S©MR ISSN 1813-3304

СИБИРСКИЕ ЭЛЕКТРОННЫЕ
МАТЕМАТИЧЕСКИЕ ИЗВЕСТИЯ

Siberian Electronic Mathematical Reports
http://semr.math.nsc.ru

Том 19, стр. ???–??? (2022) УДК 510.67
DOI 10.33048/semi.2019.16.xxx MSC 06A75, 03C10, 03C15, 03C50

SPHERICAL ORDERS, PROPERTIES AND COUNTABLE SPECTRA OF THEIR THEORIES

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Abstract. We study semantic and syntactic properties of spherical orders and their elementary theories, including finite and dense orders and their theories. It is shown that theories of dense n-spherical orders are countably categorical and decidable. The values for spectra of countable models of unary expansions of n-spherical theories are described. The Vaught conjecture is confirmed for countable constant expansions of dense n-spherical theories.

Keywords: spherical order, elementary theory, dense spherical order, countably categorical theory, spectrum of countable models, Vaught conjecture.

1. Introduction

Spherical orders [1, 2] are natural generalizations of linear and circular orders [3, 4, 5] allowing to compare tuples and their elements on n-spheres. These orders present geometric realizations of arbitrary arities of formulae along with known algebraic ones [6].

In the paper, we study some kinds of n-spherical orders including finite and dense ones, and their elementary theories. We show that theories of dense n-spherical orders are countably categorical and decidable. Based on classical Ehrenfeucht examples, we construct Ehrenfeucht expansions of dense n-spherical orders producing theories with arbitrarily finitely many $m \neq 0, 2$ countable models. The Vaught

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Kulpeshov, B.Sh., Sudoplatov, S.V., Properties of ranks for families of strongly minimal theories.

The research is supported by Committee of Science in Education and Science Ministry of the Republic of Kazakhstan, Grant No. AP08855544. The work of the second author was carried out in the framework of the State Contract of the Sobolev Institute of Mathematics, Project No. FWNF-2022-0012.

Received December, ??, 2021, published ???, ??, 2022.

conjecture is confirmed for countable constant expansions of the dense n-spherical theories.

We deals with complete first-order theories and use without specifications notions, notations and results on arities of theories [1, 2], on circular orders [3, 4, 5] and on distributions of countable models of complete theories [7].

2. Preliminaries

Recall a series of notions related to arities and aritizabilities of theories.

Definition [8]. A theory T is said to be Δ-*based*, where Δ is some set of formulae without parameters, if any formula of T is equivalent in T to a Boolean combination of formulae in Δ.

For Δ-based theories T, it is also said that T has *quantifier elimination* or *quantifier reduction* up to Δ.

Definition [7, 8]. Let Δ be a set of formulae of a theory T, and p a type of T lying in $S(T)$. The type p is said to be Δ-*based* if p is isolated by a set of formulae $\varphi^\delta \in p$, where $\varphi \in \Delta$, $\delta \in \{0, 1\}$.

The following lemma, being a corollary of Compactness Theorem, was noticed in [8].

Lemma 1. *A theory T is Δ-based if and only if, for any tuple $\bar{a}$ of any* (*some*) *weakly saturated model of T, the type* $\mathrm{tp}(\bar{a})$ *is Δ-based.*

Lemma 1 allows to reduce, in general, possibilities of Δ-basedness of a theory T to the Δ-deducibility of its types, and vice versa.

Below we consider special forms of Δ-basedness.

Definition [1]. An elementary theory T is called *unary*, or 1-*ary*, if any T-formula $\varphi(\overline{x})$ is T-equivalent to a Boolean combination of T-formulae, each of which is of one free variable, and of formulae of form $x \approx y$.

For a natural number $n \geq 1$, a formula $\varphi(\overline{x})$ of a theory T is called n-*ary*, or an n-*formula*, if $\varphi(\overline{x})$ is T-equivalent to a Boolean combination of T-formulae, each of which is of n free variables.

For a natural number $n \geq 2$, an elementary theory T is called n-*ary*, or an n-*theory*, if any T-formula $\varphi(\overline{x})$ is n-ary.

A theory T is called *binary* if T is 2-ary, it is called *ternary* if T is 3-ary, etc.

We admit the case $n = 0$ for n-formulae $\varphi(\overline{x})$, too. In such a case $\varphi(\overline{x})$ is just T-equivalent to a sentence $\forall \overline{x} \varphi(\overline{x})$.

We say that the *arity* of T is n and write $\mathrm{ar}(T) = n$ for this if T is n-ary but not $(n-1)$-ary. If T does not have any arity we put $\mathrm{ar}(T) = \infty$.

