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ORDERED SUPERASSOCIATIVE ALGEBRAS

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Abstract. This paper is contributed to a possible approach to the generalization theory of ordered semigroups. Particularly, the compatibility of partially order with an operation of type $(n+1)$ on a superassociative function is provided on the basis of partially order in arbitrary semigroups. This leads us to form an algebraic structure that generalize the concept of ordered semigroups called ordered superassociative algebras. For this purpose, the notion of pseudoorder is presented with several related properties investigated. Several homomorphism theorems are studied. As a consequence, all results obtained in arbitrary ordered semigroups can be viewed as a particular case of results obtained for our structure in this paper.

Keywords: Superassociative algebra, partial order, ordered semigroup, isomorphism theorem

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1. Introduction

It is widely accepted that the investigation of many properties of the composition of multiplace functions was started K. Menger in [19]. The multiplace function is one of the main concepts in the investigation of mathematics, it has various applications in numerous areas of sciences, for example, in theory of multi-valued logics, programming languages, and multivariable calculas. The general references for these applications are [3, 7]. New applications of multiplace functions to algebraic constructions were also found among researchers by A. I. Mal'cev, V. A. Artamonov, M. I. Burtman, B. M. Schein, and V. V. Vagner. The satisfaction of composition of functions with the axiom, always called *superassociative law*, was investigated in both elementary and advanced ways. This allows many scholars to focus on the notion of superassociative algebras.

Let n be a fixed positive integer. A *superassociative algebra* or a *Menger algebra* is a couple of a nonempty set G with an $(n+1)$-ary operation $\circ$ defined on it, satisfying the superassociative law:

$$\circ(\circ(x, y_1, \ldots, y_n), z_1, \ldots, z_n) = \circ(x, \circ(y_1, z_1, \ldots, z_n), \ldots, \circ(y_n, z_1, \ldots, z_n)),$$

for all $x, y_1, \ldots, y_n, z_1, \ldots, z_n \in G$. It is denoted by $(G, \circ)$.

Note that if $n = 1$, it is an arbitrary semigroup. By a superassociative subalgebra of G we mean a nonempty subset A of G closed under the restriction of an operation $\circ$ of G to A. Recent constrivutions on this algebra can be found, for instance, in [10, 16, 17, 23]. For applications of this algebra in the study of terms and universal algebra, the reader is refereed to [5, 18, 24].

The following are examples of concrete superassociative algebras.

Example 1. [7] *Some examples of superassociative algebras are provided.*

(1) *The set* $\mathbf{R}^+$ *of all positive real numbers with the operation* $\circ : (\mathbf{R}^+)^{n+1} \to \mathbf{R}^+$*, defined by*

$$\circ(x_0, \ldots, x_n) = x_0 \sqrt[n]{x_1 \cdots x_n},$$

forms a superassociative algebra.

(2) *The set of all real numbers* $\mathbf{R}$ *with the* $(n+1)$*-ary operation* $\circ$*, which is defined by*

$$\circ(x, y_1 \ldots, y_n) = x + \frac{y_1 + \ldots + y_n}{n},$$

for all $x, y_1 \ldots, y_n \in \mathbf{R}$*, is a superassociative algebra.*

(3) *On the Cartesian product* A^n *of a nonempty set* A*, an full* n*-ary function or an* n*-ary operation is refer to any mapping* f *from* A^n *to* A*. We use* $T(A^n, A)$ *to denote the set of all full* n*-ary functions on* A*. On this set, one consider the Menger's superposition, i.e., an* $(n+1)$*-operation* $\mathcal{O} : T(A^n, A)^{n+1} \to T(A^n, A)$ *defined by*

$$\mathcal{O}(f, g_1, \ldots, g_n)(a_1, \ldots, a_n) = f(g_1(a_1, \ldots, a_n), \ldots, g_n(a_1, \ldots, a_n)), \quad (1)$$

where $f, g_1, \ldots, g_n \in T(A^n, A), a_1, \ldots, a_n \in A$*. Clearly, the set* $T(A^n, A)$ *equipped with the Menger's superposition is a superassociative algebra due* *to the fact that the Menger's composition of full n-ary functions satisfy the superassociativity.*

If there exist elements $e_1, \ldots, e_n \in G$, called *selectors*, such that

$$\circ(x, e_1, \ldots, e_n) = x \text{ and } \circ(e_i, x_1 \ldots, x_n) = x_i$$

for all $x, x_1, \ldots, x_n \in G, i = 1, \ldots, n$, then a superassociative algebra $(G, \circ)$ is called *unitary.* It is obvious that selectors in this sense can be considered as an extension of an identity element in general semigroups by considering $n = 1$ so that $e_1 \in G$ and hence $\circ(x, e_1) = x$ and $\circ(e_1, x_1) = x_1$. Thus, e_1 is a right identity element and a left identity element, respectively.

Algebraic structures play a significant role in mathematics with applications in several branches such as data sciences, artificial intelligences, theoretical computer sciences, control engineering, information sciences, and various fields of science and technology. This offers sufficient motivation and intension for algebraists to review many concepts and results from the area of abstract algebra in wider framework of $(n + 1)$-ary algebras. The development of the theory of superassociative algebras and their applications was appeares in many publications of W. A. Dudek and V. S. Trokhimenko. Particularly, a superassociative algebra of n-ary operation is one of the outstanding directions for studying structural properties of n-ary functions in a recent year. Other results on this area, see [9, 11, 12]. Unfortunately, V.S. Trokhimenko passed away in 2020 due to the pandamic of COVID-19. However, the paper that mentioned his personal life and scientific works was commemoratively collected by W.A. Dudek in [6]. By using superassociative algebras initiated by K. Menger, various branches of mathematics have been extended, for example, H. Whitlock applied superassociative algebras to the investigation of compositions of multiplace functions of various arities, which is the main part of the theory of Menger algebras and the generalization theory of transformations in semigroups.

L. Fuchs [13] began the theory of partially ordered semigroups in 1963 which grew from original semigroups and certain problems in lattice theory. There arises a natural algebraic structure with an interconnect between lattice and algebraic aspects. Lattices can also be viewed as special ordered sets. An ordered semigroup is defined to be a semigroup together with a partial order $\leq$ that is compatible with the semigroup associative operation. It has been studied by several mathematicians, for example, N. G. Alimov and A. H. Clifford, in particular, G. Birkhoff continued the study of such structure and collected the khowledge in his book in the title "Lattice Theory see [1]. After that, N. Kehayopulu studied the algebraic properties of ordered semigroups, especially, pseudoorders and isomorphisms. There are many articles on ordered semigroups, for more information, we refer the readers to [15]. The study of quotient ordered semigroup homomorphisms (also called QO-homomorphisms) is posed by Y. Cao in [2], and by X. Y. Xie in [26]. Later, T. K. Mondal in [22] presented the notion of pseudoorders and proved the isomorphism theorems on ordered semirings. Without the relation, an order semigroups can be reduced to a general semigroup in a natural way. On the other hand, the theory of ordered algebraic hyperstructure was investigated by many mathematicians. The notion of ordered semihypergroups, which is a generalization of the notion of ordered semigroups was introduced by D. Heidari and B. Davvaz in [14]. For more results, the reader is referred to [4].

