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REGULARITY CRITERION FOR WEAK SOLUTIONS TO THE NAVIER-STOKES INVOLVING ONE VELOCITY AND ONE VORTICITY COMPONENTS

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Abstract. In this note, we are devoted to study the conditional regularity for the three dimensional Navier-Stokes in terms of the Morrey and BMO spaces. More precisely, we show that if u is a weak solution and $u_3 \in L^2(0,T;BMO(\mathbb{R}^3))$ and $\omega_3 \in L^{\frac{2}{2-r}}(0,T;\dot{\mathcal{M}}_{2,\frac{3}{r}}(\mathbb{R}^3))$ with $0 < r < 1$, then u is regular on $(0,T]$. This improves the available result by Zhang (2018) with $u_3 \in L^2(0,T;L^\infty(\mathbb{R}^3))$ and $\omega_3 \in L^{\frac{2}{2-r}}(0,T;L^{\frac{3}{r}}(\mathbb{R}^3))$ with $0 < r < 1$.

Keywords: Navier-Stokes equations, regularity criteria, Morrey space.

1. Introduction

In this note, we study the regularity condition of weak solutions to the Navier-Stokes equations in $\mathbb{R}^3 \times (0,T)$:

$$\begin{cases} \partial_t u + (u \cdot \nabla)u - \Delta u + \nabla\pi = 0, \\ \nabla \cdot u = 0, \\ u(x,0) = u_0(x), \end{cases} \tag{1}$$

Here, u is the unknown velocity vector and π is the unknown scalar pressure.

For $u_0 \in L^2(\mathbb{R}^3)$ with $\nabla \cdot u_0 = 0$ in $\mathbb{R}^3$, Leray [19] constructed global weak solutions. The smoothness of Leray's weak solutions is unknown. While the existence of regular solutions is still an open problem, there are many interesting sufficient

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conditions which guarantee that a given weak solution is smooth. A well-known condition states that if

$$u \in L^s(0,T;L^r(\mathbb{R}^3)) \text{ with } \frac{2}{s}+\frac{3}{r}=1, \quad 3 \leq r \leq \infty, \tag{2}$$

then the solution u is actually regular [6, 9, 14, 20, 22, 23, 25, 26]. A similar condition

$$\nabla u \in L^s(0,T;L^r(\mathbb{R}^3)) \text{ with } \frac{2}{s}+\frac{3}{r}=2, \quad \frac{3}{2} \leq r \leq \infty, \tag{3}$$

also implies the regularity as shown by Beirão da Veiga [1].

Many regularity results on the weak solutions to the three-dimensional Navier-Stokes equations have been well studied, for example see [2, 3, 4, 8, 10, 11, 12, 13, 15, 16, 17, 24, 27, 29, 31, 32, 33, 34] and the related references therein, where they have proved that the solution is a smooth one if the velocity, or vorticity, or the gradient of velocity, or their components are regular. In particular, Penel and Pokorný in [21], proposed the following regularity criterion:

$$\begin{aligned} u_3 &\in L^s(0,T;L^r(\mathbb{R}^3)); \quad \partial_1 u_2, \partial_2 u_1 \in L^p(0,T;L^q(\mathbb{R}^3)) \\ \text{with } \frac{2}{s}+\frac{3}{r} &= 1, \quad 3<r\leq\infty \quad \text{and } \frac{2}{p}+\frac{3}{q}=2, \quad 2\leq q\leq 3, \end{aligned}$$

and remarked that it is an interesting open problem whether we could make assumptions on $\omega_3=\partial_1 u_2-\partial_2 u_1$ instead of $\partial_1 u_2$ and $\partial_2 u_1$, where ω_3 is the third component of the vorticity $\omega=(\omega_1,\omega_2,\omega_3)=\text{rot}u \equiv \left(\partial_j u^k-\partial_k uj\right)_{1\leq j,k\leq 3}$.

