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BLOW-UP ANALYSIS FOR A CLASS OF PLATE VISCOELASTIC $p(x)-$KIRCHHOFF TYPE INVERSE SOURCE PROBLEM WITH VARIABLE-EXPONENT NONLINEARITIES

M. SHAHROUZI, J. FERREIRA, E. PIŞKIN, AND N. BOUMAZA

Abstract. In this work, we study the blow-up analysis for a class of plate viscoelastic $p(x)$-Kirchhoff type inverse source problem of the form:

$$u_{tt} + \Delta^2 u - \left(a + b\int_\Omega \frac{1}{p(x)}|\nabla u|^{p(x)}dx\right)\Delta_{p(x)}u - \int_0^t g(t-\tau)\Delta^2 u(\tau)d\tau$$
$$+ \beta|u_t|^{m(x)-2}u_t = \alpha|u|^{q(x)-2}u + f(t)\omega(x).$$

Under suitable conditions on kernel of the memory, initial data and variable exponents, we prove the blow up of solutions in two cases: linear damping term ($m(x) \equiv 2$) and nonlinear damping term ($m(x) > 2$). Precisely, we show that the solutions with positive initial energy blow up in a finite time when $m(x) \equiv 2$ and blow up at infinity if $m(x) > 2$.

Keywords: inverse source problem, blow-up, viscoelastic, p(x)-Kirchhoff type equation.

Corresponding author: Mohammad Shahrouzi.

1. Introduction

In this paper, we consider the following plate viscoelastic $p(x)-$Kirchhoff type inverse source problem:

$$\begin{aligned} & u_{tt}+\Delta^2 u-\left(a+b\int_\Omega \frac{1}{p(x)}|\nabla u|^{p(x)}dx\right)\Delta_{p(x)}u-\int_0^t g(t-\tau)\Delta^2 u(\tau)d\tau \\ & +\beta|u_t|^{m(x)-2}u_t=\alpha|u|^{q(x)-2}u+f(t)\omega(x), \quad (x,t)\in\Omega\times(0,+\infty) \end{aligned} \tag{1}$$

$$u(x,t)=\frac{\partial u}{\partial \nu}(x,t)=0, \qquad (x,t)\in\partial\Omega\times(0,+\infty) \tag{2}$$

$$u(x,0)=u_0(x), \quad u_t(x,0)=u_1(x), \qquad x\in\Omega \tag{3}$$

$$\int_\Omega u(x,t)\omega(x)dx=\phi(t), \qquad t>0 \tag{4}$$

while the pair of functions $\{u(x,t),f(t)\}$ are unknown. In this problem, $\Omega\subset R^n$ $(n\geq 1)$ is a bounded domain with smooth boundary $\partial\Omega$ and a unit outer normal ν. Here, $\Delta_{p(x)}$ is called $p(x)-$Laplace operator defined as

$$\Delta_{p(x)}u=div(|\nabla u|^{p(x)-2}\nabla u),$$

and $a,b>0$. Also, α and β are positive constants and $g(t)$, $\omega(x)$ and $\phi(t)$ are real valued functions with specific conditions that will be enunciated later.
In addition, $p(x)$, $m(x)$ and $q(x)$ are given continuous and measurable functions on $\overline{\Omega}$ such that

$$\begin{aligned} & 2<p^-\leq p(x)\leq p^+<\infty \\ & 2\leq m^-\leq m(x)\leq m^+<\infty \\ & 2<q^-\leq q(x)\leq q^+<\infty, \end{aligned} \tag{5}$$

with

$$\begin{aligned} & p^-:=essinf_{x\in\overline{\Omega}}p(x), \quad p^+:=esssup_{x\in\overline{\Omega}}p(x) \\ & m^-:=essinf_{x\in\overline{\Omega}}m(x), \quad m^+:=esssup_{x\in\overline{\Omega}}m(x) \\ & q^-:=essinf_{x\in\overline{\Omega}}q(x), \quad q^+:=esssup_{x\in\overline{\Omega}}q(x). \end{aligned}$$

The inverse source problems in waves arise in many scientific and industrial areas such as antenna design and synthesis, biomedical imaging and photo-acoustic tomography [5]. Solving the inverse problems are rather difficult, because they are nonlinear and improperly posed. It is known that there is no uniqueness for the inverse source problem at a fixed frequency due to the existence of non-radiating sources [6]. Therefore, additional information is required for the source in order to obtain a unique solution such as (4) and

$$\omega\in H_0^2(\Omega)\cap L^{p(.)}(\Omega)\cap L^{m(.)}(\Omega)\cap L^{q(.)}(\Omega), \quad \int_\Omega \omega^2(x)dx=1. \tag{6}$$

To the best of our knowledge, the stability and blow up of solutions of inverse source problems with variable-exponent nonlinearities are less investigated area. In this paper, we are going to extend previous results in the inverse problems with constant-exponent nonlinearities to our inverse source problem (1)-(4) with variable-exponent nonlinearities. Thus, firstly we point out some previous results in the inverse problems with constant-exponent nonlinearities. For example, Eden and Kalantarov [9] studied the following inverse problem

$$\begin{aligned}
&u_t - \Delta u + b(x,t,u,\nabla u) - |u|^p u = F(t)\omega(x), \quad x \in \Omega,\ t > 0\\
&u(x,t) = 0, \quad x \in \partial\Omega,\ t > 0\\
&u(x,0) = u_0(x), \quad x \in \Omega\\
&\int_\Omega u(x,t)\omega(x)dx = \phi(t), \quad t > 0.
\end{aligned}$$

They found conditions on data which guaranteed the global nonexistence of solutions when $\phi(t) \equiv 1$. Also, authors established a stability result with the opposite sign on the power type nonlinearity and $b(x,t,u,\nabla u) \equiv 0$. Next, Tahamtani and Shahrouzi [31] extended previous results to a Petrovsky inverse source problem (see also [32]). Shahrouzi in [23] studied the following damped viscoelastic inverse problem

$$\begin{aligned}
&u_{tt} - \nabla[(a_0 + a|\nabla u|^m)\nabla u] + \int_0^t e^{\lambda(t-\tau)} g(t-\tau)\Delta u(\tau)d\tau + bu_t = h(x,t,u,\nabla u)\\
&\qquad + |u|^p u + f(t)\omega(x), \quad x \in \Omega,\ t > 0\\
&u(x,t) = 0, \quad x \in \Gamma,\ t > 0\\
&u(x,0) = u_0(x), \quad x \in \Omega\\
&\int_\Omega u(x,t)\omega(x)dx = 1, \quad t > 0,
\end{aligned}$$

and proved the blow up of solutions under sufficient conditions on initial functions by using the modified concavity argument. See [24, 25, 26].

