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SPHERICAL ORDERS, PROPERTIES AND COUNTABLE SPECTRA OF THEIR THEORIES

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ABSTRACT. We study semantic and syntactic properties of spherical orders and their elementary theories, including finite and dense orders and their theories. It is shown that theories of dense n -spherical orders are countably categorical and decidable. The values for spectra of countable models of unary expansions of n -spherical theories are described. The Vaught conjecture is confirmed for countable constant expansions of dense n -spherical theories.

Keywords: spherical order, elementary theory, dense spherical order, countably categorical theory, spectrum of countable models, Vaught conjecture.

1. INTRODUCTION

Spherical orders [1, 2] are natural generalizations of linear and circular orders [3, 4, 5] allowing to compare tuples and their elements on n -spheres. These orders present geometric realizations of arbitrary arities of formulae along with known algebraic ones [6].

In the present paper we study some kinds of n -spherical orders including finite and dense ones, and their elementary theories. We show that theories of dense n -spherical orders are countably categorical and decidable. Basing on classical Ehrenfeucht examples, Ehrenfeucht expansions of dense n -spherical orders are constructed producing theories with arbitrarily finitely many countable models. The

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Vaught conjecture is confirmed for countable constant expansions of dense n -spherical theories.

We use without specifications notions, notations and results on arities of theories [1, 2], on circular orders [3, 4, 5] and on distributions of countable models of complete theories [7].

2. PRELIMINARIES

Recall a series of notions related to arities and aritizabilities of theories.

Definition [8]. A theory T is said to be Δ -based, where Δ is some set of formulae without parameters, if any formula of T is equivalent in T to a Boolean combination of formulae in Δ .

For Δ -based theories T , it is also said that T has *quantifier elimination* or *quantifier reduction* up to Δ .

Definition [7, 8]. Let Δ be a set of formulae of a theory T , and $p(\bar{x})$ a type of T lying in $S(T)$. The type $p(\bar{x})$ is said to be Δ -based if $p(\bar{x})$ is isolated by a set of formulas $\varphi^\delta \in p$, where $\varphi \in \Delta$, $\delta \in \{0, 1\}$.

The following lemma, being a corollary of Compactness Theorem, noticed in [8].

Lemma 1. *A theory T is Δ -based if and only if, for any tuple \bar{a} of any (some) weakly saturated model of T , the type $\text{tp}(\bar{a})$ is Δ -based.*

Lemma 1 allows to reduce, in general, possibilities of Δ -basedness of a theory T to the Δ -deducibility of its types, and vice versa.

Below we consider special forms of Δ -basedness.

Definition [1]. An elementary theory T is called *unary*, or *1-ary*, if any T -formula $\varphi(\bar{x})$ is T -equivalent to a Boolean combination of T -formulas, each of which is of one free variable, and of formulas of form $x \approx y$.

For a natural number $n \geq 1$, a formula $\varphi(\bar{x})$ of a theory T is called *n -ary*, or an *n -formula*, if $\varphi(\bar{x})$ is T -equivalent to a Boolean combination of T -formulas, each of which is of n free variables.

For a natural number $n \geq 2$, an elementary theory T is called *n -ary*, or an *n -theory*, if any T -formula $\varphi(\bar{x})$ is n -ary.

A theory T is called *binary* if T is 2-ary, it is called *ternary* if T is 3-ary, etc.

We will admit the case $n = 0$ for n -formulae $\varphi(\bar{x})$. In such a case $\varphi(\bar{x})$ is just T -equivalent to a sentence $\forall \bar{x} \varphi(\bar{x})$.

If T is a theory such that T is n -ary and not $(n - 1)$ -ary then the value n is called the *arity* of T and it is denoted by $\text{ar}(T)$. If T does not have any arity we put $\text{ar}(T) = \infty$.

Similarly, for a formula φ of a theory T we denote by $\text{ar}_T(\varphi)$ the natural value n if φ is n -ary and not $(n - 1)$ -ary. If a theory T is fixed we write $\text{ar}(\varphi)$ instead of $\text{ar}_T(\varphi)$.