Similarly, for a formula φ of a theory T we denote by $\mathrm{ar}_T(\varphi)$ the natural value n if φ is n-ary and not $(n-1)$-ary. If a theory T is fixed we write $\mathrm{ar}(\varphi)$ instead of $\mathrm{ar}_T(\varphi)$.

By the definition, $\mathrm{ar}(\varphi) \leq |\mathrm{FV}(\varphi)|$, where $\mathrm{FV}(\varphi)$ is the set of all free variables of the formula φ.

3. n-spherical orders

Recall [3, 4, 5] that a *circular*, or *cyclic* order relation is described by a ternary relation K_3 satisfying the following conditions:

(co1) $\forall x \forall y \forall z (K_3(x, y, z) \to K_3(y, z, x))$;

(co2) $\forall x\forall y\forall z(K_3(x,y,z)\wedge K_3(y,x,z)\leftrightarrow x\approx y\vee y\approx z\vee z\approx x)$;
(co3) $\forall x\forall y\forall z(K_3(x,y,z)\to\forall t[K_3(x,y,t)\vee K_3(t,y,z)])$;
(co4) $\forall x\forall y\forall z(K_3(x,y,z)\vee K_3(y,x,z))$.

By the definition, $\mathrm{ar}(K_3(x,y,z))=3$ if the relation has at least three element domain, where the *domain* is the set of projections of K_3 into its some/any coordinate. Hence, theories whose models have infinite circular order relations are at least 3-ary.

The following generalization of circular order produces an n-*ball*, or n-*spherical*, or n-*circular* order relation [1, 2], for $n\geq 4$, which is described by an n-ary relation K_n satisfying the following conditions:
(nso1) $\forall x_1,\ldots,x_n(K_n(x_1,x_2,\ldots,x_n)\to K_n(x_2,\ldots,x_n,x_1))$;
(nso2)
$$\forall x_1,\ldots,x_n\Bigg((K_n(x_1,\ldots,x_i,\ldots,x_j,\ldots,x_n)\wedge \wedge K_n(x_1,\ldots,x_j,\ldots,x_i,\ldots,x_n))\leftrightarrow\bigvee_{1\leq k<l\leq n}x_k\approx x_l\Bigg)$$
for any $1\leq i<j\leq n$;
(nso3)
$$\forall x_1,\ldots,x_n\Bigg(K_n(x_1,\ldots,x_n)\to \to\forall t\left(\bigvee_{i=1}^{n}K_n(x_1,\ldots,x_{i-1},t,x_{i+1},\ldots,x_n)\right)\Bigg);$$
(nso4)
$$\forall x_1,\ldots,x_n(K_n(x_1,\ldots,x_i,\ldots,x_j,\ldots,x_n)\vee \vee K_n(x_1,\ldots,x_j,\ldots,x_i,\ldots,x_n)),\ 1\leq i<j\leq n.$$

Structures $\mathcal{A}=\langle A,K_n\rangle$ with n-spherical orders K_n will be called n-*spherically ordered sets*, or n-*spherical orders*, too.

Remark 1. By the axiom (nso2) the universe A of a structure $\mathcal{A}=\langle A,K_n\rangle$ with an n-spherical order K_n is uniquely defined as the set of all coordinates of tuples in K_n. Moreover, by (nso1) it is covered by a set of coordinates of tuples in K_n with a fixed number, say A is represented as the set of second coordinates of tuples in K_n.

Similarly to the circular order, the universe A is the *domain* for the n-spherical order K_n.

Remark 2. The axioms above are naturally adapted for $n=2$ producing a linear order K_2. Here (nso2) produces the reflexivity: $\forall xK_2(x,x)$, and the antisymmetry: $\forall x_1,x_2(K_2(x_1,x_2)\wedge K_2(x_2,x_1)\to x_1\approx x_2)$, (nso1) is replaced by the transitivity:
$$\forall x_1,x_2,x_3(K_2(x_1,x_2)\wedge K_2(x_2,x_3)\to K_2(x_1,x_3)),$$
and the axioms (nso3) and (nso4) produce the linearity:
$$\forall x_1,x_2(K_2(x_1,x_2)\vee K_2(x_2,x_1)).$$

The only case $n=2$, i.e. a linear order, can admit endpoints, since for the cases $n\geq 3$ each element lays between other ones.

Now we consider a series of models illustrating n-spherical orders.