On the other hand, it is known that modeling of some physical phenomena such as flows of electro-rheological fluids, nonlinear viscoelasticity and image processing give rise to equations with nonstandard growth conditions, i.e, equations with variable exponents of nonlinearities. In direct problems, equations with nonlinearities of variable-exponent type have largely been discussed by several authors. For instance, Antontsev [1] considered the equation:

$$u_{tt} = div\big(a(x,t)|\nabla u|^{p(x,t)-2}\nabla u\big) + \alpha\Delta u_t + b(x,t)u|u|^{\sigma(x,t)-2} + f(x,t)$$

in $\Omega \subseteq R^n$, where $\alpha > 0$ is a constant and a, b, p, σ are given functions. For specific conditions on a, b, p, σ, the existence theorems for small and any finite time have been proved and blow up of solutions under some suitable conditions on data has been established. Messaoudi and Talahmeh [17], considered the following nonlinear equation with variable exponents:

$$u_{tt} - div\big(|\nabla u|^{r(.)-2}\nabla u\big) + a|u_t|^{m(.)-2}u_t = b|u|^{p(.)-2}u. \tag{7}$$

They proved a finite-time blow-up result for the solutions with negative initial energy and also certain solutions with positive energy in appropriate range of $m(.), r(.)$ and $p(.)$. In another study, Messaoudi [18] studied equation (7) with $a = 1,\ b = 0$ in the presence of damping term $-\Delta u_t$. He proved several decay results depending on the range of variable exponents m and r. Shahrouzi [27] studied the behavior of solutions to the following initial-boundary value problem with variable-exponent nonlinearities

$$u_{tt} - \Delta u - div(|\nabla u|^{m(x)}\nabla u) + \int_0^t g(t-\tau)\Delta u(\tau)d\tau + h(x,t,u,\nabla u) + \beta u_t$$
$$= |u|^{p(x)}u, \ in \ \Omega \times (0,+\infty)$$
$$\begin{cases} u(x,t) = 0, & x \in \Gamma_0, t > 0 \\ \frac{\partial u}{\partial n}(x,t) = \int_0^t g(t-\tau)\frac{\partial u}{\partial n}(\tau)d\tau - |\nabla u|^{m(x)}\frac{\partial u}{\partial n} + \alpha u, & x \in \Gamma_1, t > 0 \end{cases}$$
$$u(x,0) = u_0(x), \quad u_t(x,0) = u_1(x), \quad in \quad \Omega.$$

Under appropriate conditions, he proved a general decay result associated to solution energy. Moreover, regarding arbitrary positive initial energy, blow up of solutions has been proved. Antontsev and Ferreira [2], studied a nonlinear class viscoelastic plate equation with a lower order by perturbation of $\vec{p}(x,t)$-Laplace operator of the form

$$u_{tt} + \Delta^2 u - \Delta_{\vec{p}(x,t)}u + \int_0^t g(t-s)\Delta u(s)ds - \varepsilon\Delta u_t + f(u) = 0,$$

associated with initial and Dirichlet-Neumann boundary conditions. Here, $\Delta_{\vec{p}(x,t)}$ is the $\vec{p}(x,t)$-Laplace operator which is defined as

$$\Delta_{\vec{p}(x,t)}u = \sum_{i=1}^{n} \frac{\partial}{\partial x_i}\big(|\frac{\partial u}{\partial x_i}|^{p_i(x,t)-2}\frac{\partial u}{\partial x_i}\big), \ \vec{p}(x,t) = (p_1, p_2, \cdots, p_n).$$

They proved a blow up in finite time with negative initial energy under suitable conditions on g, f and the variable exponent of the $\vec{p}(x,t)$-Laplace operator. Recently, Antontsev et al. [3] looked into the following nonlinear Timoshenko equation with variable exponents:

$$u_{tt} + \Delta^2 u - M(\|\nabla u\|^2_{L^2(\Omega)})\Delta u + |u_t|^{p(x)-2}u_t = |u|^{q(x)-2}u,$$

and demonstrated the local existence of the solution under suitable conditions. Moreover, nonexistence of solutions was proved with negative initial energy (see also [4]).

Dai and Hao [7] studied the following equation

$$-M\big(\int_\Omega \frac{1}{p(x)}|\nabla u|^{p(x)}dx\big)div(|\nabla u|^{p(x)-2}\nabla u) = f(x,u),$$

and by means of a direct variational approach and the theory of the variable-exponent Sobolev spaces, they established conditions through which the existence and multiplicity of solutions for the problem were verified. In another study, Hamdani et al. [15] investigated the following nonlocal $p(x)-$Kirchhoff type equation

$$-(a - b\int_\Omega \frac{1}{p(x)}|\nabla u|^{p(x)}dx)div(|\nabla u|^{p(x)-2}\nabla u) = \lambda|u|^{p(x)-2}u + g(x,u),$$

and obtained a nontrivial weak solution by using the Mountain Pass theorem. Related to the inverse problems with variable exponent nonlinearities, Shahrouzi in [28] studied the general decay and blow up of solutions for the following Lamé system of inverse problem

$$\begin{aligned} &u_{tt} - \Delta_e u - div\big(|\nabla u|^{r(x)-2}\nabla u\big) + \beta u_t + h(x,t,u,\nabla u) + a|u_t|^{m(x)-2}u_t \\ &\quad = b|u|^{p(x)-2}u + f(t)\omega(x),\ (x,t) \in \Omega \times (0,\infty) \\ &u(x,t) = \frac{\partial u}{\partial \nu}(x,t) = 0,\ (x,t) \in \partial\Omega \times (0,\infty) \\ &u(x,0) = u_0(x), \quad u_t(x,0) = u_1(x),\ x \in \Omega \\ &\int_\Omega u(x,t)\omega(x)dx = \phi(t), \quad t > 0. \end{aligned}$$

The author proved the general decay of solutions when $b = 0$, $h(x,t,u,\nabla u) \equiv 0$ and the integral overdetermination tends to zero as time goes to infinity in appropriate range of variable exponents. Furthermore, in the absence of damping terms ($a = \beta = 0$) and when $\phi(t) \equiv 1$, blow up of solutions in a finite time has been proved. The relevant equations with variable-exponent nonlinearities have also been studied in [20, 22, 29, 30, 16, 14, 21, 19].

Motivated by the aforementioned works, in the present paper, we study the blow-up analysis for a class of fourth-order viscoelastic $p(x)-$Kirchhoff type inverse source problem with variable-exponent nonlinearities. We mentioned before, existence of variable-exponent nonlinearities makes the study of inverse problems difficult. However, we try to extend and improve the previous results ([24, 25, 28]) to a class of plate viscoelastic $p(x)-$Kirchhoff type inverse problems with variable-exponent nonlinearities. To the best of our knowledge, this is the first work dealing with the blow-up result for a plate viscoelastic $p(x)-$Kirchhoff type inverse source problem subject to the variable-exponent nonlinearities and various damping terms.

The rest of the paper is organized as follows. In Section 2, we recall some definitions and Lemmas about the variable-exponent Lebesgue space, $L^{p(.)}(\Omega)$, the Sobolev space, $W^{1,p(.)}(\Omega)$ and additional conditions to be used for the main results. Section 3 includes two parts. First, we prove that the solutions of (1)-(4) blow-up in a finite time with suitable conditions on initial data and variable exponents when $m(x) \equiv 2$. Next, in the second part, we show that for $m(x) \geq m^- > 2$ and under appropriate conditions on data, the solutions of (1)-(4) blow up at infinity.