Clearly, $\text{ar}(\varphi) \leq |\text{FV}(\varphi)|$, where $\text{FV}(\varphi)$ is the set of free variables of the formula φ .

3. n -SPHERICAL ORDERS

Recall [3, 4, 5] that a *circular*, or *cyclic* order relation is described by a ternary relation K_3 satisfying the following conditions:

(co1) $\forall x \forall y \forall z (K_3(x, y, z) \rightarrow K_3(y, z, x))$;

(co2) $\forall x \forall y \forall z (K_3(x, y, z) \wedge K_3(y, x, z) \leftrightarrow x \approx y \vee y \approx z \vee z \approx x)$;

- (co3) $\forall x \forall y \forall z (K_3(x, y, z) \rightarrow \forall t [K_3(x, y, t) \vee K_3(t, y, z)]);$
 (co4) $\forall x \forall y \forall z (K_3(x, y, z) \vee K_3(y, x, z)).$

Clearly, $\text{ar}(K_3(x, y, z)) = 3$ if the relation has at least three element domain. Hence, theories with infinite circular order relations are at least 3-ary.

The following generalization of circular order produces a n -ball, or n -spherical, or n -circular order relation [1, 2], for $n \geq 4$, which is described by a n -ary relation K_n satisfying the following conditions:

- (nso1) $\forall x_1, \dots, x_n (K_n(x_1, x_2, \dots, x_n) \rightarrow K_n(x_2, \dots, x_n, x_1));$

- (nso2) $\forall x_1, \dots, x_n \left((K_n(x_1, \dots, x_i, \dots, x_j, \dots, x_n) \wedge \right.$
 $\left. \wedge K_n(x_1, \dots, x_j, \dots, x_i, \dots, x_n)) \leftrightarrow \bigvee_{1 \leq k < l \leq n} x_k \approx x_l \right)$

for any $1 \leq i < j \leq n$;

- (nso3) $\forall x_1, \dots, x_n \left(K_n(x_1, \dots, x_n) \rightarrow \right.$
 $\left. \rightarrow \forall t \left(\bigvee_{i=1}^n K_n(x_1, \dots, x_{i-1}, t, x_{i+1}, \dots, x_n) \right) \right);$

- (nso4) $\forall x_1, \dots, x_n (K_n(x_1, \dots, x_i, \dots, x_j, \dots, x_n) \vee$
 $\vee K_n(x_1, \dots, x_j, \dots, x_i, \dots, x_n)), 1 \leq i < j \leq n.$

Structures $\mathcal{A} = \langle A, K_n \rangle$ with n -spherical orders will be called n -spherical orders, too.

Remark 1. The axioms above are valid for $n = 2$ producing a linear order K_2 . The only case $n = 2$ can admit endpoints.

Now we consider a series of models illustrating n -spherical orders.

Example 1. Recall that a *directed tetrahedron*, or *dirtetrahedron* [9] is a 4-tuple (A_1, A_2, A_3, A_4) of points in \mathbb{R}^3 which do not belong to a common plane. For the dirtetrahedron (A_1, A_2, A_3, A_4) we denote by K_4^4 the closure of the set of even permutations of (A_1, A_2, A_3, A_4) by cyclic permutations united with the set I_4 of 4-tuples (A_i, A_j, A_k, A_l) having at least two equal points. Thus K_4^4 is generated by three dirtetrahedrons

$$(A_1, A_2, A_3, A_4), (A_1, A_3, A_4, A_2), (A_1, A_4, A_2, A_3)$$

using cyclic permutations and I_4 .

Clearly, K_4^4 satisfies the axioms (nso1)–(nso4) for $n = 4$, producing a 4-spherical order. It contains all 4-tuple which do not produce odd permutations of (A_1, A_2, A_3, A_4) , i.e., $|K_4^4| = 4^4 - \frac{4!}{2} = 4(4^3 - 3) = 244$.