The primary objectives of the present work are summarized as follows: To propose a novel approach of ordered semigroups in terms of the generalized fixed arity of operation and to present the suitable structure consisting a nonempty set with one operation having the arity $n+1$ for a fixed natural number n and one order relation. To introduce a concept of i-compatible with respect to an $(n+1)$-operation on some sets and establish a quotient structure including the order. To present a concept of homomorphisms and isomorphisms in our main structure of type $n+1$ and to extend those results in [15, 19, 20, 21] to a generalized structure, i.e., a superassociative algebra with the partially order which we will define later.

The paper organized as follows: After the introductory words, the main results of this paper will be begin in Section 2 by first introducing the novel definition of ordered superassociative algebras and presenting some fundamental concrete examples based on the algebraic structures of ordered semigroups and superassociative algebras. It is commonly seen that the pseudoorder is a significant tool for defining a quotient structure of ordered semigroups and proving the isomorphism theorems. This leads us to present the study of pseudoorders and propose some algebraic properties of them. In the final part of the results, the concepts of homomorphisms and isomorphisms are introduced. Furthermore, important theorems concerning the fundamental homomorphism theorem and isomorphism theorems are investigated. The results obtained in [15] become then special cases. In section 3 we complete the paper with a summary discussion on the challenging problems and a recommendation for the future work.

2. Results

In order to study the generalized structure of ordered semigroups through the arity of an operation, in this section, we first introduce the notion of ordered superassociative algebras and propose their concrete examples. Then we define congruences using the concept of congruences in usual superassociative algebras. In this work, we provide a definition for pseudoorders of ordered superassociative algebras under the partial order. We begin our main results by defining an ordered superassociative algebra.

Firstly, we will use the following notation: For every non-negative integer i, j, if $i \leq j$, then the sequence $x_i, x_{i+1}, \ldots, x_j$ is well-defined. If $i > j$, then $x_i, x_{i+1}, \ldots, x_j$ is the empty symbol. We sometimes write x_i^j instead of $x_i, x_{i+1}, \ldots, x_j$.

Definition 1. *Let n be a fixed natural number. An ordered superassociative algebra or an ordered Menger algebra is a structure $(G, \circ, \leq)$ satisfying the following assertions:*

1. *$(G, \circ)$ is a superassociative algebra,*
2. *$(G, \leq)$ is a partially ordered set,*
3. *a partial order $\leq$ is compatible with $(n+1)$-operation on G, i.e., for any elements $x, y, a_1, \ldots, a_{n+1}$ in G, the following implication*

$$x \leq y \Rightarrow \circ(a_1, \ldots, a_{i-1}, x, a_{i+1}, \ldots, a_{n+1}) \leq \circ(a_1, \ldots, a_{i-1}, y, a_{i+1}, \ldots, a_{n+1})$$

holds for every $i = 1, \ldots, n+1$.

According to the Definition 1, an abstract ordered superassociative algebra can be regarded as a natural generalization of arbitrary ordered semigroups if $n = 1$, i.e., Definition 1 conincides with definition of ordered semigroups in [15].

Example 2. *Some elementary examples of ordered superassociative algebras will be presented.*

(1) *On the set* $\mathbf{R}^+$*, define the operation* $\circ : (\mathbf{R}^+)^{n+1} \to \mathbf{R}^+$*, by*

$$\circ(x_0, \ldots, x_n) = x_0 \sqrt[n]{x_1 \cdots x_n},$$

and define a partial order as $\leq$ *(usual less than or equal to). Clearly,* $\mathbf{R}^+$ *forms an ordered superassociative algebra.*

(2) *The set of all real numbers* $\mathbf{R}$ *with the* $(n+1)$*-ary operation* $\circ$*, which is defined by*

$$\circ(x, y_1 \ldots, y_n) = x + \frac{y_1 + \ldots + y_n}{n},$$

for all $x, y_1 \ldots, y_n \in \mathbf{R}$*, and the partial order* $\leq$*, is an ordered superassociative algebra.*

As in Example 2, if we put $n = 1$, then the statements (1) and (2) reduce to ordered semigroups $(\mathbf{R}^+, \cdot, \leq)$ and $(\mathbf{R}, +, \leq)$, respectively.

We define the notion of congruences on ordered superassociative algebras, using which we introduce the quotient structure of all equivalence classes for ordered superassociative algebras. After that, we make an important observation.

Definition 2. *Let* $(G, \circ, \leq)$ *be an ordered superassociative algebra and* ρ *be a binary relation on* G*. Then* ρ *is called i-compatible on* G*, for each* $i = 1, \ldots, n+1$*, if* $(x, y) \in \rho$ *implies that*

$$(\circ(a_1^{i-1}, x, a_{i+1}^{n+1}), \circ(a_1^{i-1}, y, a_{i+1}^{n+1})) \in \rho$$

holds for all $x, y, a_1, \ldots, a_{i-1}, a_{i+1}, \ldots, a_{n+1} \in G$*. Moreover,* ρ *is called compatible on* G *if it is i-compatible on* G *for every* $i \in \{1, \ldots, n+1\}$*.*

Definition 3. *Let* $(G, \circ, \leq)$ *be an ordered superassociative algebra. a binary relation* ρ *on* G *is said to be congruence on* G *if it is equivalence and compatible on* G*.*

Remark 1. *It is commonly seen that Definitions 2 and 3 are natural generalizations of congruence in an ordinary ordered semigroup by setting* $n = 1$*. Furthermore, we remark that if* $i = 1$*, we have* $(\circ(x, a_2), \circ(y, a_2)) \in \rho$*, which means that* ρ *is right congruence. On the other hand, if* $i = 2$*, we have* $(\circ(a_1, x), \circ(a_1, y)) \in \rho$*, subsequently,* ρ *is left congruence.*

Now, The characterization of congruences on ordered superassociative algebras is given .

Theorem 2. *Let* $(G, \circ, \leq)$ *be an ordered superassociative algebra. A binary equivalence relation* ρ *on* G *is congruence on* G *if and only if for every* $j = 1, \ldots, n+1$*,* $(a_j, b_j) \in \rho$ *imply that* $(\circ(a_1^{n+1}), \circ(b_1^{n+1})) \in \rho$*.*

Proof. Let ρ be congruence on a superassociative algebra $(G, \circ)$. Assume first that $(a_j, b_j) \in \rho$ for all $j = 1, \ldots, n+1$. From $(a_1, b_1) \in \rho$ and ρ is 1-compatible, then we have $(\circ(a_1, a_2, \ldots, a_{n+1}), \circ(b_1, a_2, \ldots, a_{n+1})) \in \rho$. Since $(a_2, b_2) \in \rho$ and ρ is 2-compatible, then we have $(\circ(b_1, a_2, a_3, \ldots, a_{n+1}), \circ(b_1, b_2, \ldots, a_{n+1})) \in \rho$. Similarly, for $i = 3, \ldots, n$ we have the same results. Lastly, from $(a_{n+1}, b_{n+1}) \in \rho$ and ρ is $(n+1)$-compatible, then $(\circ(b_1, b_2, \ldots, b_n, a_{n+1}), \circ(b_1, b_2, \ldots, b_n, b_{n+1})) \in \rho$. By the transitivity of ρ implies that $(\circ(a_1, \ldots, a_{n+1}), \circ(b_1, \ldots, b_{n+1})) \in \rho$. Conversely, assume that the condition is valid. Suppose now that $(a, b) \in \rho$. Since $(a_j, b_j) \in \rho$ for all $j = 1, \ldots, n+1$, by the assumption, we obtain that ρ is congruence on G. □

Remark 3. *In view of the characterization of a congruence relation for ordered superassociative algebras in Theorem 2, one can notice that it is the definition of a stable equivalence which defined by W. A. Dudek and V. S. Trokhimenko in* [8].