2. Preliminaries

In this section, we recall some notations and functionals. We denote by $\|.\|_q$ the L^q-norm over Ω and in particular, the L^2-norm is denoted $\|.\|$ in Ω. We shall assume that the functions $g(t)$, $\omega(x)$ and those appearing in the data satisfy the following conditions:

$$g(t) \geq 0, \quad g'(t) \leq 0, \quad 1 - \int_0^\infty g(t)dt = l > 0, \tag{8}$$

and

$$\begin{aligned} &u_0 \in H_0^2(\Omega) \cap L^{p(.)}(\Omega) \cap L^{m(.)}(\Omega), \qquad u_1 \in L^2(\Omega) \cap L^{q(.)}(\Omega), \\ &\int_\Omega u_0(x)\omega(x)dx = \phi(0). \end{aligned} \tag{9}$$

In order to study problem (1)-(4), we need some hypotheses and theories about Lebesgue and Sobolev spaces with variable-exponents (for details, see [8, 10, 11, 12, 13]). Let $p(x) \geq 1$ and measurable, we assume that

$$C_+(\overline{\Omega}) = \{h | h \in C(\overline{\Omega}),\ h(x) > 1 \ for\ any\ x \in \overline{\Omega}\},$$

$$h^+ = \max_{\overline{\Omega}} h(x), \quad h^- = \min_{\overline{\Omega}} h(x) \quad for\ any\ h \in C(\overline{\Omega}),$$

$$L^{p(x)}(\Omega) = \Big\{u |\ \text{u is a measurable real-valued function,}\ \int_\Omega |u(x)|^{p(x)} dx < \infty\Big\}.$$

We equip the Lebesgue space with a variable exponent, $L^{p(x)}(\Omega)$, with the following Luxembourg-type norm

$$\|u\|_{p(x)} := \inf\Big\{\lambda > 0 \Big| \int_\Omega |\frac{u(x)}{\lambda}|^{p(x)} dx \leq 1\Big\}$$

Lemma 1. [8, 13] *Let Ω be a bounded domain in R^n*
(i) the space $(L^{p(x)}(\Omega), \|.\|_{p(x)})$ is a Banach space, and its conjugate space is $L^{q(x)}(\Omega)$, where $\frac{1}{q(x)} + \frac{1}{p(x)} = 1$. For any $u \in L^{p(x)}(\Omega)$ and $v \in L^{q(x)}(\Omega)$, we have

$$|\int_\Omega uv dx| \leq (\frac{1}{p^-} + \frac{1}{q^-}) \|u\|_{p(x)} \|v\|_{q(x)};$$

(ii) If $p, q \in C_+(\overline{\Omega})$, $q(x) \leq p(x)$ for any $x \in \overline{\Omega}$, then $L^{p(x)}(\Omega) \hookrightarrow L^{q(x)}(\Omega)$, and the imbedding is continuous.

The variable-exponent Lebesgue Sobolev space $W^{1,p(x)}(\Omega)$ is defined by

$$W^{1,p(x)}(\Omega) = \{u \in L^{p(x)}(\Omega) | \nabla u \ exists\ and\ |\nabla u| \in L^{p(x)}(\Omega)\}.$$

This space is a Banach space with respect to the norm $\|u\|_{W^{1,p(x)}(\Omega)} = \|u\|_{p(x)} + \|\nabla u\|_{p(x)}$. Furthermore, let $W_0^{1,p(x)}(\Omega)$ be the closure of $C_0^\infty(\Omega)$ in $W^{1,p(x)}(\Omega)$. The dual of $W_0^{1,p(x)}(\Omega)$ is defined as $W^{-1,p'(x)}(\Omega)$, by the same way as the usual Sobolev spaces, where $\frac{1}{p(x)} + \frac{1}{p'(x)} = 1$.
If we define

$$p^*(x) = \begin{cases} \frac{Np(x)}{N-p(x)}, & p^+ < N \\ \infty, & p^+ \geq N, \end{cases}$$

then we have

Lemma 2. [8, 13] *Let Ω be a bounded domain in R^n. Then for any measurable bounded exponent $p(x)$ we have*
(i) $W^{1,p(x)}(\Omega)$ and $W_0^{1,p(x)}(\Omega)$ are separable Banach spaces;
(ii) if $q \in C_+(\overline{\Omega})$ and $q(x) < p^*(x)$ for any $x \in \overline{\Omega}$, then the imbedding $W^{1,p(x)}(\Omega) \hookrightarrow L^{q(x)}(\Omega)$ is compact and continuous;
(iii) if $p(x)$ is uniformly continuous in Ω, then there exists a constant $C > 0$, such that

$$\|u\|_{p(x)} \leq C \|\nabla u\|_{p(x)} \qquad \forall u \in W_0^{1,p(x)}(\Omega).$$

By (iii) of Lemma 2, we know that the space $W_0^{1,p(x)}(\Omega)$ has an equivalent norm given by $\|u\|_{W^{1,p(x)}(\Omega)} = \|\nabla u\|_{p(x)}$.
We recall the Young's inequality

$$ab \leq \theta a^{q(x)} + C(\theta, q(x)) b^{q'(x)}, \quad a, b \geq 0, \quad \beta > 0, \quad \frac{1}{q(x)} + \frac{1}{q'(x)} = 1, \tag{10}$$

where $C(\theta, q(x)) = \frac{1}{q'(x)}(\theta q(x))^{-\frac{q'(x)}{q(x)}}$. In special case when $\theta = \frac{1}{q(x)}$, we have from (10)

$$ab \leq \frac{a^{q(x)}}{q(x)} + \frac{b^{q'(x)}}{q'(x)}. \tag{11}$$

Adapting the conditions (6) and integral over-determination (4), by multiplying equation (1) in $\omega(x)$, the key observation is that the problem (1)-(4) is equivalent to the following direct problem

$$\begin{aligned} &u_{tt} + \Delta^2 u - \left(a + b\int_\Omega \frac{1}{p(x)}|\nabla u|^{p(x)}dx\right)\Delta_{p(x)}u - \int_0^t g(t-\tau)\Delta^2 u(\tau)d\tau \\ &\quad + \beta|u_t|^{m(x)-2}u_t = \alpha|u|^{q(x)-2}u + f(t)\omega(x),\ (x,t) \in \Omega \times (0,+\infty) \end{aligned} \tag{12}$$

$$u(x,t) = \frac{\partial u}{\partial \nu}(x,t) = 0,\ (x,t) \in \partial\Omega \times (0,+\infty) \tag{13}$$

$$u(x,0) = u_0(x), \quad u_t(x,0) = u_1(x),\ x \in \Omega \tag{14}$$

in which the unknown function $f(t)$ is replaced by

$$\begin{aligned} f(t) \quad = \quad & \phi''(t) + \int_\Omega \Delta u \Delta\omega(x)dx + \beta\int_\Omega |u_t|^{m(x)-1}\omega(x)dx \\ & + (a + b\int_\Omega \frac{1}{p(x)}|\nabla u|^{p(x)}dx)\int_\Omega |\nabla u|^{p(x)-1}\nabla\omega(x)dx \\ & - \int_0^t g(t-\tau)\int_\Omega (\Delta u(\tau) - \Delta u)\Delta\omega(x)dxd\tau \\ & - \int_0^t g(t-\tau)\int_\Omega \Delta u\Delta\omega(x)dxd\tau - \alpha\int_\Omega |u|^{q(x)-1}\omega(x)dx. \end{aligned} \tag{15}$$

At this point, we state the local existence of solutions for the problem (12)-(14), that can be established employing the Galerkin method as in [1].

Theorem 1. *(Local existence) Let $u_0 \in W_0^{1,p(.)}(\Omega)$, $u_1 \in L^2(\Omega)$ and assume that (6), (8) and (9) be satisfied. Then problem (12)-(14) has a unique weak solution such that*

$$\begin{aligned} &u \in L^\infty\Big((0,T), W_0^{1,p(.)}(\Omega)\Big) \cap L^{q(.)}((0,T),\Omega), \\ &u_t \in L^\infty\Big((0,T), L^2(\Omega)\Big) \cap L^{m(.)}((0,T),\Omega), \\ &u_{tt} \in L^\infty\Big((0,T), W_0^{-1,p'(.)}(\Omega)\Big), \end{aligned}$$

for any $T > 0$ and $\frac{1}{p(.)} + \frac{1}{p'(.)} = 1$.