After that Zhang et al. in [30], proposed the following criterion under the condition

$$\begin{aligned} \partial_3 u_3 &\in L^s(0,T;L^r(\mathbb{R}^3)); \quad \omega_3 \in L^p(0,T;L^q(\mathbb{R}^3)) \\ \text{with } \frac{2}{s}+\frac{3}{r} &= 2, \quad \frac{3}{2}<r\leq\infty \quad \text{and } \frac{2}{p}+\frac{3}{q}=2, \quad \frac{3}{2}<q\leq\infty. \end{aligned}$$

Recently, Zhang [28] studied the regularity criterion of the weak solutions involving one velocity and one vorticity component in Lebesgue space

$$\begin{aligned} u_3 &\in L^s(0,T;L^r(\mathbb{R}^3)); \quad \omega_3 \in L^p(0,T;L^q(\mathbb{R}^3)) \\ \text{with } \frac{2}{s}+\frac{3}{r} &= 1, \quad 3<r\leq\infty \quad \text{and } \frac{2}{p}+\frac{3}{q}=2, \quad \frac{3}{2}<q\leq\infty. \end{aligned} \tag{4}$$

The aim of this study is to refine (4) when $r=\infty$ and to improve the above regularity criterion (4) from Lebesgue space framework to critical Morrey space framework.

Recall that for $1<p\leq q<\infty$, the homogeneous Morrey space $\dot{\mathcal{M}}_{p,q}(\mathbb{R}^3)$ is defined as the function $f\in L^p_{\text{loc}}(\mathbb{R}^3)$ such that

$$\|f\|_{\dot{\mathcal{M}}_{p,q}} = \sup_{x\in\mathbb{R}^3, 0<R<\infty} R^{\frac{3}{q}-\frac{3}{p}} \left(\int_{|x-y|<R} |f(y)|^p\, dy\right)^{\frac{1}{p}} < \infty.$$

This space is an homogeneous space of degree $-\frac{3}{q}$ and implies the following relation ship :

$$L^q(\mathbb{R}^3)=\dot{\mathcal{M}}_{q,q}(\mathbb{R}^3)\subset\dot{\mathcal{M}}_{p,q}(\mathbb{R}^3)\subset L^p_{\text{loc}}(\mathbb{R}^3), \quad 1<p\leq q<\infty.$$

Moreover, we also need the predual of $\dot{\mathcal{M}}_{2,\frac{3}{r}}(\mathbb{R}^3)$, which is investigated by Lemarié-Rieusset [18] will be used in the following section.

Lemma 1. *For $0 \leq r < \frac{3}{2}$, let the space $\mathcal{M}(\dot{B}^2_{r,1} \mapsto L^2)$ as the space of functions which are locally square integrable on $\mathbb{R}^3$ and such that pointwise multiplication with these functions maps boundedly the Besov space $\dot{B}^r_{2,1}(\mathbb{R}^3)$ to $L^2(\mathbb{R}^3)$. The norm in $\mathcal{M}(\dot{B}^r_{2,1} \mapsto L^2)$ is given by the operator norm of pointwise multiplication:*

$$\|f\|_{\mathcal{M}(\dot{B}^r_{2,1} \mapsto L^2)} = \sup_{\|g\|_{\dot{B}^r_{2,1}} \leq 1} \|fg\|_{L^2}.$$

Then, f belongs to $\mathcal{M}(\dot{B}^r_{2,1} \mapsto L^2)$ if and only if f belongs to $\dot{\mathcal{M}}_{2,\frac{3}{r}}(\mathbb{R}^3)$ (with equivalence of norms).

Before stating our result, let us recall the definitions of weak solutions to (1).

Definition 1. *Let $u_0 \in L^2(\mathbb{R}^3)$ with $\nabla \cdot u_0 = 0$ in $\mathbb{R}^3$. The function u is called a Leray weak solution of (1) in $(0,T)$ if u satisfies the following properties:*

- **(i):** $u \in L^\infty(0,T;L^2(\mathbb{R}^3)) \cap L^2(0,T;H^1(\mathbb{R}^3))$;
- **(ii):** $\partial_t u + (u\cdot\nabla)u - \Delta u + \nabla\pi = 0$ *in* $\mathcal{D}'((0,T)\times\mathbb{R}^3)$;
- **(iii):** $\nabla \cdot u = 0$ *in* $\mathcal{D}'((0,T)\times\mathbb{R}^3)$;
- **(iv):** *the energy inequality holds*

$$\|u(t)\|^2_{L^2} + 2\int_0^t \|\nabla u(\tau)\|^2_{L^2}\, d\tau \leq \|u_0\|^2_{L^2}, \quad \text{for all } t \in [0,T]. \tag{5}$$

Our main result reads as follows.