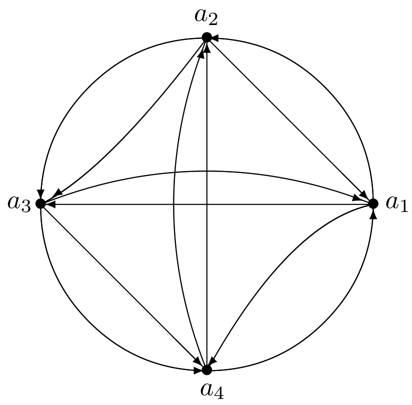

FIG. 1

Example 1. Recall that a *directed tetrahedron*, or *dirtetrahedron* [9] is a 4-tuple (A_1, A_2, A_3, A_4) of points in $\mathbb{R}^3$ which do not belong to a common plane. For the dirtetrahedron (A_1, A_2, A_3, A_4) we define the structure $\langle\{A_1, A_2, A_3, A_4\}, K_4^4\rangle$, where the 4-ary relation K_4^4 on the 4-element set $\{A_1, A_2, A_3, A_4\}$ is the closure of the set of even permutations of (A_1, A_2, A_3, A_4) with fixed A_1 by the cyclic permutation $(1, 2, 3, 4)$ (as it follows from Axiom nso1), united with the set I_4 of 4-tuples (A_i, A_j, A_k, A_l) having at least two equal points (as it follows from Axiom nso2). Thus K_4^4 is generated by three dirtetrahedrons

$$(A_1, A_2, A_3, A_4), (A_1, A_3, A_4, A_2), (A_1, A_4, A_2, A_3)$$

using cyclic permutations and I_4.

We immediately check that K_4^4 satisfies the axioms (nso1)–(nso4) for $n = 4$, producing a 4-spherical order. It consists of all 4-tuples on $\{A_1, A_2, A_3, A_4\}$ described above forbidding $\frac{4!}{2}$ possibilities for odd permutations and producing $|K_4^4| = 4^4 - \frac{4!}{2} = 4(4^3 - 3) = 244$.

Replacing points A_1, A_2, A_3, A_4 by elements a_1, a_2, a_3, a_4, respectively, we obtain an isomorphic copy for K_4^4 with generating 4-tuples represented in Figure 1.

Example 2. Similarly Example 1, taking an n-tuple $(A_1, A_2, \ldots, A_n)$ of points in $\mathbb{R}^{n-1}$, $n \geq 5$, which do not belong to a common hyperplane we obtain a *directed n-hedron.* For the directed n-hedron $(A_1, A_2, \ldots, A_n)$ we define the structure $\langle\{A_1, A_2, \ldots, A_n\}, K_n^n\rangle$, where the n-ary relation K_n^n on the n-element set $\{A_1, A_2, \ldots, A_n\}$ is the closure of the set of even permutations of $(A_1, A_2, \ldots, A_n)$ with fixed A_1 by the cyclic permutation $(1, 2, \ldots, n)$ (as it follows from Axiom nso1), united with the set I_n of n-tuples $(A_{i_1}, A_{i_2}, \ldots, A_{i_n})$ having at least two equal points (as it follows from Axiom nso2).

We check that K_n^n satisfies the axioms (nso1)–(nso4) in the general case, producing an n-spherical order. Indeed, (nso1) holds since K_n^n is closed under cyclic permutations, (nso2) is true as K_n^n forbids odd permutations of tuples with pairwise distinct coordinates and fixed first coordinate, (nso3) is satisfied since an additional element t either repeats some coordinate of forms an even permutation of $(A_1, A_2, \ldots, A_n)$ with respect to given elements $x_1, x_2, \ldots, x_n$, and (nso4) holds since for any pairwise distinct elements $x_1, x_2, \dots, x_n$ either $(x_1, x_2 \dots, x_n)$ is an even permutation of $(A_1, A_2, \dots, A_n)$ or an arbitrary transposition for $(x_1, x_2 \dots, x_n)$ is an even permutation of $(A_1, A_2, \dots, A_n)$.

By the definition the relations K_n^n are generated from the tuple $(A_1, A_2, \dots, A_n)$ by its even permutations with fixed A_1, cyclic permutations and identifications of coordinates.

Similarly to K_4^4 we obtain

$$|K_n^n| = n^n - \frac{n!}{2}. \tag{1}$$

Indeed, by the definition $|I_n| = n^n - n!$. Now we calculate the number of possible permutations of $(A_1, A_2, \dots, A_n)$. Since we fix A_1 there are $\frac{(n-1)!}{2}$ of even permutations. As we consider the closure by the permutation $(1, 2, \dots, n)$, we multiply $\frac{(n-1)!}{2}$ by n obtaining $\frac{n!}{2}$ possibilities. Thus, $|K_n^n| = n^n - n! + \frac{n!}{2} = n^n - \frac{n!}{2}$.

The formula (1) is valid for 2-element linear orders and 3-element circular orders. Thus we have $|K_2^2| = 3$, $|K_3^3| = 24$, $|K_4^4| = 244$, $|K_5^5| = 3065$, etc.