3. Blow-up

In this section, we are going to prove the blow-up result for certain solutions with positive initial energy. At first, by using concavity method [9, 23, 29], we prove that the solutions of (1)-(4) blow-up in a finite time with suitable conditions on initial data and variable exponents when $m(x) \equiv 2$. Next, in the second part, by using modified method inspired by [33], we show that for $m(x) \geq m^- > 2$ and under appropriate conditions on data, the solutions of (1)-(4) blow up at infinity.

3.1. **Blow-up result with $m(x)\equiv 2$.** In order to prove the blow up of solutions with $m(x)\equiv 2$, we use the following change variable

$$v(x,t)=e^{-\lambda t}u(x,t). \tag{16}$$

A direct computation by substituting (16) into the problem (1)-(4) yields

$$\begin{aligned}
&v_{tt}+(2\lambda+\beta)v_t+\lambda(\lambda+\beta)v+\Delta^2 v-\int_0^t g_1(t-\tau)\Delta^2 v(\tau)d\tau\\
&-(a+b\int_\Omega \frac{e^{\lambda p(x)t}}{p(x)}|\nabla v|^{p(x)}dx)div\big(e^{\lambda(p(x)-2)t}|\nabla v|^{p(x)-2}\nabla v\big)\\
&=\alpha e^{\lambda(q(x)-2)t}|v|^{q(x)-2}v+e^{-\lambda t}f(t)\omega(x),\ (x,t)\in\Omega\times(0,\infty)
\end{aligned}\tag{17}$$

$$v(x,t)=\frac{\partial v}{\partial\nu}=0,\ (x,t)\in\partial\Omega\times(0,\infty) \tag{18}$$

$$v(x,0)=u_0(x),\quad v_t(x,0)=u_1(x)-\lambda u_0(x),\qquad x\in\Omega, \tag{19}$$

$$\int_\Omega v(x,t)\omega(x)dx=e^{-\lambda t}\phi(t),\qquad t>0, \tag{20}$$

where $g_1(s)=e^{-\lambda s}g(s)$ and the value of the parameter λ will be prescribed later. Similarly, adapting to the condition (6) and integral overdetermination, the inverse problem (17)-(20) is equivalent to the direct problem (17)-(19) when the unknown function $f(t)$ is replaced by

$$\begin{aligned}
f(t)\ &=\ \phi''(t)+\beta\phi'(t)+e^{\lambda t}\int_\Omega \Delta v\Delta\omega(x)dx\\
&\quad +a\int_\Omega e^{\lambda(p(x)-1)t}|\nabla v|^{p(x)-1}\nabla\omega(x)dx\\
&\quad +b(\int_\Omega \frac{e^{\lambda p(x)t}}{p(x)}|\nabla v|^{p(x)}dx)(\int_\Omega e^{\lambda(p(x)-1)t}|\nabla v|^{p(x)-1}\nabla\omega(x)dx)\\
&\quad -e^{\lambda t}\int_0^t g_1(t-\tau)\int_\Omega \Delta v\Delta\omega(x)dxd\tau\\
&\quad -e^{\lambda t}\int_0^t g_1(t-\tau)\int_\Omega (\Delta v(\tau)-\Delta v)\Delta\omega(x)dxd\tau\\
&\quad -\alpha\int_\Omega e^{\lambda(q(x)-1)}|v|^{q(x)-1}\omega(x)dx.
\end{aligned}\tag{21}$$

The energy function related with problem (17)-(19) is given by

$$E_\lambda(t)=\alpha\int_\Omega \frac{e^{\lambda(q(x)-2)t}}{q(x)}|v|^{q(x)}dx-a\int_\Omega \frac{e^{\lambda(p(x)-2)t}}{p(x)}|\nabla v|^{p(x)}dx-\frac{1}{2}I(t), \tag{22}$$

where

$$\begin{aligned}
I(t)=\|v_t\|^2+\lambda(\lambda+\beta)\|v\|^2+(1-\int_0^t g_1(s)ds)\|\Delta v\|^2+(g_1\odot\Delta v)(t)\\
+b\big(\int_\Omega \frac{e^{\lambda(p(x)-1)t}}{p(x)}|\nabla v|^{p(x)}dx\big)^2,
\end{aligned}$$

and $(g_1\odot\Delta v)(t)=\int_0^t g_1(t-\tau)\|\Delta v(\tau)-\Delta v\|^2d\tau$.
Now we are in a position to state our blow-up result as follows:

Theorem 2. *Let the conditions (5), (6) and (8), (9) be satisfied and suppose that the functions $\phi''(t), \phi'(t)$ and $\phi(t)$ are continuous and bounded such that for constants M_1 and $M_2(\lambda)$:*

$$|\phi''(t)+\beta\phi'(t)| \le M_1 \quad \text{and} \quad |\phi'(t)-\lambda\phi(t)| \le M_2(\lambda).$$

Moreover, assume that

$$q^- > \max\{4p^+ - 2,\ 3+\frac{2(p^+)^2(p^+-1)}{(p^-)^2},\ \frac{4p^+(p^-+2)}{(p^-)^2}\}, \tag{23}$$

$$l = 1-\int_0^\infty g(s)ds \ge \frac{6}{q^-+2}, \quad \alpha \ge \frac{2(q^+-1)}{q^+} \tag{24}$$

$$E_\lambda(0) \ge \frac{D_1}{\lambda(q^- -3)} + \frac{2D_2}{q^-}, \tag{25}$$

where

$$\begin{aligned} D_1 &= M_1M_2(\lambda) + M_2^2(\lambda)\big[\frac{1}{2(1-l)} + \frac{1-l}{2\lambda(q^- -3)} + \frac{1-l}{2}\big]\|\Delta\omega\|^2 \\ &\quad + a\int_\Omega \frac{M_2^{p(x)}(\lambda)}{p(x)\lambda^{p(x)-1}}|\nabla\omega|^{p(x)}dx + \alpha\int_\Omega \frac{M_2^{q(x)}(\lambda)}{q(x)}|\omega|^{q(x)}dx \\ &\quad + \frac{b}{4}\Big(\int_\Omega \frac{M_2^{p(x)}(\lambda)}{p(x)}(p(x)-1)^{p(x)-1}|\nabla\omega|^{p(x)}dx\Big)^2, \end{aligned}$$

$$\begin{aligned} D_2 &= M_1 + (\frac{l^2-2l+3}{2(1-l)})\|\Delta\omega\|^2 + a\int_\Omega \frac{|\nabla\omega|^{p(x)}}{p(x)}dx + \int_\Omega \frac{(\alpha|\omega(x)|)^{q(x)}}{q(x)}dx \\ &\quad + \frac{b}{4}\Big(\int_\Omega \frac{(p(x)-1)^{p(x)-1}}{p(x)}|\nabla\omega|^{p(x)}dx\Big)^2. \end{aligned}$$

Then, for sufficiently large λ, there exists a finite time t^* such that the solution of the problem (1)-(4) blows up in a finite time, that is

$$\|u(t)\| \to +\infty \quad as \quad t \to t^*. \tag{26}$$

To prove the blow-up result in this case, we need the following Lemmas.