Theorem 1. *Let $u_0 \in L^2(\mathbb{R}^3)$ with $\nabla \cdot u_0 = 0$ and $T > 0$. Suppose that u is a Leray weak solution of (1) in $(0,T)$. If*

$$u_3 \in L^2(0,T;BMO(\mathbb{R}^3)) \quad \text{and} \quad \omega_3 \in L^{\frac{2}{2-r}}(0,T;\dot{\mathcal{M}}_{2,\frac{3}{r}}(\mathbb{R}^3)) \quad \text{with} \quad 0 < r < 1,$$

then u is smooth in $(0,T]$. Here, BMO is the space of functions of bounded mean oscillations.

Remark 1. *It is easy to see that the Theorem 1 is a refined improvement of that Theorem 1 in [28] due to the well-known embedding $L^{\frac{3}{r}}(\mathbb{R}^3) \subset \dot{\mathcal{M}}_{2,\frac{3}{r}}(\mathbb{R}^3)$ for $0 \leq r < \frac{3}{2}$. Furthermore, our purpose is to extend these results to the marginal space BMO, which is larger than L^∞. It is a natural way to extend the space widely and improve the previous results.*

2. Proof of Theorem 1.

Proof. Taking inner product of the first equation of (1) with $-\Delta u$ and integrating by parts, we see that

$$\frac{1}{2}\frac{d}{dt}\|\nabla u(t)\|^2_{L^2} + \|\Delta u\|^2_{L^2} = \int_{\mathbb{R}^3} (u\cdot\nabla)u \cdot \Delta u dx = I. \tag{6}$$

Firstly, with the aid of the divergence free condition $\sum_{j=1}^{3} \partial_j u_j = 0$ and integration by parts, we decomposition I as :

$$
\begin{aligned}
I &= \int_{\mathbb{R}^3} \sum_{i,j,k=1}^{3} u_j \partial_j u_i \partial_{kk}^2 u_i dx = -\int_{\mathbb{R}^3} \sum_{i,j,k=1}^{3} \partial_k (u_j \partial_j u_i) \partial_k u_i dx \\
&= -\int_{\mathbb{R}^3} \sum_{i,j,k=1}^{3} \partial_k u_j \partial_j u_i \partial_k u_i dx - \frac{1}{2} \int_{\mathbb{R}^3} \sum_{i,j,k=1}^{3} u_j \partial_j (\partial_k u_i \partial_k u_i) dx \\
&= -\int_{\mathbb{R}^3} \sum_{i,j,k=1}^{3} \partial_k u_j \partial_j u_i \partial_k u_i dx.
\end{aligned}
$$

Following [28], we classify the terms $\partial_k u_j \partial_j u_i \partial_k u_i$, $1 \leq i, j, k \leq 3$ as

(1) If $k = j = 3$ or $j = i = 3$ or $k = i = 3$, we then invoke the divergence free condition to replace $\partial_3 u_3$ by $-(\partial_1 u_1 + \partial_2 u_2)$;

(2) Otherwise, at least two indices belong to $\{1, 2\}$. Thus I will be

$$
\begin{aligned}
I &= \sum_{i,j,kl=1}^{3} \int_{\mathbb{R}^3} \eta_{11ijkl} \partial_1 u_1 \partial_i u_j \partial_k u_l dx + \sum_{i,j,kl=1}^{3} \int_{\mathbb{R}^3} \eta_{12ijkl} \partial_1 u_2 \partial_i u_j \partial_k u_l dx \\
&\quad + \sum_{i,j,kl=1}^{3} \int_{\mathbb{R}^3} \eta_{21ijkl} \partial_2 u_1 \partial_i u_j \partial_k u_l dx + \sum_{i,j,kl=1}^{3} \int_{\mathbb{R}^3} \eta_{22ijkl} \partial_2 u_2 \partial_i u_j \partial_k u_l dx \\
&= I_1 + I_2 + I_3 + I_4,
\end{aligned}
$$

where η_{mnijkl} $(1 \leq m, n \leq 2)$, are suitable integers. Next, we want to represent $\partial_m u_n$ $(1 \leq m, n \leq 2)$ by u_3 and ω_3. Denoting by $\Delta_h = \partial_1 \partial_1 + \partial_2 \partial_2$ the horizontal Laplacian, we have