Using the fact that ρ is congruence on an ordered superassociative algebra G, the quotient set G/ρ can be obtained follows:

$$G/\rho = \{[a]_\rho \mid a \in G\}$$

where

$$[a]_\rho = \{b \in G \mid (a, b) \in \rho\}.$$

To apply an operation for the quotient set G/ρ, a product for equivalence classes must be prepared.

Definition 4. *An $(n+1)$-ary operation $\otimes$ on the quotient set G/ρ is defined by*

$$\otimes([a_1]_\rho, [a_1]_\rho, \ldots, [a_{n+1}]_\rho) = [\circ(a_1, a_1, \ldots, a_{n+1})]_\rho.$$

To ensure that an operation $\otimes$ of type $(n+1)$ is a well-defined, suppose now that $[a_1]_\rho = [b_1]_\rho, \ldots, [a_{n+1}]_\rho = [b_{n+1}]_\rho$. It implies that $(a_j, b_j) \in \rho$ for all $j \in \{1, \ldots, n+1\}$. Since ρ is congruence, by Theorem 2, we obtain $(\circ(a_1^{n+1}), \circ(b_1^{n+1})) \in \rho$ and so $[\circ(a_1^{n+1})]_\rho = [\circ(b_1^{n+1})]_\rho$. This follows that $\otimes([a_1]_\rho, \ldots, [a_{n+1}]_\rho) = \otimes([b_1]_\rho, \ldots, [b_{n+1}]_\rho)$. As a result, we prove the following theorem.

Theorem 4. *Let ρ be congruence on an ordered superassociative algebra $(G, \circ, \leq)$. The quotient set G/ρ together with one $(n+1)$-ary operation $\otimes$ defined in Definition 4 forms a superassociative algebra.*

Proof. The superassociativity of an operation $\otimes$ follows directly from the fact that the usual operation $\circ$ is defined on a superassociative algebra G. In fact, we now let $[x]_\rho, [y_1]_\rho, \ldots, [y_n]_\rho, [z_1]_\rho, \ldots, [z_n]_\rho \in G/\rho$. Then we obtain

$$\begin{aligned}
&\otimes(\otimes([x]_\rho, [y_1]_\rho, \ldots, [y_n]_\rho), [z_1]_\rho, \ldots, [z_n]_\rho) \\
&= \otimes([\circ(x, y_1, \ldots, y_n)]_\rho, [z_1]_\rho, \ldots, [z_n]_\rho) \\
&= [\circ(\circ(x, y_1, \ldots, y_n), z_1, \ldots, z_n)]_\rho \\
&= [\circ(x, \circ(y_1, z_1, \ldots, z_n), \ldots, \circ(y_n, z_1, \ldots, z_n))]_\rho \\
&= \otimes([x]_\rho, [\circ(y_1, z_1, \ldots, z_n)]_\rho, \ldots, [\circ(y_n, z_1, \ldots, z_n)]_\rho) \\
&= \otimes([x]_\rho, \otimes([y_1]_\rho, [z_1]_\rho, \ldots, [z_n]_\rho), \ldots, \otimes([y_n]_\rho, [z_1]_\rho, \ldots, [z_n]_\rho)).
\end{aligned}$$

Therefore G/ρ is a superassociative algebra with respect to $\otimes$. □

If ρ is a congruence relation on an ordered superassociative algebra G, then the superassociative algebra G/ρ in Theorem 4 is called a *quotient superassociative algebra of G by ρ.*

Note that Theorem 4 shows only the fact that the quotient set G/ρ with one superassociative operation forms a superassociative algebra, but not to mention the order. From the above, the following problem arises: Is there a probable congruence ρ on a superassociative algebra G/ρ for being an ordered superassociative algebra. This leads us to improve and answer this challenging problem in the following by using a novel concept of pseudoorders.

Definition 5. *Let $(G, \circ, \leq)$ be an ordered superassociative algebra. A binary relation ρ on G is called pseudoorder if it satisfies the following statements:*

(1) $\leq \subseteq \rho$,
(2) *for every $x, y, z \in \rho$, $(x, y) \in \rho$ and $(y, z) \in \rho$ imply $(x, z) \in \rho$,*

(3) *the binary relation* ρ *compatible with* $(n+1)$*-ary operation on* G*. That is, for every elements* $x, y, a_1, \ldots, a_{n+1}$ *in* G*,* $(x, y) \in \rho$ *implies*
$$(\circ(a_1^{i-1}, x, a_{i+1}^{n+1}), \circ(a_1^{i-1}, y, a_{i+1}^{n+1})) \in \rho$$
for all $i \in \{1, \ldots, n+1\}$.

The next theorem is of fundamental importance. It will be used repeatedly in the next time.

Theorem 5. *Let* $(G, \circ, \leq)$ *be an ordered superassociative algebra. If* ρ *is a pseudoorder on* G*, then the relation* $\overline{\rho}$ *which is defined by*
$$\overline{\rho} := \{(x, y) \mid (x, y) \in \rho, (y, x) \in \rho\}$$
is congruence on G*.*

Proof. We first show that $\overline{\rho}$ is an equivalence relation on G. Let $x \in G$. Since $(x, x) \in \leq \subseteq \overline{\rho}$, then $(x, x) \in \overline{\rho}$. If $(x, y) \in \overline{\rho}$, then $(x, y) \in \rho$ and $(y, x) \in \rho$. This means that $(y, x) \in \overline{\rho}$. If $(x, y) \in \overline{\rho}$ and $(y, z) \in \overline{\rho}$, then $(x, y) \in \rho, (y, x) \in \rho, (y, z) \in \rho, (z, y) \in \rho$ and thus $(x, z) \in \rho, (z, x) \in \rho$. It follows that $(x, z) \in \overline{\rho}$. Next, let $(x, y) \in \overline{\rho}$ and $a_1, \ldots, a_{n+1}$ be arbitrary elements in G. Since $(x, y) \in \rho, (y, x) \in \rho$ and ρ is a pseudoorder, then for each $i = 1, \ldots, n+1$, we have
$$(\circ(a_1^{i-1}, x, a_{i+1}^{n+1}), \circ(a_1^{i-1}, y, a_{i+1}^{n+1})) \in \rho$$
and
$$(\circ(a_1^{i-1}, y, a_{i+1}^{n+1}), \circ(a_1^{i-1}, x, a_{i+1}^{n+1})) \in \rho.$$
Thus $(\circ(a_1^{i-1}, x, a_{i+1}^{n+1}), \circ(a_1^{i-1}, y, a_{i+1}^{n+1})) \in \overline{\rho}$. □

Let $(G, \circ, \leq)$ be an ordered superassociative algebra and ρ be a pseudoorder on G. It follows by applying the results of Theorem 4 and 5 that the quotient set of G by congruence $\overline{\rho}$, i.e., $G/\overline{\rho}$ with an $(n+1)$-operation $\otimes$ is a superassociative algebra. We now define a relation $\leq_{\overline{\rho}}$ on $G/\overline{\rho}$ as follows:
$$\leq_{\overline{\rho}} := \{([x]_{\overline{\rho}}, [y]_{\overline{\rho}}) \mid \exists a \in [x]_{\overline{\rho}}, \exists b \in [y]_{\overline{\rho}}, (a, b) \in \rho\}.$$

It is not hard to prove that the following three statements are coincide, i.e.,
$$[x]_{\overline{\rho}} \leq_{\overline{\rho}} [y]_{\overline{\rho}} \Leftrightarrow \forall a \in [x]_{\overline{\rho}}, \forall b \in [y]_{\overline{\rho}}, (a, b)\rho \Leftrightarrow (x, y) \in \rho.$$

As a consequence, we have the following important theorem.