Replacing points A_1, A_2, A_3, A_4 by elements a_1, a_2, a_3, a_4 , respectively, we obtain an isomorphic copy for K_4^4 with generating 4-tuples represented in Figure 1.

Example 2. Similarly Example 1, taking a n -tuple (A_1, A_2, \dots, A_n) of points in \mathbb{R}^{n-1} , $n \geq 5$, which do not belong to a common hyperplane we obtain a *directed n -hedron*. We denote by K_n^n the closure of the set of even permutations of (A_1, A_2, \dots, A_n) by cyclic permutations united with the set I_n of n -tuples $(A_{i_1}, A_{i_2}, \dots, A_{i_n})$ having at least two equal points.

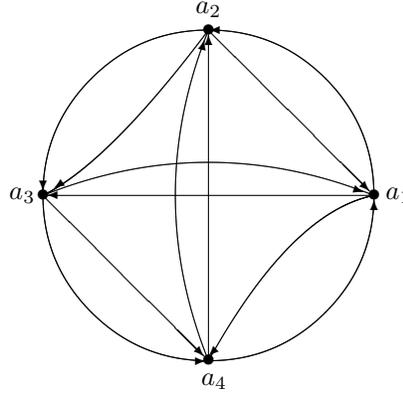


FIG. 1

Clearly, K_n^n satisfies the axioms (ns01)–(ns04) in the general case, producing a n -spherical order.

By the definition the relations K_n^n are generated from the tuple (A_1, A_2, \dots, A_n) by its even permutations, cyclic permutations and identifications of coordinates.

Similarly to K_4^4 we obtain $|K_n^n| = n^n - \frac{n!}{2}$. This formula is valid for 2-element linear orders and 3-element circular orders. Thus we have $|K_2^2| = 3$, $|K_3^3| = 24$, $|K_4^4| = 244$, $|K_5^5| = 3065$, etc.

For instance, K_5^5 is generated by twelve directed pentahedrons

$$\begin{aligned} &(A_1, A_2, A_3, A_4, A_5), (A_1, A_2, A_4, A_5, A_3), (A_1, A_2, A_5, A_3, A_4), \\ &(A_1, A_3, A_2, A_5, A_4), (A_1, A_3, A_4, A_2, A_5), (A_1, A_3, A_5, A_4, A_2), \\ &(A_1, A_4, A_2, A_3, A_5), (A_1, A_4, A_3, A_5, A_2), (A_1, A_4, A_5, A_2, A_3), \\ &(A_1, A_5, A_2, A_4, A_3), (A_1, A_5, A_3, A_2, A_4), (A_1, A_5, A_4, A_3, A_2) \end{aligned}$$

using cyclic permutations and I_5 .

Remark 2. Constructing n -spherical orders on n -element sets we start to use even permutations since $n = 4$ because for $n = 2$ there are no nonidentical even permutations and for $n = 3$ these permutations are reduced to cyclic permutations.

We denote by $S_n := S(A_1, A_2, \dots, A_n)$ the unique $(n - 1)$ -dimensional sphere containing the points A_1, A_2, \dots, A_n in Example 2, and by $H_n := H(A_1, A_2, \dots, A_n)$ the union of hyperplanes containing $(n - 1)$ -element subsets of $\{A_1, A_2, \dots, A_n\}$.

Example 3. Any relation K_n^n can be extended till a n -spherical order K_n^{n+k} by new points $B_1, \dots, B_k \in S_n \setminus H_n$ and generating n -tuples $(A_{i_1}, \dots, A_{i_r}, B_{j_1}, \dots, B_{j_s})$ with $i_1 < \dots < i_r$, $j_1 < \dots < j_s$ with respect to even permutations, cyclic permutations and identifications of coordinates.