$$
\begin{aligned}
\Delta_h u_1 &= \partial_1 \partial_1 u_1 + \partial_2 \partial_2 u_1 = \partial_1 (-\partial_2 u_2 - \partial_3 u_3) + \partial_2 \partial_2 u_1 \\
&= -\partial_2 (\partial_1 u_2 - \partial_2 u_1) - \partial_1 \partial_3 u_3 \\
&= -\partial_2 \omega_3 - \partial_1 \partial_3 u_3
\end{aligned}
$$

and

$$
\begin{aligned}
\Delta_h u_2 &= \partial_1 \partial_1 u_2 + \partial_2 \partial_2 u_2 = \partial_1 \partial_1 u_2 + \partial_2 (-\partial_1 u_1 - \partial_3 u_3) \\
&= \partial_1 (\partial_1 u_2 - \partial_2 u_1) - \partial_2 \partial_3 u_3 \\
&= \partial_1 \omega_3 - \partial_2 \partial_3 u_3.
\end{aligned}
$$

Based on the computations above, we can use the two-dimension Riesz transformation $\mathcal{R}_m = \frac{\partial_m}{\sqrt{-\Delta_h}}$ to denote the term $\partial_m u_n$ $(1 \leq m, n \leq 2)$,

$$
\begin{aligned}
\partial_m u_1 &= \frac{\partial_m}{\sqrt{-\Delta_h}} \frac{\partial_2}{\sqrt{-\Delta_h}} \omega_3 + \frac{\partial_m}{\sqrt{-\Delta_h}} \frac{\partial_1}{\sqrt{-\Delta_h}} \partial_3 u_3 \\
&= \mathcal{R}_m \mathcal{R}_2 \omega_3 + \mathcal{R}_m \mathcal{R}_1 \partial_3 u_3, \qquad (7)
\end{aligned}
$$

and

$$
\begin{aligned}
\partial_m u_2 &= -\frac{\partial_m}{\sqrt{-\Delta_h}} \frac{\partial_1}{\sqrt{-\Delta_h}} \omega_3 + \frac{\partial_m}{\sqrt{-\Delta_h}} \frac{\partial_2}{\sqrt{-\Delta_h}} \partial_3 u_3 \\
&= -\mathcal{R}_m \mathcal{R}_1 \omega_3 + \mathcal{R}_m \mathcal{R}_2 \partial_3 u_3. \qquad (8)
\end{aligned}
$$

By (7), the term I_1 could be turned into

$$I_1 = \sum_{i,j,k,l=1}^{3} \int_{\mathbb{R}^3} \eta_{11ijkl}\partial_1 u_1 \partial_i u_j \partial_k u_l dx$$
$$= \sum_{i,j,k,l=1}^{3} \int_{\mathbb{R}^3} \eta_{11ijkl} \left(\mathcal{R}_1\mathcal{R}_2\omega_3 + \mathcal{R}_1\mathcal{R}_1\partial_3 u_3\right) \partial_i u_j \partial_k u_l dx$$
$$= \sum_{i,j,k,l=1}^{3} \int_{\mathbb{R}^3} \eta_{11ijkl}\mathcal{R}_1\mathcal{R}_2\omega_3 \partial_i u_j \partial_k u_l dx - \sum_{i,j,k,l=1}^{3} \int_{\mathbb{R}^3} \eta_{11ijkl}\mathcal{R}_1\mathcal{R}_1 u_3 \partial_3 \left(\partial_i u_j \partial_k u_l\right) dx$$
$$= \sum_{i,j,k,l=1}^{3} \int_{\mathbb{R}^3} \eta_{11ijkl}\mathcal{R}_1\mathcal{R}_2\omega_3 \partial_i u_j \partial_k u_l dx$$
$$- \sum_{i,j,k,l=1}^{3} \int_{\mathbb{R}^3} \eta_{11ijkl}\mathcal{R}_1\mathcal{R}_1 u_3 \left(\partial_3\partial_i u_j \partial_k u_l + \partial_i u_j \partial_3\partial_k u_l\right) dx = I_{11} + I_{12}.$$