Theorem 6. *Let* $(G, \circ, \leq)$ *be an ordered superassociative algebra. If* ρ *is a pseudoorder on* G*, then* $(G/\overline{\rho}, \otimes, \leq_{\overline{\rho}})$ *forms an ordered superassociative algebra.*

Proof. By defining a congruence relation $\overline{\rho}$, then by Theorem 4, we obtain that $(G/\overline{\rho}, \otimes)$ is a superassociative algebra. Now, we first prove that the relation $\leq_{\overline{\rho}}$ is a partial order on $G/\overline{\rho}$. Let $[x]_{\overline{\rho}} \in G/\overline{\rho}$. Since $x \in G$, then $x \leq x$. Because of $\leq \subseteq \rho$, $(x, x) \in \rho$ and that $[x]_{\overline{\rho}} \leq_{\overline{\rho}} [x]_{\overline{\rho}}$. Let $[x]_{\overline{\rho}}, [y]_{\overline{\rho}} \in G/\overline{\rho}$ be such that $[x]_{\overline{\rho}} \leq_{\overline{\rho}} [y]_{\overline{\rho}}$ and $[y]_{\overline{\rho}} \leq_{\overline{\rho}} [x]_{\overline{\rho}}$. Then we have $(x, y) \in \rho$ and $(y, x) \in \rho$ and thus $(x, y) \in \overline{\rho}$. Hence $[x]_{\overline{\rho}} = [y]_{\overline{\rho}}$. Let $[x]_{\overline{\rho}}, [y]_{\overline{\rho}}, [z]_{\overline{\rho}}$ be elements in $G/\overline{\rho}$ such that $[x]_{\overline{\rho}} \leq_{\overline{\rho}} [y]_{\overline{\rho}}$ and $[y]_{\overline{\rho}} \leq_{\overline{\rho}} [z]_{\overline{\rho}}$. Then $(x, y) \in \rho$ and $(y, z) \in \rho$. So $(x, z) \in \rho$ and that $[x]_{\overline{\rho}} \leq_{\overline{\rho}} [z]_{\overline{\rho}}$. Therefore, $(G/\overline{\rho}, \leq_{\overline{\rho}})$ is a partially ordered set. In order to show that the order $\leq_{\overline{\rho}}$ compatible with an $(n+1)$-operation $\otimes$ on $G/\overline{\rho}$, let $j = 1, \ldots, n+1$, $[x]_{\overline{\rho}}, [y]_{\overline{\rho}}$, and $[z_j]_{\overline{\rho}}$ be arbitrary equivalence classes in $G/\overline{\rho}$. Assume now that $[x]_{\overline{\rho}} \leq_{\overline{\rho}} [y]_{\overline{\rho}}$. Then we have $(x, y) \in \rho$. Since ρ is a pseudoorder on G, we get
$$(\circ(z_1^{i-1}, x, z_{i+1}^{n+1}), \circ(z_1^{i-1}, y, z_{i+1}^{n+1})) \in \rho$$

for all $i = 1, \ldots, n+1$. This means that

$$[\circ(z_1^{i-1}, x, z_{i+1}^{n+1})]_{\overline{\rho}} \leq_{\overline{\rho}} [\circ(z_1^{i-1}, y, z_{i+1}^{n+1})]_{\overline{\rho}}$$

for every $i = 1, \ldots, n+1$. By the definition of an operation $\otimes$ on $G/\overline{\rho}$, we get

$$\begin{aligned} &\otimes([z_1]_{\overline{\rho}}, \ldots, [z_{i-1}]_{\overline{\rho}}, [x]_{\overline{\rho}}, [z_{i+1}]_{\overline{\rho}}, \ldots, [z_{n+1}]_{\overline{\rho}}) \\ \leq_{\overline{\rho}} &\otimes([z_1]_{\overline{\rho}}, \ldots, [z_{i-1}]_{\overline{\rho}}, [y]_{\overline{\rho}}, [z_{i+1}]_{\overline{\rho}}, \ldots, [z_{n+1}]_{\overline{\rho}}) \ . \end{aligned}$$

The proof is absolutely finished. □

Some algebraic properties of a quotient set $G/\overline{\rho}$ with one multiplication and one partial order are discussed in the next theorem.

Theorem 7. *Let $(G, \circ, \leq)$ be an ordered superassociative algebra and ρ a pseudoorder on G. Let A be the set of all pseudoorders μ on G such that $\rho \subseteq \mu$. Let B be the set of all pseudoorders on $G/\overline{\rho}$. For $\mu \in A$, we define a relation μ' on $G/\overline{\rho}$ by*

$$\mu' := \{([x]_{\overline{\rho}}, [y]_{\overline{\rho}}) \mid \exists a \in [x]_{\overline{\rho}}, \exists b \in [y]_{\overline{\rho}}, (a, b) \in \mu\}.$$

Then the following two statements hold:

(1) $|A| = |B|$.
(2) $\mu_1 \subseteq \mu_2 \Leftrightarrow \mu_1' \subseteq \mu_2'$ *for all* $\mu_1, \mu_2 \in A$.

Proof. Note that $\mu' = \{([x]_{\overline{\rho}}, [y]_{\overline{\rho}}) \mid \forall a \in [x]_{\overline{\rho}}, \forall b \in [y]_{\overline{\rho}}, (a, b) \in \mu\} = (x, y) \in \mu$. First, we prove that μ' is a pseudoorder on $G/\overline{\rho}$. If $([x]_{\overline{\rho}}, [y]_{\overline{\rho}}) \in \leq_{\overline{\rho}}$, then $(x, y) \in \rho \subseteq \mu$. So $\leq_{\overline{\rho}} \subseteq \mu'$. Let $([x]_{\overline{\rho}}, [y]_{\overline{\rho}}) \in \mu', ([y]_{\overline{\rho}}, [z]_{\overline{\rho}}) \in \mu'$. Then we have $(x, y) \in \rho \subseteq \mu$ and $(y, z) \in \rho \subseteq \mu$ so $(x, z) \in \mu$. Hence $([x]_{\overline{\rho}}, [z]_{\overline{\rho}}) \in \leq_{\overline{\rho}}$. Let $j = 1, \ldots, n+1$, $[x]_{\overline{\rho}}, [y]_{\overline{\rho}}$, and $[z_j]_{\overline{\rho}}$ be arbitrary elements in $G/\overline{\rho}$. If $([x]_{\overline{\rho}}, [y]_{\overline{\rho}}) \in \mu'$, then we have $(x, y) \in \mu$. Since μ is a pseudoorder on G, we obtain

$$(\circ(z_1^{i-1}, x, z_{i+1}^{n+1}), \circ(z_1^{i-1}, y, z_{i+1}^{n+1})) \in \mu$$

for all $i = 1, \ldots, n+1$. Then we have

$$([\circ(z_1^{i-1}, x, z_{i+1}^{n+1})]_{\overline{\rho}}, [\circ(z_1^{i-1}, y, z_{i+1}^{n+1})]_{\overline{\rho}}) \in \mu'$$