The structures with the relations K_n^{n+k} generalize the structures with the relations K_n^n in Example 2. Since K_n^{n+k} admits even permutations only and there are A_{n+k}^n n -tuples with distinct coordinates we obtain

$$|K_n^{n+k}| = (n + k)^n - \frac{A_{n+k}^n}{2}.$$

In particular, we have $|K_2^3| = 3^2 - \frac{3!}{2} = 6$, $|K_2^4| = 4^2 - \frac{4!}{4} = 10$, $|K_2^5| = 5^2 - \frac{5!}{12} = 15$, etc.; $|K_3^4| = 4^3 - \frac{4!}{2} = 52$, $|K_3^5| = 5^3 - \frac{5!}{4} = 95$, $|K_3^6| = 6^3 - \frac{6!}{12} = 156$, etc.; $|K_4^5| = 5^4 - \frac{5!}{2} = 565$, $|K_4^6| = 6^4 - \frac{6!}{4} = 1116$, $|K_4^7| = 7^4 - \frac{7!}{12} = 1981$, etc.

Remark 3. The construction of finite n -spherical orders shows that these orders of given finite cardinality are unique up to isomorphism.

Remark 4. The process of adding B_1, \dots, B_k in Example 3 can be continued arbitrarily many (unlimit and limit) steps obtaining λ -element n -spherical orders K_n^λ on S_n , for λ with $n \leq \lambda \leq 2^\omega$, and on appropriate elementary extensions S_n^λ of S_n , for $\lambda > 2^\omega$. All these orders on spheres are called *spherical models*.

The definition of K_n^λ implies that this relation does not reduced to Boolean combinations of relations with less arities, then the arity of the formula $K_n^\lambda(x_1, \dots, x_n)$ equals n : $\text{ar}(K_n^\lambda(x_1, \dots, x_n)) = n$.

The formula witnessing the identification of coordinates for the formula $K_n(x_1, \dots, x_n)$ of n -spherical order will be denoted by $I_n(x_1, \dots, x_n)$. Clearly,

$$\text{ar}(I_n(x_1, \dots, x_n)) = 1,$$

since $I_n(x_1, \dots, x_n)$ is equivalent to the formula

$$\bigvee_{1 \leq i < j \leq n} \left(x_i \approx x_j \wedge \bigwedge_{k \notin \{i, j\}} x_k \approx x_k \right).$$

Thus, $\text{ar}(K_n(x_1, \dots, x_n) \wedge \neg I_n(x_1, \dots, x_n)) = n$.

4. COUNTABLY CATEGORICAL n -SPHERICAL ORDERS

Definition. A n -spherical order K_n , $n \geq 2$, is called *dense* if it contains at least two elements and for each $(a_1, a_2, a_3, \dots, a_n) \in K_n$ with $a_1 \neq a_2$ there is $b \notin \{a_1, a_2, \dots, a_n\}$ with

$$\models K_n(a_1, b, a_3, \dots, a_n) \wedge K_n(b, a_2, a_3, \dots, a_n).$$

Clearly, dense n -spherical orders K_n witness the strict order property producing unstable structures $\langle A, K_n \rangle$.

The following theorem generalizes the known result on countable categoricity of dense linear orders [10, Proposition 3.1.7].

Theorem 1. *If \mathcal{A} and \mathcal{B} are countable dense n -spherical orders, $n \geq 2$, without endpoints for $n = 2$, then $\mathcal{A} \simeq \mathcal{B}$.*

Proof. The case $n = 2$ is considered in [10, Proposition 3.1.7]. Therefore we assume that $n \geq 3$, and for this case we slightly modify the arguments for that proposition.

Let $A = \{a_m \mid m \in \omega\}$, $B = \{b_m \mid m \in \omega\}$. We consider the set G consisting of maps $g: A_1 \rightarrow B_1$ satisfying the following conditions:

- 1) A_1 and B_1 are finite subsets of A and B , respectively;
- 2) $g: \mathcal{A}(A_1) \simeq \mathcal{B}(B_1)$ if $A_1 \neq \emptyset$;
- 3) if $|A_1| = 2m > 0$ then $\{a_0, \dots, a_{m-1}\} \subseteq A_1$ and $\{b_0, \dots, b_{m-1}\} \subseteq B_1$;
- 4) if $|A_1| = 2m + 1$ then $\{a_0, \dots, a_m\} \subseteq A_1$ and for $m > 0$, $\{b_0, \dots, b_{m-1}\} \subseteq B_1$.