Lemma 3. *Under the conditions of Theorem 2, the unknown function $f(t)$, defined by (21), satisfies*

$$\begin{aligned} e^{-2\lambda t}|\phi'(t)-\lambda\phi(t)|f(t) &\le \frac{a\lambda(p^+-1)}{p^+}\int_\Omega e^{\lambda(p(x)-2)t}|\nabla v|^{p(x)}dx \\ &\quad + (1-l)\|\Delta v\|^2 + \frac{\lambda(q^- -3)}{2}(g_1 \odot \Delta v)(t) \\ &\quad + \frac{2b}{(p^-)^2}\Big(\int_\Omega e^{\lambda(p(x)-1)t}|\nabla v|^{p(x)}dx\Big)^2 \\ &\quad + \frac{\alpha(q^+-1)}{q^+}\int_\Omega e^{\lambda(q(x)-2)t}|v|^{q(x)}dx + e^{-2\lambda t}D_1. \end{aligned} \tag{27}$$

Proof. By using (21), we have

$$\begin{aligned}
& e^{-2\lambda t}|\phi'(t)-\lambda\phi(t)|f(t) \\
= & e^{-2\lambda t}(\phi''(t)+\beta\phi'(t))|\phi'(t)-\lambda\phi(t)| + e^{-\lambda t}|\phi'(t)-\lambda\phi(t)|\int_\Omega \Delta v\Delta\omega(x)dx \\
& +ae^{-2\lambda t}|\phi'(t)-\lambda\phi(t)|\int_\Omega |\nabla(e^{\lambda t}v)|^{p(x)-1}\nabla\omega(x)dx \\
& +be^{-2\lambda t}|\phi'(t)-\lambda\phi(t)|(\int_\Omega \frac{|\nabla(e^{\lambda t}v)|^{p(x)}}{p(x)}dx)(\int_\Omega |\nabla(e^{\lambda t}v)|^{p(x)-1}\nabla\omega(x)dx) \\
& -e^{-\lambda t}|\phi'(t)-\lambda\phi(t)|\int_0^t g_1(t-\tau)\int_\Omega \Delta v(\tau)\Delta\omega(x)dxd\tau \\
& -\alpha e^{-\lambda t}|\phi'(t)-\lambda\phi(t)|\int_\Omega e^{\lambda(q(x)-2)t}|v|^{q(x)-1}\omega(x)dx. \qquad (28)
\end{aligned}$$

At this point, by using the Young's inequality (10) and Cauchy-Schwarz inequality and (5), we estimate the terms on the right-hand side of (28) as follows

(29)
$$e^{-\lambda t}|\phi'(t)-\lambda\phi(t)|.|\int_\Omega \Delta v\Delta\omega dx| \leq \frac{1-l}{2}\|\Delta v\|^2 + \frac{e^{-2\lambda t}|\phi'(t)-\lambda\phi(t)|^2}{2(1-l)}\|\Delta\omega\|^2.$$

$$\begin{aligned}
& e^{-\lambda t}|\phi'(t)-\lambda\phi(t)|.|\int_\Omega e^{\lambda(p(x)-2)t}|\nabla v|^{p(x)-1}\nabla\omega(x)dx| \\
\leq\ & e^{-2\lambda t}\int_\Omega \frac{\lambda(p(x)-1)}{p(x)}e^{\lambda p(x)t}|\nabla v|^{p(x)}dx \\
& +e^{-2\lambda t}\int_\Omega \frac{|\phi'(t)-\lambda\phi(t)|^{p(x)}}{p(x)\lambda^{p(x)-1}}|\nabla\omega(x)|^{p(x)}dx \\
\leq\ & \frac{\lambda(p^+-1)}{p^+}\int_\Omega e^{\lambda(p(x)-2)t}|\nabla v|^{p(x)}dx \\
& +e^{-2\lambda t}\int_\Omega \frac{|\phi'(t)-\lambda\phi(t)|^{p(x)}}{p(x)\lambda^{p(x)-1}}|\nabla\omega(x)|^{p(x)}dx. \qquad (30)
\end{aligned}$$

$$\begin{aligned}
& e^{-\lambda t}|\phi'(t)-\lambda\phi(t)|.(\int_\Omega \frac{e^{\lambda(p(x)-1)t}}{p(x)}|\nabla v|^{p(x)}dx)(\int_\Omega e^{\lambda(p(x)-1)t}|\nabla v|^{p(x)-1}\nabla\omega(x)dx) \\
\leq\ & e^{-2\lambda t}(\int_\Omega \frac{e^{\lambda p(x)t}}{p(x)}|\nabla v|^{p(x)}dx)(\int_\Omega \frac{1}{p(x)}e^{\lambda p(x)t}|\nabla v|^{p(x)}dx \\
& +\int_\Omega \frac{|\phi'(t)-\lambda\phi(t)|^{p(x)}}{p(x)}(p(x)-1)^{p(x)-1}|\nabla\omega(x)|^{p(x)}dx) \\
\leq\ & \frac{2}{(p^-)^2}\big(\int_\Omega e^{\lambda(p(x)-1)t}|\nabla v|^{p(x)}dx\big)^2 \\
& +\frac{e^{-2\lambda t}}{4}\big(\int_\Omega \frac{|\phi'(t)-\lambda\phi(t)|^{p(x)}}{p(x)}(p(x)-1)^{p(x)-1}|\nabla\omega(x)|^{p(x)}dx\big)^2. \qquad (31)
\end{aligned}$$

$$\begin{aligned}
& e^{-\lambda t}|\phi'(t)-\lambda\phi(t)|\int_0^t g_1(t-\tau)\int_\Omega \Delta v(\tau)\Delta\omega(x)dxd\tau \\
= \ & e^{-\lambda t}|\phi'(t)-\lambda\phi(t)|\int_0^t g_1(t-\tau)(\int_\Omega (\Delta v(\tau)-\Delta v)\Delta\omega(x)dx \\
& +\int_\Omega \Delta v\Delta\omega(x)dx)d\tau \\
\leq \ & \frac{\lambda(q^- -3)}{2}(g_1\odot\Delta v)(t)+\frac{1-l}{2}\|\Delta v\|^2 \\
& +e^{-2\lambda t}|\phi'(t)-\lambda\phi(t)|^2(\frac{1-l}{2}+\frac{1-l}{2\lambda(q^- -3)})\|\Delta\omega\|^2, \quad (32)
\end{aligned}$$

where the fact $\int_0^\infty g_1(s)ds < \int_0^\infty g(s)ds = 1-l$ has been used.

$$\begin{aligned}
& e^{-\lambda t}|\phi'(t)-\lambda\phi(t)|\int_\Omega e^{\lambda(q(x)-2)t}|v|^{q(x)-1}\omega(x)dx \\
\leq \ & e^{-2\lambda t}\Big(\int_\Omega \frac{q(x)-1}{q(x)}e^{\lambda q(x)t}|v|^{q(x)}dx+\int_\Omega \frac{|\phi'(t)-\lambda\phi(t)|^{q(x)}}{q(x)}|\omega(x)|^{q(x)}dx\Big) \\
\leq \ & \frac{q^+-1}{q^+}\int_\Omega e^{\lambda(q(x)-2)t}|v|^{q(x)}dx+e^{-2\lambda t}\int_\Omega \frac{|\phi'(t)-\lambda\phi(t)|^{q(x)}}{q(x)}|\omega(x)|^{q(x)}dx.
\end{aligned}$$