For instance, K_5^5 is generated by twelve directed pentahedrons

$$(A_1, A_2, A_3, A_4, A_5), (A_1, A_2, A_4, A_5, A_3), (A_1, A_2, A_5, A_3, A_4),$$
$$(A_1, A_3, A_2, A_5, A_4), (A_1, A_3, A_4, A_2, A_5), (A_1, A_3, A_5, A_4, A_2).$$
$$(A_1, A_4, A_2, A_3, A_5), (A_1, A_4, A_3, A_5, A_2), (A_1, A_4, A_5, A_2, A_3),$$
$$(A_1, A_5, A_2, A_4, A_3), (A_1, A_5, A_3, A_2, A_4), (A_1, A_5, A_4, A_3, A_2)$$

using cyclic permutations and I_5.

Remark 3. Constructing n-spherical orders on n-element sets we start to use even permutations since $n = 4$ because for $n = 2$ there are no nonidentical even permutations and for $n = 3$ these permutations are reduced to cyclic permutations.

We denote by $S_n := S(A_1, A_2, \dots, A_n)$ the unique $(n-1)$-dimensional sphere containing the points $A_1, A_2, \dots, A_n$ in Example 2, and by $H_n := H(A_1, A_2, \dots, A_n)$ the union of hyperplanes containing $(n-1)$-element subsets of $\{A_1, A_2, \dots, A_n\}$.

Example 3. Any relation K_n^n can be extended till an n-spherical order K_n^{n+k} by adding new points $A_{n+1}, \dots, A_{n+k} \in S_n \setminus H_n$, where the relation K_n^{n+k} is defined as follows. We choose an arbitrary n elements $A_{j_1}, \dots, A_{j_n}$ from $\{A_1, \dots, A_{n+k}\}$ where $j_1 < j_2 < \dots < j_n$ and consider all even permutations τ of $\{2, 3, \dots, n\}$. Let $\sigma = (1, 2, 3, \dots, n)$ and $\rho = \sigma^i \tau$ for some $i < n$. Then K_n^{n+k} contains $(A_{j_{\rho(1)}}, \dots, A_{j_{\rho(n)}}) \in K_n^{n+k}$ as well as all tuples with at least two equal coordinates. Thus we define the structures $\langle \{A_1, \dots, A_n, A_{n+1}, \dots, A_{n+k}\}, K_n^{n+k} \rangle$ generalizing the structures with the relations K_n^n in Example 2. Since there are $A_{n+k}^n = \frac{(n+k)!}{k!}$ n-tuples with distinct coordinates, by simple combinatorial calculation, we obtain

$$|K_n^{n+k}| = (n+k)^n - \frac{A_{n+k}^n}{2}.$$

In particular, we have $|K_2^3| = 3^2 - \frac{3!}{2} = 6$, $|K_2^4| = 4^2 - \frac{4!}{4} = 10$, $|K_2^5| = 5^2 - \frac{5!}{12} = 15$, etc.; $|K_3^4| = 4^3 - \frac{4!}{2} = 52$, $|K_3^5| = 5^3 - \frac{5!}{4} = 95$, $|K_3^6| = 6^3 - \frac{6!}{12} = 156$, etc.; $|K_4^5| = 5^4 - \frac{5!}{2} = 565$, $|K_4^6| = 6^4 - \frac{6!}{4} = 1116$, $|K_4^7| = 7^4 - \frac{7!}{12} = 1981$, etc.

Remark 4. The construction of finite n-spherical orders shows that these orders of given finite cardinality are unique up to isomorphism.

Remark 5. The process of adding $B_1, \ldots, B_k$ in Example 3 can be continued arbitrarily many (unlimit and limit) steps obtaining λ-element n-spherical orders K_n^λ on S_n, for λ with $n \leq \lambda \leq 2^\omega$, and on appropriate elementary extensions S_n^λ of S_n, for $\lambda > 2^\omega$.

In particular, taking a sphere S in $\mathbb{R}^3$ we can interpret a 4-spherical order taking the set of all 4-tuples (A_1, A_2, A_3, A_4), where A_4 is situated inside the spherical triangle $A_1A_2A_3$ [10]. It can be naturally spread for group trigonometries based on ordered groups [11].

All these orders on spheres are called *spherical models*.

The definition of K_n^λ implies that this relation cannot be reduced to Boolean combinations of relations with less arities, then the arity of the formula $K_n^\lambda(x_1, \ldots, x_n)$ equals n: $\mathrm{ar}(K_n^\lambda(x_1, \ldots, x_n)) = n$.

The formula witnessing the identification of coordinates for the formula $K_n(x_1, \ldots, x_n)$ of n-spherical order will be denoted by $I_n(x_1, \ldots, x_n)$. Clearly,

$$\mathrm{ar}(I_n(x_1, \ldots, x_n)) = 1,$$

since $I_n(x_1, \ldots, x_n)$ is equivalent to the formula

$$\bigvee_{1 \leq i < j \leq n} \left(x_i \approx x_j \wedge \bigwedge_{k \notin \{i,j\}} x_k \approx x_k \right).$$

Thus, $\mathrm{ar}(K_n(x_1, \ldots, x_n) \wedge \neg I_n(x_1, \ldots, x_n)) = n$.