for every $i = 1, \ldots, n+1$. Hence μ' is a pseudoorder on $G/\overline{\rho}$. Define a mapping $f : A \to B$ by $f(\mu) = \mu'$ for all $\mu \in A$. Clearly, f is well-defined. Indeed: Let $\mu_1, \mu_2 \in A$ be such that $\mu_1 = \mu_2$. If $([x]_{\overline{\rho}}, [y]_{\overline{\rho}}) \in \mu_1'$, then $(x, y) \in \mu_1 = \mu_2$, so $([x]_{\overline{\rho}}, [y]_{\overline{\rho}}) \in \mu_2'$. Thus $\mu_1' \subseteq \mu_2'$. Similarly, $\mu_2' \subseteq \mu_1'$. To show that f is injective, let $\mu_1, \mu_2 \in A$ be such that $f(\mu_1) = f(\mu_2)$. Then $\mu_1' = \mu_2'$. Assume that $(x, y) \in \mu_1$. Then $([x]_{\overline{\rho}}, [y]_{\overline{\rho}}) \in \mu_1' = \mu_2'$, we have $(x, y) \in \mu_2$. So $\mu_1 \subseteq \mu_2$. For $\mu_2 \subseteq \mu_1$, it is similar. The fact that f is surjective follows form this: Let $\delta \in B$. We define a relation μ on S as follows:

$$\mu := \{(x, y) \mid ([x]_{\overline{\rho}}, [y]_{\overline{\rho}}) \in \delta\}.$$

Since δ is a pseudoorder on $G/\overline{\rho}$, we have that μ is a pseudoorder on G. Moreover, $\rho \subseteq \mu$, since, $(x, y) \in \rho$ implies $([x]_{\overline{\rho}}, [y]_{\overline{\rho}}) \in \leq_{\overline{\rho}} \subseteq \delta$ so $(x, y) \in \mu$. We see that $\mu' = \delta$ since $([x]_{\overline{\rho}}, [y]_{\overline{\rho}}) \in \mu' \Leftrightarrow (x, y) \in \mu \Leftrightarrow ([x]_{\overline{\rho}}, [y]_{\overline{\rho}}) \in \delta$. Therefore $f(\mu) = \delta$. The proof of the second part is obvious. □

Now, we present the definition of several kinds of a mapping between any two ordered superassociative algebras.

Definition 6. *If $(G, \circ, \leq_G)$ and $(K, *, \leq_K)$ are ordered superassociative algebras, then a mapping $\alpha : G \to K$ is said to be*

(1) *isotone* if $x, y \in G$, $x \leq_G y$ implies $\alpha(x) \leq_K \alpha(y)$,
(2) *reverse isotone* if $x, y \in G$, $\alpha(x) \leq_K \alpha(y)$ implies $x \leq_G y$,
(3) *homomorphism* if it is isotone and

$$\alpha(\circ(x_1, \ldots, x_{n+1})) = *(\alpha(x_1), \ldots, \alpha(x_{n+1})),$$

for all $x_1, \ldots, x_{n+1} \in G$,
(4) *isomorphism if it is homomorphism, surjective, and reverse isotone. In particular, we say that* G *and* K *are isomorphic and we write* $G \cong K$ *if there is isomorphism from* G *to* K.

Theorem 8. *Let* ρ *be a pseudoorder on an ordered superassociative algebra* $(G, \circ, \leq)$. *Then a mapping* $\rho^\sharp : G \to G/\overline{\rho}$, *defined by* $\rho^\sharp(a) = [a]_{\overline{\rho}}$ *for all* $a \in G$, *is a surjective homomorphism from* G *to* G/ρ.

Proof. Clearly, $\rho^\sharp$ is surjective. Suppose first that $x_1, \ldots, x_{n+1} \in G$. Then we obtain

$$\begin{aligned} \rho^\sharp(\circ(x_1, \ldots, x_{n+1})) &= [\circ(x_1, \ldots, x_{n+1})]_\rho \\ &= \otimes([x_1]_\rho, \ldots, [x_{n+1}]_\rho) \\ &= \otimes(\rho^\sharp(x_1), \ldots, \rho^\sharp(x_{n+1})). \end{aligned}$$

This shows that $\rho^\sharp$ is a surjective homomorphism from G to G/ρ. □

In the semigroup theory, we know that the kernel $ker\alpha$ of a homomorphism α from a semigroup S into another semigroup S is congruence of S, and $S/\ker\alpha$ becomes a quotient semigroup. Analogous to this concept we define the kernel of an ordered superassociative algebra homomorphism, and show that the kernel becomes a congruence relation, moreover, it will be a pseudoorder. Let $(G, \circ, \leq_G)$ and $(K, *, \leq_K)$ be two ordered superassociative algebras and $\alpha : G \to K$ a homomorphism. Define kernel of α by

$$ker\alpha = \{(a, b) \in G \times G \mid \alpha(a) = \alpha(b)\}.$$

Lemma 1. *Let* $(G, \circ, \leq_G)$ *and* $(K, *, \leq_K)$ *be two ordered superassociative algebras and* $\alpha : G \to K$ *a homomorphism. Then* $ker\alpha$ *is congruence on* G.

Proof. It is easy to see that $ker\alpha$ is an equivalence relation on G. Here we show that $ker\alpha$ is compatible with respect to $(n+1)$-operation. Suppose that $(a_j, b_j) \in ker\alpha$ for every $j = 1, \ldots, n+1$. Then $\alpha(a_j) = \alpha(b_j)$ for all $j = 1, \ldots, n+1$. So

$$\begin{aligned} \alpha(\circ(a_1, \ldots, a_{n+1})) &= *(\alpha(a_1), \ldots, \alpha(a_{n+1})) \\ &= *(\alpha(b_1), \ldots, \alpha(b_{n+1})) \\ &= \alpha(\circ(b_1, \ldots, b_{n+1})). \end{aligned}$$

Thus $(\circ(a_1, \ldots, a_{n+1}), \circ(b_1, \ldots, b_{n+1})) \in ker\alpha$, and hence $ker\alpha$ is congruence on G. □

The following lemma will play an essential role for the construction of a pseudoorder on a superassociative algebra induced by homomorphisms of ordered superassociative algebras.

Lemma 2. *Let* α *be a homomorphism from an ordered superassociative algebra* $(G, \circ, \leq_G)$ *to* $(K, *, \leq_K)$. *Then the relation*

$$\tilde{\alpha} = \{(x, y) \in G \times G \mid \alpha(x) \leq_K \alpha(y)\}$$

is a pseudoorder on G. *Furthermore,* $ker\alpha = \overline{\tilde{\alpha}}$.