(33)

Combining estimations (29)-(33) with (28) and by using hypotheses of Theorem 2 about $\phi''(t),\phi'(t)$ and $\phi(t)$, we derive inequality (27) and proof of Lemma 3 is completed. □

Lemma 4. *Under the conditions of Theorem 2, the energy functional $E_\lambda(t)$, defined by (22), satisfies*

$$E_\lambda(t) \geq E_\lambda(0)-\frac{D_1}{\lambda(q^- -3)} \qquad \forall t\in R^+, \quad (34)$$

Proof. A multiplication of equation (17) by v_t and integrating over Ω give

$$\begin{aligned}
E'_\lambda(t) = \ & (2\lambda+\beta)\|v_t\|^2 - a\int_\Omega \frac{\lambda(p(x)-2)}{p(x)}e^{\lambda(p(x)-2)t}|\nabla v|^{p(x)}dx \\
& -b\lambda\Big(\int_\Omega \frac{e^{\lambda(p(x)-1)t}}{p(x)}|\nabla v|^{p(x)}dx\Big)^2-\frac{1}{2}(g_1'\odot\Delta v)(t)+\frac{1}{2}g_1(t)\|\Delta v\|^2 \\
& -b(\int_\Omega \frac{e^{\lambda(p(x)-1)t}}{p(x)}|\nabla v|^{p(x)}dx)(\int_\Omega \frac{\lambda(p(x)-2)}{p(x)}e^{\lambda(p(x)-1)t}|\nabla v|^{p(x)}dx) \\
& +\alpha\int_\Omega \frac{\lambda(q(x)-2)}{q(x)}e^{\lambda(q(x)-2)t}|v|^{q(x)}dx-e^{-2\lambda t}(\phi'(t)-\lambda\phi(t))f(t) \\
\geq \ & (2\lambda+\beta)\|v_t\|^2-\frac{a\lambda(p^+-2)}{p^+}\int_\Omega e^{\lambda(p(x)-2)t}|\nabla v|^{p(x)}dx \\
& -\frac{b\lambda(p^+-1)}{(p^-)^2}\Big(\int_\Omega e^{\lambda(p(x)-1)t}|\nabla v|^{p(x)}dx\Big)^2 \\
& +\frac{\alpha(q^- -2)}{q^-}\int_\Omega e^{\lambda(q(x)-2)t}|v|^{q(x)}dx \\
& -e^{-2\lambda t}|\phi'(t)-\lambda\phi(t)|f(t), \quad (35)
\end{aligned}$$

where conditions (5) and (8) have been used.
Employing (22), we obtain from (35) the following inequality for some $\varepsilon > 0$

$$\begin{aligned} E'_\lambda(t) - \varepsilon E_\lambda(t) \geq & \frac{\alpha[\lambda(q^- - 2) - \varepsilon]}{q^-} \int_\Omega e^{\lambda(q(x)-2)t} |v|^{q(x)} dx + \frac{\varepsilon\lambda(\lambda+\beta)}{2}\|v\|^2 \\ & + (2\lambda + \beta + \frac{\varepsilon}{2})\|v_t\|^2 + \frac{\varepsilon}{2}(1 - \int_0^t g_1(s)ds)\|\Delta v\|^2 + \frac{\varepsilon}{2}(g_1 \odot \Delta v)(t) \\ & + \frac{a[\varepsilon - \lambda(p^+ - 2)]}{p^+} \int_\Omega e^{\lambda(p(x)-2)t} |\nabla v|^{p(x)} dx \\ & + b\Big[\frac{\varepsilon}{2(p^+)^2} - \frac{\lambda(p^+ - 1)}{(p^-)^2})\Big]\Big(\int_\Omega e^{\lambda(p(x)-1)t} |\nabla v|^{p(x)} dx\Big)^2 \\ & - e^{-2\lambda t} |\phi'(t) - \lambda\phi(t)| f(t), \end{aligned} \tag{36}$$

where (5) has been used.
Thanks to the Lemma 3 and taking into account (27) and set $\varepsilon := \lambda(q^- - 3)$, then we get

$$\begin{aligned} E'_\lambda(t) - \lambda(q^- - 3)E_\lambda(t) \geq & \frac{\alpha}{q^-}[\lambda - \frac{q^-(q^+ - 1)}{q^+}] \int_\Omega e^{\lambda(q(x)-2)t} |v|^{q(x)} dx \\ & + [\frac{\lambda(q^- - 3)}{2}(1 - \int_0^t g_1(s)ds) + l - 1]\|\Delta v\|^2 \\ & + \frac{\lambda a}{p^+}(q^- - 2p^+) \int_\Omega e^{\lambda(p(x)-2)t} |\nabla v|^{p(x)} dx \\ & + b\Big[\frac{\lambda(q^- - 3)}{2(p^+)^2} - \frac{\lambda(p^+ - 1)}{(p^-)^2} - \frac{2}{(p^-)^2}\Big]\Big(\int_\Omega e^{\lambda(p(x)-1)t} |\nabla v|^{p(x)} dx\Big)^2 \\ & - e^{-2\lambda t} D_1, \end{aligned} \tag{37}$$

where D_1 is satisfied in Theorem 2.
By using (23) and for sufficiently large λ, we deduce from (37)

$$E'_\lambda(t) - \lambda(q^- - 3)E_\lambda(t) \geq -e^{-2\lambda t} D_1 \geq -D_1.$$

Integrating over $(0, t)$, we observe that

$$E_\lambda(t) \geq E_\lambda(0) - \frac{D_1}{\lambda(q^- - 3)}, \quad \forall t \geq 0,$$

and proof of Lemma 4 is complete. □

Now, we are in a position to prove the Theorem 2 by using Lemma 3 and Lemma 4.

Proof of Theorem 2. For obtaining the blow-up result, we apply concavity method by defining the following functional

$$\psi(t) = \|v(t)\|^2, \tag{38}$$

then

$$\psi'(t) = 2\int_\Omega v v_t dx, \tag{39}$$

$$\psi''(t) = 2\int_\Omega v v_{tt} dx + 2\|v_t\|^2. \tag{40}$$

A multiplication of equation (17) by v and integrating over Ω give

$$
\begin{aligned}
\int_\Omega v v_{tt} dx &= -(2\lambda+\beta)\int_\Omega v v_t dx - \lambda(\lambda+\beta)\|v\|^2 - (1-\int_0^t g_1(s)ds)\|\Delta v\|^2 \\
&\quad - a\int_\Omega e^{\lambda(p(x)-2)t}|\nabla v|^{p(x)}dx + \alpha\int_\Omega e^{\lambda(q(x)-2)t}|v|^{q(x)}dx \\
&\quad - b(\int_\Omega \frac{e^{\lambda p(x)t}}{p(x)}|\nabla v|^{p(x)}dx)(\int_\Omega e^{\lambda(p(x)-2)t}|\nabla v|^{p(x)}dx) \\
&\quad + \int_0^t g_1(t-\tau)\int_\Omega (\Delta v(\tau)-\Delta v)\Delta v dx d\tau + e^{-2\lambda t}f(t). \qquad (41)
\end{aligned}
$$