4. Countably categorical n-spherical orders

Definition. An n-spherically ordered set $\langle A, K_n \rangle$, where $n \geq 2$, is called *dense* if it contains at least two elements and for each $(a_1, a_2, a_3, \ldots, a_n) \in K_n$ with $a_1 \neq a_2$ there is $b \in A \setminus \{a_1, a_2, \ldots, a_n\}$ such that

$$\models K_n(a_1, b, a_3, \ldots, a_n) \wedge K_n(b, a_2, a_3, \ldots, a_n).$$

Clearly, dense n-spherical orders K_n witness the strict order property producing unstable structures $\langle A, K_n \rangle$, since fixing $n - 2$ distinct coordinates in the relation K_n we obtain a dense linear order.

The following theorem generalizes the known result on countable categoricity of dense linear orders [12, Proposition 3.1.7].

Theorem 1. *If $\mathcal{A}$ and $\mathcal{B}$ are countable dense n-spherical orders, $n \geq 2$, without endpoints for $n = 2$, then $\mathcal{A} \simeq \mathcal{B}$.*

Proof. The case $n = 2$ is considered in [12, Proposition 3.1.7]. Therefore we assume that $n \geq 3$, and for this case we slightly modify the arguments for that proposition.

Let $A = \{a_m \mid m \in \omega\}$, $B = \{b_m \mid m \in \omega\}$. We consider the set G consisting of maps g: $A_1 \to B_1$ satisfying the following conditions:

1) A_1 and B_1 are finite subsets of A and B, respectively;

2) g: $\mathcal{A}(A_1) \xrightarrow{\sim} \mathcal{B}(B_1)$ if $A_1 \neq \emptyset$, i.e., g is an isomorphism between a substructure $\mathcal{A}(A_1)$ of $\mathcal{A}$ with the universe A_1 and a substructure $\mathcal{B}(B_1)$ of $\mathcal{B}$ with the universe B_1;

3) if $|A_1| = 2m > 0$ then $\{a_0, \ldots, a_{m-1}\} \subseteq A_1$ and $\{b_0, \ldots, b_{m-1}\} \subseteq B_1$;

4) if $|A_1| = 2m+1$ then $\{a_0, \ldots, a_m\} \subseteq A_1$ and for $m > 0$, $\{b_0, \ldots, b_{m-1}\} \subseteq B_1$.

Since $\emptyset \in G$ then $G \neq \emptyset$. Let $g: A_1 \to B_1$ belong to G and $|A_1| = 2n$. In view of 3) there is $a \in A \setminus A_1$ with $\{a_0, \ldots, a_m\} \subseteq A_1 \cup \{a\}$. Now we find an element $b \in B \setminus B_1$ such that $K_n(b, g(c_2), \ldots, g(c_n)) \Leftrightarrow K_n(a, c_2, \ldots, c_n)$ for all $c_2, \ldots, c_n \in A_1$. The element b exists since $\mathcal{B}$ is dense and the condition 2) for g holds. Clearly, $g \cup \{\langle a, b\rangle\} \in G$. If $|A_1| = 2n+1$ then we replace $\mathcal{A}$ and $\mathcal{B}$ each other and find a pair $\langle a, b\rangle \notin g$ with $g \cup \{\langle a, b\rangle\} \in G$. Thus the partial order $\langle G, \subseteq\rangle$ does not have maximal elements. Hence G contains an infinite chain $X \subseteq G$. In view of 2)–4) the union $\bigcup X$ is an isomorphism of $\mathcal{A}$ onto $\mathcal{B}$.

Theorem 1 immediately implies:

Corollary 1. *For any dense n-spherical order $\mathcal{A}$ its theory* $\mathrm{Th}(\mathcal{A})$ *is countably categorical.*

Remark 6. Extending dense n-spherical orders by finitely many new discrete elements preserves countable categoricity. It means that there are finitely many possibilities for $(a_1, a_2, a_3, \ldots, a_n) \in K_n$ such that $a_1 \neq a_2$ and there are finitely many $b \notin \{a_1, a_2\}$ with

$$\models K_n(a_1, b, a_3, \ldots, a_n) \wedge K_n(b, a_2, a_3, \ldots, a_n).$$

Similarly linear orders having infinite discrete parts loose the countable categoricity obtaining a definable successor function.

