$*.*

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