Since $\emptyset \in G$ then $G \neq \emptyset$. Let $g: A_1 \rightarrow B_1$ belong to G and $|A_1| = 2n$. In view of 3) there is $a \in A \setminus A_1$ with $\{a_0, \dots, a_m\} \subseteq A_1 \cup \{a\}$. Now we find an element $b \in B \setminus B_1$ such that $K_n(b, g(c_2), \dots, g(c_n)) \Leftrightarrow K_n(a, c_2, \dots, c_n)$ for all

$c_2, \dots, c_n \in A_1$. The element b exists since \mathcal{B} is dense and the condition 2) for g holds. Clearly, $g \cup \{\langle a, b \rangle\} \in G$. If $|A_1| = 2n + 1$ then we replace \mathcal{A} and \mathcal{B} each other and find a pair $\langle a, b \rangle \notin g$ with $g \cup \{\langle a, b \rangle\} \in G$. Thus the partial order $\langle G, \subseteq \rangle$ does not have maximal elements. Hence G contains an infinite chain $X \subseteq G$. In view of 2)–4) the union $\bigcup X$ is an isomorphism of \mathcal{A} onto \mathcal{B} .

Theorem 1 immediately implies:

Corollary 1. *For any dense n -spherical order \mathcal{A} its theory $\text{Th}(\mathcal{A})$ is countably categorical.*

Remark 5. Extensions of dense n -spherical orders by finitely many new discrete elements preserve the countable categoricity. It means that there are finitely many possibilities for $(a_1, a_2, a_3, \dots, a_n) \in K_n$ such that $a_1 \neq a_2$ and there are finitely many $b \notin \{a_1, a_2\}$ with

$$\models K_n(a_1, b, a_3, \dots, a_n) \wedge K_n(b, a_2, a_3, \dots, a_n).$$

Similarly linear orders having infinite discrete parts lose the countable categoricity obtaining a definable successor function.

Similarly theories of dense linear orders, in view of Lemma 1 theories T_n of dense n -spherical orders $\langle A, K_n \rangle$, $n \geq 2$, admit the quantifier elimination since complete types are forced by collections of quantifier free formulas. Besides the theories T_n are finitely axiomatizable. Using Corollary 1 and the arguments for [10, Proposition 8.3.1] we obtain the following its generalization:

Theorem 2. *For any natural $n \geq 2$ the theory T_n of dense n -spherical order is decidable.*

5. EHRENFUCHT THEORIES BASED ON DENSE n -SPHERICAL ORDERS

In this section we modify classical Ehrenfeucht examples on linear dense orders [7, 11] to Ehrenfeucht theories based on dense n -spherical orders and producing arbitrarily finitely many countable models.

For a theory T , we denote by $I(T, \lambda)$ the number of pairwise non-isomorphic models of T in a power λ . The value $I(T, \lambda)$ is called the λ -spectrum of T . If $\lambda = \omega$ then the λ -spectrum of T is called the *countable spectrum* of T .

Definition [12]. A theory T is *Ehrenfeucht* if $1 < I(T, \omega) < \omega$, i.e., the countable spectrum of T is finite and greater than 1.

Let $\mathcal{A}_n = \langle A_n, K_n \rangle$ be a countable dense n -spherical order, $n \geq 2$. Let T_n^m be the theory of a structure \mathcal{M}_n^m , formed from the structure \mathcal{A}_n by adding pairwise distinct constants c_k , $k \in \omega$, such that $\models K_n(c_{i_1}, \dots, c_{i_n})$ for any $i_1 < \dots < i_n$, and unary predicates P_0, \dots, P_{m-3} which form a partition of the set A_n , with

$$\begin{aligned} & \models \forall x_1, \dots, x_n \left(\bigwedge_{1 \leq j < k \leq n} \neg(x_j \approx x_k) \wedge K_n(x_1, \dots, x_n) \rightarrow \right. \\ & \left. \rightarrow \exists t \left(\bigwedge_{j=1}^n \neg(t \approx x_j) \wedge K_n(x_1, t, x_3, \dots, x_n) \wedge K_n(t, x_2, x_3, \dots, x_n) \wedge P_i(t) \right) \right), \\ & i = 0, \dots, m - 3. \end{aligned}$$