Proof. Let $(x, y) \in \leq_G$. Since $x \leq_G y$ and α is isotone, we get $\alpha(x) \leq_K \alpha(y)$. This means that $(x, y) \in \tilde{\alpha}$. Let $(x, y) \in \tilde{\alpha}, (y, z) \in \tilde{\alpha}$. Then $\alpha(x) \leq_K \alpha(y)$ and $\alpha(y) \leq_K \alpha(z)$ and thus $\alpha(x) \leq_K \alpha(z)$. It follows that $(x, z) \in \tilde{\alpha}$. Let $z_1, \ldots, z_{n+1}$ be elements in G. Assume that $(x, y) \in \tilde{\alpha}$. Then $\alpha(x) \leq_K \alpha(y)$. Since α is a homomorphism and K is an ordered superassociative algebra, then we have

$$\begin{array}{l} \alpha(\circ(z_1, \ldots, z_{i-1}, x, z_{i+1}, \ldots, z_{n+1})) \\ \quad = \quad *(\alpha(z_1), \ldots, \alpha(z_{i-1}), \alpha(x), \alpha(z_{i+1}), \ldots, \alpha(z_{n+1})) \\ \quad \leq_K \quad *(\alpha(z_1), \ldots, \alpha(z_{i-1}), \alpha(y), \alpha(z_{i+1}), \ldots, \alpha(z_{n+1})) \\ \quad = \quad \alpha(\circ(z_1, \ldots, z_{i-1}, y, z_{i+1}, \ldots, z_{n+1})). \end{array}$$

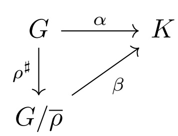
This shows that $\tilde{\alpha}$ is a pseudoorder on G. Finally, let $x, y \in G$. We have

$$\begin{array}{rcl} (x, y) \in ker\alpha & \Leftrightarrow & \alpha(x) = \alpha(y) \\ & \Leftrightarrow & \alpha(x) \leq_K \alpha(y) \text{ and } \alpha(y) \leq_K \alpha(x) \\ & \Leftrightarrow & (x, y) \in \tilde{\alpha} \text{ and } (y, x) \in \tilde{\alpha} \\ & \Leftrightarrow & (x, y) \in \overline{\tilde{\alpha}}. \end{array}$$

The proof is completed. □

The next theorem shows that every quotient ordered superassociative algebra of an ordered superassociative algebra $(G, \circ, \leq_G)$ is a homomorphic image of G. We can remark early that it generalizes the results which mentioned in [15].

Theorem 9. *Let α be homomorphism of an ordered superassociative algebra $(G, \circ, \leq_G$ $)$ into an ordered superassociative algebra $(K, *, \leq_K)$. Let ρ a pseudoorder be such that $\rho \subseteq \tilde{\alpha}$. Then there exists a unigue homomorphism β of $G/\overline{\rho}$ into K such that the following diagram*



is commutative, i.e., $\beta \circ \rho^\sharp = \alpha$. Conversely, if ρ is a pseudoorder on G for which there exists a homomorphism $\beta : G/\overline{\rho} \to K$ such that $\beta \circ \rho^\sharp = \alpha$, then $\rho \subseteq \tilde{\alpha}$.

Proof. For each equivalence class $[a]_{\overline{\rho}}$ of $G/\overline{\rho}$, define a mapping $\beta : G/\overline{\rho} \to K$ by $\beta([a]_{\overline{\rho}}) = \alpha(a)$ where $a \in G$. It is clear that a mapping β is well-defined. Indeed, if $[a]_{\overline{\rho}}, [b]_{\overline{\rho}} \in G/\overline{\rho}$ and $[a]_{\overline{\rho}} = [b]_{\overline{\rho}}$, then $(a, b) \in \overline{\rho}$. So $(a, b) \in \rho$ and $(b, a) \in \rho$. Since $\rho \subseteq \tilde{\alpha}$, we have $(a, b) \in \tilde{\alpha}$ and $(b, a) \in \tilde{\alpha}$. That is, $\alpha(a) \leq_K \alpha(b)$ and $\alpha(b) \leq_K \alpha(a)$. Hence $\alpha(a) = \alpha(b)$. To prove that β is homomorphism, let $[a_1]_{\overline{\rho}}, \ldots, [a_{n+1}]_{\overline{\rho}} \in G/\overline{\rho}$. Then

$$\begin{array}{rcl} \beta(\otimes([a_1]_{\overline{\rho}}, \ldots, [a_{n+1}]_{\overline{\rho}})) & = & \beta([\circ(a_1, \ldots, a_{n+1})]_{\overline{\rho}}) \\ & = & \alpha(\circ(a_1, \ldots, a_{n+1})) \\ & = & *(\alpha(a_1), \ldots, \alpha(a_{n+1})) \\ & = & *(\beta([a_1]_{\overline{\rho}}), \ldots, \beta([a_{n+1}]_{\overline{\rho}})). \end{array}$$

Moreover, assume that $[a]_{\overline{\rho}} \leq_{\overline{\rho}} [b]_{\overline{\rho}}$. Then $(a, b) \in \rho \subseteq \tilde{\alpha}$ and so $\alpha(a) \leq_K \alpha(b)$. To prove that the diagram commutes, let a be an arbitrary element in G. Actually, we obtain $(\beta \circ \rho^\sharp)(a) = \beta([a]_{\overline{\rho}}) = \alpha(a)$. Hence $\beta \circ \rho^\sharp = \alpha$. Assume now that a mapping $\gamma : G/\overline{\rho} \to K$ is a homomorphism satisfying $\gamma \circ \rho^\sharp = \alpha$. If $[x]_{\overline{\rho}} \in G/\overline{\rho}$, then $\gamma([x]_{\overline{\rho}}) = \gamma(\rho^\sharp(x)) = (\gamma \circ \rho^\sharp)(x) = \alpha(x) = (\beta \circ \rho^\sharp) = \beta(\rho^\sharp(x)) = \beta([x]_{\overline{\rho}})$. The uniqueness of β is obtained. For the converse, assume that the conditions hold. Then $\rho \subseteq \tilde{\alpha}$. Indeed: $(a,b) \in \rho \Rightarrow [a]_{\overline{\rho}} \leq_{\overline{\rho}} [b]_{\overline{\rho}} \Rightarrow \beta([a]_{\overline{\rho}}) \leq_K \beta([a]_{\overline{\rho}}) \Rightarrow (\beta \circ \rho^\sharp)(a) \leq_K (\beta \circ \rho^\sharp)(b) \Rightarrow \alpha(a) \leq_K \alpha(b) \Rightarrow (a,b) \in \tilde{\alpha}$. □

Following the result of Theorem 9, we obtain:

Corollary 1. *Let $(G, \circ, \leq_G)$, $(K, *, \leq_K)$ be ordered superassociative algebras, $\alpha : G \to K$ a homomorphism, and ρ a pseudoorder on G. Then $\rho \subseteq \tilde{\alpha}$ if and only if there exists a unique homomorphism $\beta : G/\overline{\rho} \to K$ such that $\beta \circ \rho^\sharp = \alpha$.*

According to the $(n+1)$-superassociative operation on G and K, if we set $n = 1$, then Theorem 9 and Corollary 1 coincide with he results of ordered semigroups occurring in the paper of N. Kehayopulu and M. Tsingelis [15].

We have the following corollaries, which can be proved independently.

Corollary 2. *If G and K are ordered superassociative algebras, $\alpha : G \to K$ a homomorphism and reverse isotone mapping of G into K, then $G \cong im\alpha$.*

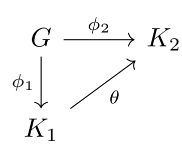
Proof. The proof is clear. □

Corollary 3. *Let G and K be ordered superassociative algebras and $\alpha : G \to K$ a homomorphism. Then $G/\ker\alpha \cong im\alpha$.*

Proof. From Theorem 9, put $\rho = \tilde{\alpha}$. This means that $\tilde{\alpha} \subseteq \tilde{\alpha}$. Suppose that there is a mapping $\beta : G/\tilde{\alpha} \to K$ defined by $\beta([a]_{\overline{\rho}}) = \alpha(a)$ for all $a \in G$ in which β is homomorphism. We show that β is reverse isotone, let $a, b \in G$ be such that $\beta([a]_{\overline{\tilde{\alpha}}}) \leq_K \beta([b]_{\overline{\tilde{\alpha}}})$. Then by Lemma 2, we have $(a,b) \in \tilde{\alpha}$ and thus $([a]_{\overline{\tilde{\alpha}}}, [b]_{\overline{\tilde{\alpha}}}) \in \leq_{\overline{\tilde{\alpha}}}$ since $\tilde{\alpha}$ is a pseudoorder on G. By Corollary 2, we obtain $G/\tilde{\alpha} \cong im\beta$. It is not difficult to prove that $im\beta = im\alpha$. In fact, we let $u \in im\beta$, then there exists $[a]_\rho \in G/\rho$ such that $\beta([a]_\rho) = u$. From $\beta \circ \rho^\sharp = \alpha$, it implies that $\alpha(a) = u$ and so $u \in im\alpha$. The converse is valid. Again by applying Lemma 2, we have $\overline{\tilde{\alpha}} = ker\alpha$. Therefore, $G/\ker\alpha \cong im\alpha$. □

Any two homomorphisms of ordered superassociative algebras can be induced a homomorphism under certain conditions by the following.