Now we estimate I_{11} firstly. By using Hölder's inequality, the interpolation inequality $\|f\|_{\dot{B}^r_{2,1}} \leq C\|f\|_{L^2}^{1-r}\|\nabla f\|_{L^2}^r$ for $0 < r < 1$, and the Young inequality, we can estimate I_{11} as

$$\begin{aligned} |I_{11}| &\leq \left\|\mathcal{R}_2\mathcal{R}_1\omega_3 \cdot \nabla u\right\|_{L^2} \left\|\nabla u\right\|_{L^2} \\ &\leq C \left\|\omega_3\right\|_{\dot{\mathcal{M}}_{2,\frac{3}{r}}} \left\|\nabla u\right\|_{\dot{B}^r_{2,1}} \left\|\nabla u\right\|_{L^2} \\ &\leq C \left\|\omega_3\right\|_{\dot{\mathcal{M}}_{2,\frac{3}{r}}} \left\|\nabla u\right\|_{L^2}^{2-r} \left\|\Delta u\right\|_{L^2}^{r} \\ &\leq C \left\|\omega_3\right\|_{\dot{\mathcal{M}}_{2,\frac{3}{r}}}^{\frac{2}{2-r}} \left\|\nabla u\right\|_{L^2}^2 + \frac{1}{4}\left\|\Delta u\right\|_{L^2}^2 \end{aligned} \tag{9}$$

where we have used the continuity of the operator $\mathcal{R}_i\mathcal{R}_j$ on Morrey spaces $\dot{\mathcal{M}}_{p,\frac{3}{r}}(\mathbb{R}^2)$ for the values $p \geq 2$ and $0 < r < 1$ (see e.g. [7]).

In order to handle I_{12}, we recall the following property of Hardy space $\mathcal{H}^1$ and BMO [5]:

$$\int_{\mathbb{R}^3} fgh dx \leq C \left\|f\right\|_{BMO} \left\|gh\right\|_{\mathcal{H}^1} \leq C \left\|f\right\|_{BMO} \left\|g\right\|_{L^2} \left\|h\right\|_{L^2}, \tag{10}$$

for any $\nabla \cdot g = 0$ and $\nabla \times h = 0$. Because of the Riesz transformation being bounded in $BMO(\mathbb{R}^2)$ to $BMO(\mathbb{R}^2)$, and according to above inequality (10) and Young inequality, we can show

$$\begin{aligned} |I_{12}| &\leq \left\|u_3\right\|_{BMO} \left\|\partial_i u \cdot \nabla\partial_i u\right\|_{\mathcal{H}^1} \\ &\leq C \left\|u_3\right\|_{BMO} \left\|\nabla u\right\|_{L^2} \left\|\Delta u\right\|_{L^2} \\ &\leq \frac{1}{4}\left\|\Delta u\right\|_{L^2}^2 + C\left\|u_3\right\|_{BMO}^2 \left\|\nabla u\right\|_{L^2}^2. \end{aligned}$$

Inserting (9) and (10) into (6), we see that

$$\frac{d}{dt}\left\|\nabla u(t)\right\|_{L^2}^2 + \left\|\Delta u\right\|_{L^2}^2 \leq C\left(\left\|u_3\right\|_{BMO}^2 + \left\|\omega_3\right\|_{\dot{\mathcal{M}}_{2,\frac{3}{r}}}^{\frac{2}{2-r}}\right)\left\|\nabla u\right\|_{L^2}^2.$$

Taking the Gronwall inequality into consideration, one shows that

$$\|\nabla u(t)\|_{L^2}^2 + \int_0^t \|\Delta u(\tau)\|_{L^2}^2 \, d\tau \\ \leq \|\nabla u_0\|_{L^2}^2 \exp\left(C \int_0^T \|u_3(t)\|_{BMO}^2 + \|\omega_3(t)\|_{\dot{\mathcal{M}}_{2,\frac{3}{r}}}^{\frac{2}{2-r}} \, dt \right) < \infty,$$

for any $t \in [0, T)$, with constant C independent of T . Thus,

$$u \in L^\infty(0, T; H^1(\mathbb{R}^3)) \subseteq L^\infty(0, T; L^6(\mathbb{R}^3)) \,,$$

which implies that u is smooth. This completes the proof of Theorem 1. □

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