Similarly the theories of dense linear orders, in view of Lemma 1, the theories T_n of dense n-spherical orders $\langle A, K_n\rangle$, $n \geq 2$, admit the quantifier elimination since complete types are forced by collections of quantifier free formulae. Besides the theories T_n are finitely axiomatizable. Using Corollary 1 and the arguments for [12, Proposition 8.3.1] we obtain the following its generalization:

Theorem 2. *For any natural $n \geq 2$ the theory T_n of a dense n-spherical order is decidable.*

5. Ehrenfeucht theories based on dense n-spherical orders

In this section we modify classical Ehrenfeucht examples on linear dense orders [7, 13] to Ehrenfeucht theories based on dense n-spherical orders and producing arbitrarily finitely many countable models.

For a theory T, we denote by $I(T, \lambda)$ the number of pairwise non-isomorphic models of T in a power λ. The value $I(T, \lambda)$ is called the *λ-spectrum* of T. If $\lambda = \omega$ then the λ-spectrum of T is called the *countable spectrum* of T.

Definition [14]. A theory T is *Ehrenfeucht* if $1 < I(T, \omega) < \omega$, i.e., the countable spectrum of T is finite and greater than 1.

Let $\mathcal{A}_n = \langle A_n, K_n\rangle$ be a countable dense n-spherical order, $n \geq 2$. Let T_n^m be the theory of a structure $\mathcal{M}_n^m$, formed from the structure $\mathcal{A}_n$ by adding pairwise distinct constants c_k, $k \in \omega$, such that $\models K_n(c_{i_1}, \ldots, c_{i_n})$ for any $i_1 < \ldots < i_n$, and unary predicates $P_0, \ldots, P_{m-3}$ which form a partition of the set A_n, with

$$\models \forall x_1, \ldots, x_n \left(\bigwedge_{1 \leq j < k \leq n} \neg(x_j \approx x_k) \wedge K_n(x_1, \ldots, x_n) \to \right.$$

$$\left. \to \exists t \left(\bigwedge_{j=1}^{n} \neg(t \approx x_j) \wedge K_n(x_1, t, x_3, \ldots, x_n) \wedge K_n(t, x_2, x_3, \ldots, x_n) \wedge P_i(t) \right) \right),$$

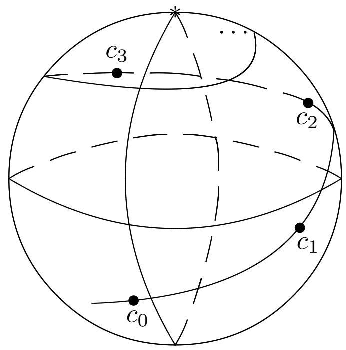

FIG. 2

$i = 0, \ldots, m-3$.

We denote by $p_\infty(x)$ the type which is forced by the set of formulae

$$K_n(c_{i_1}, c_{i_2}, \ldots, c_{i_{n-1}}, x),$$

$i_1 < i_2 < \ldots < i_{n-1}$. The type $p_\infty(x)$ has $m-2$ completions $p^i_\infty(x)$ which are forced by $p_\infty(x) \cup \{P_i(x)\}$, $i = 0, \ldots, m-3$.

The theory T^m_n has exactly m pairwise non-isomorphic countable models:

(a) a prime model $\mathcal{M}_m$ which omit the type $p_\infty(x)$; $(\lim\limits_{k\to\infty} c_k = \infty)$;

(b) prime models $\mathcal{M}^i_m$ over realizations of types $p^i_\infty(x)$, $i = 0, \ldots, m-3$, with a limit element for the constants c_k, realizing $p^i_\infty(x)$;

(c) a saturated model $\overline{\mathcal{M}}_m$, without limit elements for the constants c_k, realizing $p^i_\infty(x)$.

Replacing constants c_k by unary predicates $U_k = \{c_k\}$ we obtain theories $\widehat{T}^m_n$ instead of T^m_n such that $\widehat{T}^m_n$ are unary expansions of $T_n = \mathrm{Th}(\mathcal{A}_n)$, i.e. expansions of T_n by unary predicates. It is easy to see that this transformation preserves the number of pairwise non-isomorphic countable models:

$$I(\widehat{T}^m_n, \omega) = m = I(T^m_n, \omega).$$

Corollary 1 and the arguments above producing values for $I(\widehat{T}^m_n, \omega)$ imply the following:

Theorem 3. *For any $m \in \omega \setminus \{0, 2\}$, $n \in \omega \setminus \{0, 1\}$ there is a unary expansion $\widehat{T}^m_n$ of the theory T_n of dense n-spherical order $\mathcal{A}_n$ such that $I(\widehat{T}^m_n, \omega) = m$.*

The theories T^m_2 in Theorem 3 are classical Ehrenfeucht examples, and the theories T^m_3 are their adaptations for dense circular orders.

Figure 2 illustrates a model of the theory T^3_4. Constants c_k, $k \in \omega$, are situated on a spiral with a top limit vertex $*$.