Theorem 10. *Let $(G, \circ, \leq_G), (K_1, *_1, \leq_{K_1})$ and $(K_2, *_2, \leq_{K_2})$ be ordered superassociative algebras, ϕ_1 and ϕ_2 homomorphisms from G to K_1 and G to K_2, respectively. If ϕ_1 is surjective and $\tilde{\phi_1} \subseteq \tilde{\phi_2}$, then there exists a unique homomorphism θ from K_1 to K_2 such that the following diagram commutes*



i.e., $\theta \circ \phi_1 = \phi_2$.

Proof. Let k be an arbitrary element in K_1. Since ϕ_1 is surjective, there exists $a_k \in G$ such that $\phi_1(a_k) = k$. Define $\theta : K_1 \to K_2$ by $\theta(k) = \phi_2(a_k)$. This is well-defined, if $x, y \in K_1$ and $x = y$. Then $\phi_1(a_x) = \phi_1(a_y)$ so that $\phi_1(a_x) \leq_{K_1} \phi_1(a_y)$ and $\phi_1(a_y) \leq_{K_1} \phi_1(a_x)$. Since $\tilde{\phi_1} \subseteq \tilde{\phi_2}$, we have $(a_x, a_y) \in \tilde{\phi_1} \subseteq \tilde{\phi_2}$ and $(a_y, a_x) \in \tilde{\phi_1} \subseteq \tilde{\phi_2}$. Then $\phi_2(a_x) \leq_{K_2} \phi_2(a_y)$ and $\phi_2(a_y) \leq_{K_2} \phi_2(a_x)$. These imply that $\phi_2(a_x) = \phi_2(a_y)$ and so $\theta(x) = \theta(y)$. It is obvious that $\theta \circ \phi_1 = \phi_2$ since

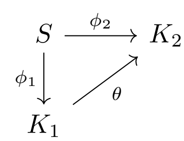
$(\theta\circ\phi_1)(a)=\theta(\phi_1(a))=\theta(a_1)=\phi_2(a)$. Next, let $k_j\in K_1$ for every $j=1,\dots,n+1$. Then we have

$$\begin{aligned}\theta(*_1(k_1,\dots,k_{n+1})) &= \theta(*_1(\phi_1(a_{k_1}),\dots,\phi_1(a_{k_{n+1}})))\\ &= \theta(\phi_1(\circ(a_{k_1},,\dots,a_{k_{n+1}})))\\ &= \phi_2(\circ(a_{k_1},,\dots,a_{k_{n+1}}))\\ &= *_2(\phi_2(a_{k_1}),\dots,\phi_2(a_{k_{n+1}}))\\ &= *_2(\theta(\phi_1(a_{k_1})),\dots,\theta(\phi_1(a_{k_{n+1}})))\\ &= *_2(\theta(k_1),\dots,\theta(k_{n+1})).\end{aligned}$$

To prove that θ is isotone, let $x,y\in K_1$ be such that $x\leq_{K_1} y$. Since ϕ_1 is isotone, then we have $\phi_1(a_x)\leq_{K_2}\phi_1(a_y)$. This means that $(a_x,a_y)\in\tilde{\phi_1}$. Since $\tilde{\phi_1}\subseteq\tilde{\phi_2}$, $(a_x,a_y)\in\tilde{\phi_2}$ and thus $\phi_2(a_x)\leq_{K_2}\phi_2(a_y)$. Hence $\theta(x)\leq_{K_2}\theta(y)$. Finally, suppose that $\eta:K_1\to K_2$ satisfies $\eta\circ\phi_1=\phi_2$. Then $\eta(k)=\eta(\phi_1(a_k))=(\eta\circ\phi_1)(a_k)=\phi_2(a_k)=(\theta\circ\phi_1)(a_k)=\theta(\phi_1(a_k))=\theta(k)$ for all $k\in K_1$. □

In case $n=1$, then Theorem 10 reduce to the situation for ordered semigroups as follows:

Corollary 4. *Let ϕ_1 and ϕ_2 be homomorphisms from an ordered semigroup $(S,\cdot,\leq)$ to ordered semigroups $(K_1,*_1,\leq_{K_1})$ and $(K_2,*_2,\leq_{K_2})$, respectively, such that ϕ_1 is surjective and $\tilde{\phi_1}\subseteq\tilde{\phi_2}$. Then there exists a unique homomorphism θ of K_1 and K_2 such that the following diagram commutes*



i.e., $\theta\circ\phi_1=\phi_2$.

Let ρ and σ be pseudoorders on an ordered superassociative algebra $(G,\circ,\leq_G)$. Then we define a binary relation σ/ρ on $G/\overline{\rho}$ by

$$\sigma/\rho:=\{([a]_{\overline{\rho}},[b]_{\overline{\rho}})\mid \exists x\in[a]_{\overline{\rho}},\exists y\in[b]_{\overline{\rho}},(x,y)\in\sigma\}.$$

It is observed that

$$([a]_{\overline{\rho}},[b]_{\overline{\rho}})\in\sigma/\rho\Leftrightarrow\forall x\in[a]_{\overline{\rho}},\forall y\in[b]_{\overline{\rho}},(x,y)\in\sigma\Leftrightarrow(a,b)\in\sigma.$$

Using a relation on the quotient set which defined previously, one can prove suddenly that it becomes a pseudoorder. Then the last theorem is provided.

Theorem 11. *Let G be an ordered superassociative algebra, ρ,σ pseudoorders on G with $\rho\subseteq\sigma$. Then the following two assertions hold:*

1. *σ/ρ is a pseudoorder on $G/\overline{\rho}$,*
2. *$(G/\overline{\rho})/(\overline{\sigma/\rho})\cong G/\overline{\sigma}$.*