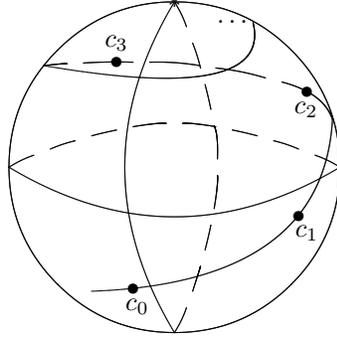


FIG. 2

We denote by $p_\infty(x)$ the type which is forced by the set of formulae

$$K_n(c_{i_1}, c_{i_2}, \dots, c_{i_{n-1}}, x),$$

$i_1 < i_2 < \dots < i_{n-1}$. The type $p_\infty(x)$ has $m - 2$ completions $p_\infty^i(x)$ which are forced by $p_\infty(x) \cup \{P_i(x)\}$, $i = 0, \dots, m - 3$.

The theory T_n^m has exactly m pairwise non-isomorphic countable models:

- (a) a prime model \mathcal{M}_m which omit the type $p_\infty(x)$; ($\lim_{k \rightarrow \infty} c_k = \infty$);
- (b) prime models \mathcal{M}_m^i over realizations of types $p_\infty^i(x)$, $i = 0, \dots, m - 3$, with a limit element for the constants c_k , realizing $p_\infty^i(x)$;
- (c) a saturated model $\overline{\mathcal{M}}_m$, without limit elements for the constants c_k , realizing $p_\infty^i(x)$.

Replacing constants c_k by unary predicates $U_k = \{c_k\}$ we obtain theories \widehat{T}_n^m instead of T_n^m such that \widehat{T}_n^m are unary expansions of $T_n = \text{Th}(\mathcal{A}_n)$, i.e. expansions of T_n by unary predicates. Clearly, this transformation preserves the number of pairwise non-isomorphic countable models:

$$I(\widehat{T}_n^m, \omega) = m = I(T_n^m, \omega).$$

Corollary 1 and the arguments above producing values for $I(\widehat{T}_n^m, \omega)$ imply the following:

Theorem 3. *For any $m \in \omega \setminus \{0, 2\}$, $n \in \omega \setminus \{0, 1\}$ there is a unary expansion \widehat{T}_n^m of the theory T_n of dense n -spherical order \mathcal{A}_n such that $I(\widehat{T}_n^m, \omega) = m$.*

The theories T_2^m in Theorem 3 are classical Ehrenfeucht examples, and the theories T_3^m are their adaptations for dense circular orders.

Figure 2 illustrates a model of the theory T_4^3 . Constants c_k , $k \in \omega$, are situated on a spiral with a top limit vertex $*$.

6. CONSTANT EXPANSIONS OF DENSE n -SPHERICAL ORDERS AND THEIR COUNTABLE SPECTRA

Now we consider possibilities for distributions of countable sequences of constants expanding n -spherical theories $T_n = \text{Th}(\mathcal{A}_n)$, where $\mathcal{A}_n = \langle A_n, K_n \rangle$ are dense n -spherical orders, $n \geq 2$. These distributions produce distributions of countable models of these expansions $T \supseteq T_n$ and possibilities for values $I(T, \omega)$.

The distributions for $n = 2$ are described in [7, 13].

Recall [14] that a linearly ordered structure \mathcal{M} is *o-minimal* if any definable (with parameters) subset of M is a finite union of singletons and open intervals (a, b) , where $a \in M \cup \{-\infty\}$, $b \in M \cup \{+\infty\}$. A theory T is *o-minimal* if each model of T is *o-minimal*.