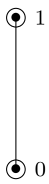

FIG. 3

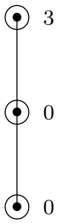

FIG. 4

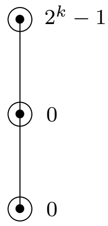

FIG. 5

6. Constant expansions of dense n-spherical orders and their countable spectra

Now we consider possibilities for distributions of countable sequences of constants expanding n-spherical theories $T_n = \mathrm{Th}(\mathcal{A}_n)$, where $\mathcal{A}_n = \langle A_n, K_n \rangle$ are dense n-spherical orders, $n \geq 2$. These distributions produce distributions of countable models of these expansions $T \supseteq T_n$ and possibilities for values $I(T, \omega)$.

The distributions for $n = 2$ are described in [7, 15].

Recall [16] that a linearly ordered structure $\mathcal{M}$ is *o-minimal* if any definable (with parameters) subset of M is a finite union of singletons and open intervals (a, b), where $a \in M \cup \{-\infty\}$, $b \in M \cup \{+\infty\}$. A theory T is *o-minimal* if each model of T is o-minimal.

As examples of Ehrenfeucht o-minimal theories, we mention the theories $T^1 \rightleftharpoons \mathrm{Th}((\mathbb{Q}; <, c_n)_{n \in \omega})$ and $T^2 \rightleftharpoons \mathrm{Th}((\mathbb{Q}; <, c_n, c'_n)_{n \in \omega})$, where $<$ is an ordinary strict order on the set $\mathbb{Q}$ of rationals, the constants c_n form a strictly increasing sequence, and the constants c'_n form a strictly decreasing sequence, $c_n < c'_n$, $n \in \omega$.

The theory T^1 is an Ehrenfeucht's example [13] with $I(T^1, \omega) = 3$. It has two almost prime models and one limit model:

• a prime model with the empty set of realizations of the type $p(x)$ isolated by the set $\{c_n < x \mid n \in \omega\}$ of formulae;

• a prime model over a realization of the type $p(x)$, with the least realization of that type;

• one limit model over the type $p(x)$, with the set of realizations of $p(x)$ forming an open convex set.

The Hasse diagram for the Rudin–Keisler preorder $\leq_{\mathrm{RK}}$ and values of the function IL of distributions of numbers of limit models for $\sim_{\mathrm{RK}}$-classes of T^1 is represented in Figure 3. This Hasse diagram equals the Hasse diagram for the theory T^3_4.[1]

The theory T^2 has the six pairwise non-isomorphic countable models:

• a prime model with the empty set of realizations of the type $p(x)$ isolated by the set $\{c_n < x \mid n \in \omega\} \cup \{x < c'_n \mid n \in \omega\}$;

• a prime model over a realization of $p(x)$, with a unique realization of this type;

• a prime model over a realization of the type $q(x, y)$ isolated by the set $p(x) \cup p(y) \cup \{x < y\}$; here the set of realizations of $p(x)$ forms a closed interval $[a, b]$;

[1] Hasse diagrams for distributions of countable models are defined and studied for the general case [7], for quite o-minimal Ehrenfeucht theories [17], and for disjoint unions of Ehrenfeucht theories [18].

• three limit models over the type $q(x,y)$, in which the sets of realizations of $q(x,y)$ are convex sets of forms $(a,b]$, $[a,b)$, (a,b) respectively.

In Figure 4 we represent the Hasse diagram of Rudin–Keisler preorders $\leq_{\rm RK}$ and values of distribution functions IL of numbers of limit models on $\sim_{\rm RK}$-equivalence classes for the theory T^2.

The following theorem shows that the number of countable models of Ehrenfeucht o-minimal theories is exhausted by combinations of these numbers for the theories T^1 and T^2.

Theorem 4. [15] *Let T be an o-minimal theory in a countable language. Then either T has 2^ω countable models or T has exactly $3^r\cdot 6^s$ countable models, where r and s are natural numbers. Moreover, for any $r,s\in\omega$ there is an o-minimal theory T with exactly $3^r\cdot 6^s$ countable models.*

Theorem 4 immediately implies:

Corollary 2. *Let T be a countable constant expansion of the 2-spherical theory T_2. Then either T has 2^ω countable models or T has exactly $3^r\cdot 6^s$ countable models, where r and s are natural numbers. Moreover, for any $r,s\in\omega$ there is an aforesaid theory T with exactly $3^r\cdot 6^s$ countable models.*

Since any 3-spherical structure is obtained by placing a linear order on a circle, the arguments for the 2-spherical theory T_2 are valid for the 3-spherical theory T_3 obtaining the following:

Corollary 3. *Let T be a countable constant expansion of the 3-spherical theory T_3. Then either T has 2^ω countable models or T has exactly $3^r\cdot 6^s$ countable models, where r and s are natural numbers. Moreover, for any $r,s\in\omega$ there is an aforesaid theory T with exactly $3^r\cdot 6^s$ countable models.*

Considering the dense n-spherical Ehrenfeucht theories T_n, $n\geq 4$, we obtain both the possibilities $3^k\cdot 6^s$ of countable models, with appropriate sequences of constants, and the following new possibilities.