By virtue of the Young's inequality (10) with $\theta = \frac{1-l}{2}$, $q(x) = q'(x) = 2$, we obtain

$$
\begin{aligned}
&|\int_\Omega \Delta v \int_0^t g_1(t-\tau)(\Delta v(\tau)-\Delta u)d\tau dx| \\
\le\ & \frac{1-l}{2}\|\Delta v\|^2 + \frac{1}{2(1-l)}\int_\Omega \Big(\int_0^t g_1(t-\tau)|\Delta v(\tau)-\Delta v|d\tau\Big)^2 dx \\
=\ & \frac{1-l}{2}\|\Delta v\|^2 + \frac{1}{2(1-l)}\int_\Omega \Big(\int_0^t \frac{g_1(t-\tau)}{\sqrt{g_1(t-\tau)}}\sqrt{g_1(t-\tau)}|\Delta v(\tau)-\Delta v|d\tau\Big)^2 dx \\
\le\ & \frac{1-l}{2}\|\Delta v\|^2 + \frac{1}{2(1-l)}(\int_0^t g_1(s)ds)\int_\Omega\int_0^t g_1(t-\tau)|\Delta v(\tau)-\Delta v|^2 d\tau dx \\
\le\ & \frac{1-l}{2}\|\Delta v\|^2 + \frac{1}{2}(g_1 \odot \Delta v)(t),
\end{aligned}
$$

(42)

where $\int_0^t g_1(s)ds < \int_0^\infty g_1(s)ds < \int_0^\infty g(s)ds = 1-l$.
Combining (42) with (41) and by using (5), we deduce

$$
\begin{aligned}
\int_\Omega v v_{tt} dx &\ge -(2\lambda+\beta)\int_\Omega v v_t dx - \lambda(\lambda+\beta)\|v\|^2 \\
&\quad -[(1-\int_0^t g_1(s)ds) + \frac{1-l}{2}]\|\Delta v\|^2 \\
&\quad -\frac{1}{2}(g_1 \odot \Delta v)(t) - a\int_\Omega e^{\lambda(p(x)-2)t}|\nabla v|^{p(x)}dx \\
&\quad -\frac{b}{p^-}(\int_\Omega e^{\lambda(p(x)-1)t}|\nabla v|^{p(x)}dx)^2 \\
&\quad +\alpha\int_\Omega e^{\lambda(q(x)-2)t}|v|^{q(x)}dx + e^{-2\lambda t}f(t). \qquad (43)
\end{aligned}
$$

At this point, similar to Lemma 3.1 (when $|\phi'(t) - \lambda\phi(t)| := 1$), one can observe the following estimation of the last term on the right-hand side of (43):

$$
\begin{aligned}
e^{-2\lambda t}f(t) &\le (1-l)\|\Delta v\|^2 + \frac{a(p^+-1)}{p^+}\int_\Omega e^{\lambda(p(x)-2)t}|\nabla v|^{p(x)}dx \\
&\quad +\frac{1}{2}(g_1 \odot \Delta v)(t) + \frac{2b}{(p^-)^2}\Big(\int_\Omega e^{\lambda(p(x)-1)t}|\nabla v|^{p(x)}dx\Big)^2 \\
&\quad +\frac{q^+-1}{q^+}\int_\Omega e^{\lambda(q(x)-2)t}|v|^{q(x)}dx + e^{-2\lambda t}D_2, \qquad (44)
\end{aligned}
$$

where D_2 satisfies Theorem 2.
Therefore by utilizing (44) and (22) into (43), we obtain for $\delta > 0$

$$\begin{aligned}\int_\Omega vv_{tt}dx \quad \geq \quad & \delta E_\lambda(t) - (2\lambda+\beta)\int_\Omega vv_t dx + (\frac{\delta}{2}-1)\lambda(\lambda+\beta)\|v\|^2 \\ & +[(\frac{\delta}{2}-1)(1-\int_0^t g_1(s)ds) - \frac{3(1-l)}{2}]\|\Delta v\|^2 \\ & +\frac{\delta}{2}\|v_t\|^2 + (\frac{\delta}{2}-1)(g_1 \odot \Delta v)(t) \\ & +a(\frac{\delta}{p^+} - \frac{p^+-1}{p^+} - 1)\int_\Omega e^{\lambda(p(x)-2)t}|\nabla v|^{p(x)}dx \\ & +b(\frac{\delta}{2(p^+)^2} - \frac{2}{(p^-)^2} - \frac{1}{p^-})(\int_\Omega e^{\lambda(p(x)-1)t}|\nabla v|^{p(x)}dx)^2 \\ & +[\alpha(1-\frac{\delta}{q^-}) - \frac{q^+-1}{q^+}]\int_\Omega e^{\lambda(q(x)-2)t}|v|^{q(x)}dx - e^{-2\lambda t}D_2.\end{aligned} \tag{45}$$

Thus, by using the fact that

$$1-\int_0^t g_s ds \geq 1-\int_0^\infty g_1(s)ds \geq 1-\int_0^\infty g(s)ds = l,$$

we choose $\delta := \frac{q^-}{2}$ and apply the conditions of Theorem 2 to obtain from (45)

$$\begin{aligned}\int_\Omega vv_{tt}dx \quad \geq \quad & \frac{q^-}{2}E_\lambda(t) + \frac{q^-}{4}\|v_t\|^2 + \frac{(q^- -4)}{4}\lambda(\lambda+\beta)\|v\|^2 \\ & -(2\lambda+\beta)\int_\Omega vv_t dx - D_2.\end{aligned} \tag{46}$$

Now, by using Lemma 4 and (25), we get from (46)

$$\int_\Omega vv_{tt}dx \geq \frac{q^-}{4}\|v_t\|^2 - (2\lambda+\beta)\int_\Omega vv_t dx. \tag{47}$$

By substituting (38)-(40) in (47) we get

$$\psi''(t) \geq \frac{(q^- +4)}{2}\|v_t\|^2 - (2\lambda+\beta)\psi'(t),$$

thus

$$\psi(t)\psi''(t) \geq \frac{(q^- +4)}{8}(\psi'(t))^2 - (2\lambda+\beta)\psi(t)\psi'(t), \tag{48}$$

where inequality $(\psi'(t))^2 \leq 4\|v_t\|^2\|v\|^2$ has been used.
Hence, the concavity argument (see [9]) gives us

$$\lim_{t\to t^*}\psi(t) = \infty,$$

which yields solutions of problem (17)-(19) blow up in a finite time t^*. Since this system is equivalent to (1)-(4), the proof of Theorem 2 is complete.

Remark. Under the conditions of Theorem 2, if we choose initial data appropriately such that

$$\psi'(0) - \frac{8(2\lambda+\beta)}{q^- -4}\psi(0) > 0,$$

then we obtain an upper bound for the lifetime of the solutions as

$$t^* < \frac{1}{2\lambda+\beta}\ln\frac{(q^- - 4)\psi'(0)}{(q^- - 4)\psi'(0) - 8(2\lambda+\beta)\psi(0)}.$$

3.2. **Blow-up result with $m(x) > 2$.** In this part, we suppose that $2 < m^- \leq m(x) \leq m^+ < +\infty$ and we shall prove that the solutions of problem (1)-(4) blow up at infinity. By constructing a proper auxiliary functional and using modified method inspired by [33], blow up at infinity has been proved when the variable exponents and initial data satisfy appropriate conditions and the initial energy is positive.
Firstly, we define

$$\begin{aligned} E(t) &= \frac{1}{2}\|u_t\|^2 + \frac{1}{2}(1 - \int_0^t g(s)ds)\|\Delta u\|^2 + a\int_\Omega \frac{1}{p(x)}|\nabla u|^{p(x)}dx \\ &\quad + \frac{b}{2}\Big(\int_\Omega \frac{1}{p(x)}|\nabla u|^{p(x)}dx\Big)^2 + \frac{1}{2}(g\odot\Delta u)(t) \\ &\quad -\alpha\int_\Omega \frac{1}{q(x)}|u|^{q(x)}dx. \end{aligned} \tag{49}$$