As examples of Ehrenfeucht *o-minimal* theories, we mention the theories $T^1 \equiv \text{Th}((\mathbb{Q}; <, c_n)_{n \in \omega})$ and $T^2 \equiv \text{Th}((\mathbb{Q}; <, c_n, c'_n)_{n \in \omega})$, where $<$ is an ordinary strict order on the set \mathbb{Q} of rationals, constants c_n form a strictly increasing sequence, and constants c'_n form a strictly decreasing sequence, $c_n < c'_n$, $n \in \omega$.

The theory T^1 is an Ehrenfeucht's example [11] with $I(T^1, \omega) = 3$. It has two almost prime models and one limit model:

- a prime model with empty set of realizations of type $p(x)$ isolated by the set $\{c_n < x \mid n \in \omega\}$ of formulas;
- a prime model over a realization of the type $p(x)$, with the least realization of that type;
- one limit model over the type $p(x)$, with the set of realizations of $p(x)$ forming an open convex set.

The Hasse diagram for the Rudin–Keisler preorder \leq_{RK} and values of the function IL of distributions of numbers of limit models for \sim_{RK} -classes of T^1 is represented in Figure 3. This Hasse diagram equals the Hasse diagram for the theory T_4^3 .

The theory T^2 has six pairwise non-isomorphic countable models:

- a prime model with empty set of realizations of type $p(x)$ isolated by the set $\{c_n < x \mid n \in \omega\} \cup \{x < c'_n \mid n \in \omega\}$;
- a prime model over a realization of $p(x)$, with a unique realization of this type;
- a prime model over a realization of type $q(x, y)$ isolated by the set $p(x) \cup p(y) \cup \{x < y\}$; here the set of realizations of $p(x)$ forms a closed interval $[a, b]$;
- three limit models over the type $q(x, y)$, in which the sets of realizations of $q(x, y)$ are convex sets of forms (a, b) , $[a, b)$, (a, b) respectively.

In Figure 4 we represent the Hasse diagram of Rudin–Keisler preorders \leq_{RK} and values of distribution functions IL of numbers of limit models on \sim_{RK} -equivalence classes for the theory T^2 .

The following theorem shows that the number of countable models of Ehrenfeucht *o-minimal* theories is exhausted by combinations of these numbers for the theories T^1 and T^2 .



FIG. 3



FIG. 4

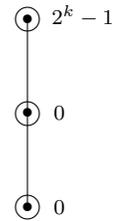


FIG. 5

Theorem 4. [13] *Let T be an o-minimal theory in a countable language. Then either T has 2^ω countable models or T has exactly $3^r \cdot 6^s$ countable models, where r and s are natural numbers. Moreover, for any $r, s \in \omega$ there is an o-minimal theory T with exactly $3^r \cdot 6^s$ countable models.*

Theorem 4 immediately implies:

Corollary 2. *Let T be a countable constant expansion of the 2-spherical theory T_2 . Then either T has 2^ω countable models or T has exactly $3^r \cdot 6^s$ countable models, where r and s are natural numbers. Moreover, for any $r, s \in \omega$ there is an aforesaid theory T with exactly $3^r \cdot 6^s$ countable models.*

The arguments for the 2-spherical theory T_2 are valid for the 3-spherical theory T_3 obtaining the following:

Corollary 3. *Let T be a countable constant expansion of the 3-spherical theory T_3 . Then either T has 2^ω countable models or T has exactly $3^r \cdot 6^s$ countable models, where r and s are natural numbers. Moreover, for any $r, s \in \omega$ there is an aforesaid theory T with exactly $3^r \cdot 6^s$ countable models.*

Considering the dense n -spherical Ehrenfeucht theories T_n , $n \geq 4$, we obtain both the possibilities $3^k \cdot 6^s$ of countable models, with appropriate sequences of constants, and the following new possibilities.