Considering a consistent non-isolated set $p(x)$ of formulae

$$K^\delta_n(c_{j_1},\dots,c_{j_{i-1}},x,c_{j_{i+1}},\dots,c_{j_n})\wedge\bigwedge_{k\neq i}\neg x\approx c_{j_k},$$

$\delta=\delta_{j_1,\dots,j_{i-1},x,j_{i+1},\dots,j_n}\in\{0,1\}$, with pairwise distinct $c_{j_1},\dots,c_{j_{i-1}},c_{j_{i+1}},\dots,c_{j_n}$, we observe that each such an additional formula divides the domain of previous ones into two parts with respect to a $(n-1)$-dimensional plane containing points $c_{j_1},\dots,c_{j_{i-1}},c_{j_{i+1}},\dots,c_{j_n}$. In view of quantifier elimination for T_n any consistent set $p(x)$ forces a complete type. Thus a limit part of the domain being a set of solutions for $p(x)$ is defined by the limits for $c_{j_1},\dots,c_{j_{i-1}},c_{j_{i+1}},\dots,c_{j_n}$.

We have the following possibilities for these limits:

• $p(x)$ is omitted;

• $p(x)$ has a unique realization a_∞ being a common limit of the sequences;

• $p(x)$ has infinitely many realizations independently including / not including $n-1$ limits with respect to coordinates of $(c_{j_1},\dots,c_{j_{i-1}},c_{j_{i+1}},\dots,c_{j_n})$.

Hence there are $2^{n-1}+2$ possibilities for countable models, where 3 models are prime over finite sets (prime over $\emptyset$, prime over $\{a_\infty\}$, and prime over limits for $c_{j_1},\dots,c_{j_{i-1}},c_{j_{i+1}},\dots,c_{j_n}$) and $2^{n-1}-1$ limit models.

If some constants in $c_{j_1}, \ldots, c_{j_{i-1}}, c_{j_{i+1}}, \ldots, c_{j_n}$ are fixed, with $k > 1$ independent moving sequences the total number of countable models related to $p(x)$ equals 2^k+2 including 3 almost prime models and $2^k - 1$ limit models, see Figure 5.

Since distinct types $p(x)$ are independent the total number of possibilities for countable models is obtained by the multiplications of values $2^k + 2$ for various $k \in n \setminus \{1\}$, if there are finitely many non-isolated 1-types, and there are 2^ω countable models otherwise. Thus we have the following:

Theorem 5. *Let T be a countable constant expansion of the n-spherical theory T_n, $n \geq 3$. Then either T has 2^ω countable models or T has exactly $\prod\limits_{k\in n\setminus\{1\}} (2^k + 2)^{r_k}$ countable models, where r_k are natural numbers. Moreover, for any $r_0, \ldots, r_{n-1} \in \omega$ there is an aforesaid theory T with exactly $\prod\limits_{k\in n\setminus\{1\}} (2^k + 2)^{r_k}$ countable models.*

Theorem 5 confirms the Vaught conjecture for countable constant expansions T of n-spherical theories T_n. In particular,

$$\text{either } I(T, \omega) = 2^\omega \text{ or } I(T, \omega) = 3^{r_1} \cdot 6^{r_2} \cdot 10^{r_3}$$

for $T \supseteq T_4$,

$$\text{either } I(T, \omega) = 2^\omega \text{ or } I(T, \omega) = 3^{r_1} \cdot 6^{r_2} \cdot 10^{r_3} \cdot 18^{r_4}$$

for $T \supseteq T_5$, etc.

Similarly to the quite o-minimal Ehrenfeucht theories distributions of countable models of constant expansions of n-spherical theories are given by Hasse diagrams for disjoint unions of theories [17, 18], based on the diagrams represented in Figures 3–5.

7. Conclusion

We studied spherical orders, which generalize known linear and circular orders. Semantic and syntactic properties of spherical orders and their elementary theories, including finite and dense orders and their theories are investigated. It is shown that theories of dense n-spherical orders are countably categorical and decidable. The values for spectra of countable models of unary expansions of n-spherical theories are described. The Vaught conjecture is confirmed for countable constant expansions of dense n-spherical theories. It would be natural to study various modifications and kinds of spherical orders and their theories.

Acknowledgement. The authors thank the anonymous referee for the constructive critics and suggestions contributing to the improvement of presentation.

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