Proof. First, we prove that σ/ρ is a pseudoorder on $G/\overline{\rho}$ with respect to the multiplication $\otimes$ and the order $\leq_{\overline{\rho}}$. Suppose that $([x]_{\overline{\rho}},[y]_{\overline{\rho}})\in\leq_{\overline{\rho}}$. Then $(x,y)\in\rho\subseteq\sigma$ and so $([x]_{\overline{\rho}},[y]_{\overline{\rho}})\in\sigma/\rho$. Let $([x]_{\overline{\rho}},[y]_{\overline{\rho}})\in\sigma/\rho$ and $([y]_{\overline{\rho}},[z]_{\overline{\rho}})\in\sigma/\rho$. Then $(x,y)\in\sigma$ and $(y,x)\in\sigma$. So $(x,z)\in\sigma$. This means that $([x]_{\overline{\rho}},[z]_{\overline{\rho}})\in\sigma/\rho$. For every $j=1,\dots,n+1$, let $a_j\in G$ and $([x]_{\overline{\rho}},[y]_{\overline{\rho}})\in\sigma/\rho$. Since $(x,y)\in\sigma$ and σ is a pseudoorder on G, we have $(\circ(a_1,\dots,a_{i-1},x,a_{i+1},\dots,a_{n+1}),\circ(a_1,\dots,a_{i-1},y,a_{i+1},\dots,a_{n+1}))\in\sigma$. Hence $([\circ(a_1,\dots,a_{i-1},x,a_{i+1},\dots,a_{n+1})]_{\overline{\rho}},[\circ(a_1,\dots,a_{i-1},y,a_{i+1},\dots,a_{n+1})]_{\overline{\rho}})\in$

σ/ρ. Now, it comes to the second part of the proof for this theorem. Form now on, we write ordered superassociative algebras $(G/\overline{\rho}, \otimes_{\overline{\rho}}, \leq_{\overline{\rho}})$ and $(G/\overline{\sigma}, \otimes_{\overline{\sigma}}, \leq_{\overline{\sigma}})$ instead of $G/\overline{\rho}$ and $G/\overline{\sigma}$ without multiplications and orders, respectively. Define a mapping $\alpha : G/\overline{\rho} \to G/\overline{\sigma}$ by $\alpha([x]_{\overline{\rho}}) = [x]_{\overline{\sigma}}$ for all $[x]_{\overline{\rho}} \in G/\overline{\rho}$. To ensure that this above defining is well-defined, let $x, y \in G$ be such that $[x]_{\overline{\rho}} = [y]_{\overline{\rho}}$. Then $(x, y) \in \rho \subseteq \sigma$ and $(y, x) \in \rho \subseteq \sigma$. So $(x, y) \in \overline{\sigma}$ and that $[x]_{\overline{\sigma}} = [y]_{\overline{\sigma}}$. To show that α is homomorphism, let $[x_j]_{\overline{\rho}} \in G/\overline{\rho}$ for every $j = 1, \ldots, n+1$. Then we have

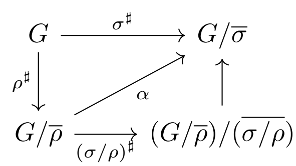
$$\begin{aligned}
\alpha(\otimes_{\overline{\rho}}([x_1]_{\overline{\rho}}, \ldots, [x_{n+1}]_{\overline{\rho}})) &= \alpha([\circ(x_1, \ldots, x_{n+1})]_{\overline{\rho}}) \\
&= [\circ(x_1, \ldots, x_{n+1})]_{\overline{\sigma}} \\
&= \otimes_{\overline{\sigma}}([x_1]_{\overline{\sigma}}, \ldots, [x_{n+1}]_{\overline{\sigma}}) \\
&= \otimes_{\overline{\sigma}}(\alpha([x_1]_{\overline{\rho}}), \ldots, \alpha([x_{n+1}]_{\overline{\rho}})).
\end{aligned}$$

Moreover, α is isotone, since, for all $[x]_{\overline{\rho}}, [y]_{\overline{\rho}} \in G/\overline{\rho}$, we have $[x]_{\overline{\rho}} \leq_{\overline{\rho}} [y]_{\overline{\rho}}$. Then $(x, y) \in \rho \subseteq \sigma$ and so $[x]_{\overline{\sigma}} \leq_{\overline{\sigma}} [y]_{\overline{\sigma}}$. By Corollary 3, $(G/\overline{\rho})/ker\alpha \cong im\alpha$. Because $\tilde{\alpha} = \{([x]_{\overline{\rho}}, [y]_{\overline{\rho}}) \mid \alpha([x]_{\overline{\rho}}) \leq_{\overline{\sigma}} \alpha([y]_{\overline{\rho}})\}$, then we have

$$\begin{aligned}
([x]_{\overline{\rho}}, [y]_{\overline{\rho}}) \in \tilde{\alpha} &\Leftrightarrow \alpha([x]_{\overline{\rho}}) \leq_{\overline{\sigma}} \alpha([y]_{\overline{\rho}}) \\
&\Leftrightarrow [x]_{\overline{\sigma}} \leq_{\overline{\sigma}} [x]_{\overline{\sigma}} \\
&\Leftrightarrow (x, y) \in \sigma \\
&\Leftrightarrow ([x]_{\overline{\rho}}, [y]_{\overline{\rho}}) \in \sigma/\rho.
\end{aligned}$$

This shows that $\tilde{\alpha} = \sigma/\rho$. It turns out by Lemma 2 that $ker\alpha = \overline{\tilde{\alpha}} = \overline{\sigma/\rho}$. Obviously, $im\alpha = G/\overline{\sigma}$. Indeed: $im\alpha = \{\alpha([x]_{\overline{\rho}}) \mid x \in G\} = \{[x]_{\overline{\sigma}} \mid x \in G\} = G/\overline{\sigma}$. Therefore, $(G/\overline{\rho})/(\overline{\sigma/\rho}) \cong G/\overline{\sigma}$. The proof is completely finished. □

The relationship between the quotient ordered superassociative algebras is presented by the following diagram.



As an immediate consequence of Theorem 11, we have the following generalization:

Corollary 5. *Let G be ordered superassociative algebra and let $\rho_1, \rho_2, \ldots, \rho_{m+1}$ be pseudoorders on G such that $\rho_1 \subseteq \rho_2 \subseteq \ldots \subseteq \rho_{m+1}$. Then for each $i = 1, \ldots, m$, the relation*

$$\rho_{i+1}/\rho_i = \{([a]_{\overline{\rho_i}}, [b]_{\overline{\rho_i}}) \in G/\overline{\rho_i} \times G/\overline{\rho_i} \mid \exists x \in [a]_{\overline{\rho_i}}, \exists y \in [b]_{\overline{\rho_i}}, (x, y) \in \rho_{i+1}\}$$

is a pseudoorder on $G/\overline{\rho_i}$ and

$$(G/\overline{\rho_i})/(\overline{\rho_{i+1}/\rho_i}) \cong G/\overline{\rho_{i+1}}.$$

We note that if $m = 1$, then Corollary 5 and Theorem 11 are identical. Particularly, the second isomorphism theorem for ordered semigroups is a partucular case of this result if $n = m = 1$.

3. Discussion

The paper was established to investigate a new algebraic structure named as ordered superassociative algebras. It turns out that every ordered superassociative algebra can be reduced to arbitrary ordered semigroups in a natural way. To investigate the construction of quotient ordered superassociative algebras in sense of their equlvalence classes, it is essential to have the notion of a pseudoorder. The concept of i-compatible, congruences and pseudoorders of ordered superassociative algebras that we studied supports the concept of left and right congruence and pseudoorders of ordered semigroups in case $n = 1$. We also made an attention to define the homomorphisms and isomorphisms between any two ordered superassociative algebras in analog with those of ordinary ordered semigroups and discussed several connections among the structures. Also, we constructed the isomorphism theorems for ordered superassociative algebras. Finally, we list below some open problems and topics of current work.

(1) Based on the concept of several kinds of ideals on ordered semigroups, find a generalization of ideals on ordered superassociative algebras.
(2) Try to define $(n+1)$-pseudoorders instead of binary pseudoorders for ordered superassociative algebras. Study their fundamental and advanced properties.

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