Considering a consistent nonisolated set $p(x)$ of formulae

$$K_n^\delta(c_{j_1}, \dots, c_{j_{i-1}}, x, c_{j_{i+1}}, \dots, c_{j_n}) \wedge \bigwedge_{k \neq i} \neg x \approx c_{j_k},$$

$\delta = \delta_{j_1, \dots, j_{i-1}, x, j_{i+1}, \dots, j_n} \in \{0, 1\}$, with pairwise distinct $c_{j_1}, \dots, c_{j_{i-1}}, c_{j_{i+1}}, \dots, c_{j_n}$, we observe that each such an additional formula divides the domain of previous ones into two parts with respect to a $(n-1)$ -dimensional plane containing points $c_{j_1}, \dots, c_{j_{i-1}}, c_{j_{i+1}}, \dots, c_{j_n}$. In view of quantifier elimination for T_n any consistent set $p(x)$ forces a complete type. Thus a limit part of domain being a set of solutions for $p(x)$ is defined by the limits for $c_{j_1}, \dots, c_{j_{i-1}}, c_{j_{i+1}}, \dots, c_{j_n}$.

We have the following possibilities for these limits:

- $p(x)$ is omitted;
- $p(x)$ has a unique realization a_∞ being a common limit of the sequences;
- $p(x)$ has infinitely many realizations independently including / not including $n-1$ limits with respect to coordinates of $(c_{j_1}, \dots, c_{j_{i-1}}, c_{j_{i+1}}, \dots, c_{j_n})$.

Hence there are $2^{n-1} + 2$ possibilities for countable models, where 3 models are prime over finite sets (prime over \emptyset , prime over $\{a_\infty\}$, and prime over limits for $c_{j_1}, \dots, c_{j_{i-1}}, c_{j_{i+1}}, \dots, c_{j_n}$) and $2^{n-1} - 1$ limit models.

If some constants in $c_{j_1}, \dots, c_{j_{i-1}}, c_{j_{i+1}}, \dots, c_{j_n}$ are fixed, with $k > 1$ independent moving sequences the total number of countable models related to $p(x)$ equals $2^k + 2$ including 3 almost prime models and $2^k - 1$ limit models, see Figure 5.

Since distinct types $p(x)$ are independent the total number of possibilities for countable models is obtained by the multiplications of values $2^k + 2$ for various $k \in n \setminus \{1\}$, if there are finitely many nonisolated 1-types, and there are 2^ω countable models otherwise. Thus we have the following:

Theorem 5. *Let T be a countable constant expansion of the n -spherical theory T_n , $n \geq 3$. Then either T has 2^ω countable models or T has exactly $\prod_{k \in n \setminus \{1\}} (2^k + 2)^{r_k}$*

countable models, where r_k are natural numbers. Moreover, for any $r_0, \dots, r_{n-1} \in \omega$ there is an aforesaid theory T with exactly $\prod_{k \in n \setminus \{1\}} (2^k + 2)^{r_k}$ countable models.

Theorem 5 confirms the Vaught conjecture for countable constant expansions T of n -spherical theories T_n . In particular,

$$\text{either } I(T, \omega) = 2^\omega \text{ or } I(T, \omega) = 3^{r_1} \cdot 6^{r_2} \cdot 10^{r_3}$$

for $T \supseteq T_4$,

$$\text{either } I(T, \omega) = 2^\omega \text{ or } I(T, \omega) = 3^{r_1} \cdot 6^{r_2} \cdot 10^{r_3} \cdot 18^{r_4}$$

for $T \supseteq T_5$, etc.

Similarly to the quite o -minimal Ehrenfeucht theories distributions of countable models of constant expansions of n -spherical theories are given by Hasse diagrams for disjoint unions of theories [15, 16], based on the diagrams represented in Figures 3–5.

7. CONCLUSION

We studied spherical orders, which generalize known linear and circular orders. Semantic and syntactic properties of spherical orders and their elementary theories, including finite and dense orders and their theories are investigated. It is shown that theories of dense n -spherical orders are countably categorical and decidable. The values for spectra of countable models of unary expansions of n -spherical theories are described. The Vaught conjecture is confirmed for countable constant expansions of dense n -spherical theories. It would be natural to study various modifications and kinds of spherical orders and their theories.

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