By definition of $E(t)$ and using (8), we deduce

$$E'(t) \leq -\beta\int_\Omega |u_t|^{m(x)}dx + f(t)\phi'(t). \tag{50}$$

We are in a position that state blow-up result as follows:

Theorem 3. *Let the conditions (5) (with $m^- > 2$), (6) and (8), (9) be satisfied and suppose that the functions $\phi''(t), \phi'(t)$ and $\phi(t)$ are continuous and bounded such that there exist constants M_3 and M_4*

$$|\phi''(t)| \leq M_3 \quad \textit{and} \quad |\phi'(t) - m^+\phi(t)| \leq M_4.$$

Moreover, Assume that

$$\begin{aligned} &\max\{2p^+ - 1,\ \frac{3-l}{l},\ \frac{2(p^+)^2(p^- + 2)}{(p^-)^2}\} < m^- < q^-, \\ &m^+ > \frac{m^-}{\sqrt{2(m^- + 2)}}, \end{aligned} \tag{51}$$

and suppose that $E(0) > 0$ is a given initial energy level. If we choose initial data u_0 and u_1 such that satisfying

$$\int_\Omega u_0u_1dx > m^+E(0) + \frac{m^+D_3}{m^-},$$

where D_3 will be enunciate in Lemma 5. Then, for sufficiently large α and sufficiently small β, the solution of the problem (1)-(4) blows up at infinity, that is

$$\|u(t)\| \to +\infty \quad as \quad t \to +\infty. \tag{52}$$

Before going to prove of Theorem 3, we state and prove the following Lemma which will be used later:

Lemma 5. *Under the conditions of Theorem 3, for any $\varepsilon > 0$ the unknown function $f(t)$, defined by (21), satisfies*

$$\begin{aligned} |\phi(t)-\varepsilon\phi'(t)|f(t) \quad \leq \quad & \frac{l(m^- - 2)}{6}\|\Delta u\|^2 + \frac{a(p^+ - 1)}{p^+}\int_\Omega |\nabla u|^{p(x)}dx \\ & + \frac{2b}{(p^-)^2}\Big(\int_\Omega |\nabla u|^{p(x)}dx\Big)^2 \\ & + \frac{3(1-l)}{2l}(g\odot\Delta u)(t) + \frac{\beta(m^+ - 1)}{m^+}\int_\Omega |u_t|^{m(x)}dx \\ & + \frac{(q^+ - 1)}{q^+}\int_\Omega |u|^{q(x)}dx + D_3, \end{aligned} \tag{53}$$

where

$$\begin{aligned} D_3 \quad = \quad & M_3M_4 + M_4^2\Big(\frac{3(l^2-2l+2)}{l(m^- - 2)} + \frac{l}{6}\Big)\|\Delta\omega\|^2 \\ & + \beta\int_\Omega \frac{M_4^{m(x)}}{m(x)}|\omega(x)|^{m(x)}dx + \int_\Omega \frac{(\alpha M_4)^{q(x)}}{q(x)}|\omega(x)|^{q(x)}dx \\ & + \frac{b}{4}\Big(\int_\Omega \frac{M_4^{p(x)}}{p(x)}(p(x)-1)^{p(x)-1}|\nabla\omega(x)|^{p(x)}dx\Big)^2. \end{aligned}$$

Proof. Recalling (15), by virtue of Cauchy and Yang inequalities, we estimate the terms on the RHS of (15) as follows:

$$|(\phi(t)-\varepsilon\phi'(t))\int_\Omega \Delta u\Delta\omega dx| \leq \frac{l(m^- - 2)}{12}\|\Delta u\|^2 + \frac{3|\phi(t)-\varepsilon\phi'(t)|^2}{l(m^- - 2)}\|\Delta\omega\|^2. \tag{54}$$

$$\begin{aligned} & |(\phi(t)-\varepsilon\phi'(t))(a + b\int_\Omega \frac{1}{p(x)}|\nabla u|^{p(x)}dx)\int_\Omega |\nabla u|^{p(x)-1}\nabla\omega(x)dx| \\ = \quad & |(a + b\int_\Omega \frac{1}{p(x)}|\nabla u|^{p(x)}dx)\int_\Omega |\nabla u|^{p(x)-1}(\phi(t)-\varepsilon\phi'(t))\nabla\omega(x)dx| \\ \leq \quad & \underbrace{a\,|\int_\Omega |\nabla u|^{p(x)-1}(\phi(t)-\varepsilon\phi'(t))\nabla\omega(x)dx|}_{I_1} \\ & + b\,\underbrace{\Big(\int_\Omega \frac{1}{p(x)}|\nabla u|^{p(x)}dx\Big)|\int_\Omega |\nabla u|^{p(x)-1}(\phi(t)-\varepsilon\phi'(t))\nabla\omega(x)dx|}_{I_2}. \end{aligned} \tag{55}$$

For I_1 and I_2 we have

$$\begin{aligned} I_1 \quad \leq \quad & \int_\Omega \frac{p(x)-1}{p(x)}|\nabla u|^{p(x)}dx + \int_\Omega \frac{|\phi(t)-\varepsilon\phi'(t)|^{p(x)}}{p(x)}|\nabla\omega(x)|^{p(x)}dx \\ \leq \quad & \frac{p^+ - 1}{p^+}\int_\Omega |\nabla u|^{p(x)}dx + \int_\Omega \frac{|\phi(t)-\varepsilon\phi'(t)|^{p(x)}}{p(x)}|\nabla\omega(x)|^{p(x)}dx. \end{aligned} \tag{56}$$

$$\begin{aligned}
I_2 &\le \Big(\int_\Omega \frac{1}{p(x)}|\nabla u|^{p(x)}dx\Big)\Big(\int_\Omega \frac{1}{p(x)}|\nabla u|^{p(x)}dx \\
&\quad + \int_\Omega \frac{|\phi(t)-\varepsilon\phi'(t)|^{p(x)}}{p(x)}(p(x)-1)^{p(x)-1}|\nabla\omega(x)|^{p(x)}dx\Big) \\
&\le \frac{1}{(p^-)^2}\Big(\int_\Omega |\nabla u|^{p(x)}dx\Big)^2 \\
&\quad +\Big(\int_\Omega \frac{1}{p(x)}|\nabla u|^{p(x)}dx\Big)\Big(\int_\Omega \frac{|\phi(t)-\varepsilon\phi'(t)|^{p(x)}}{p(x)}(p(x)-1)^{p(x)-1}|\nabla\omega(x)|^{p(x)}dx\Big) \\
&\le \frac{2}{(p^-)^2}\Big(\int_\Omega |\nabla u|^{p(x)}dx\Big)^2 \\
&\quad +\frac{1}{4}\Big(\int_\Omega \frac{|\phi(t)-\varepsilon\phi'(t)|^{p(x)}}{p(x)}(p(x)-1)^{p(x)-1}|\nabla\omega(x)|^{p(x)}dx\Big)^2. \qquad (57)
\end{aligned}$$

By combining (56) and (57), we get

$$\begin{aligned}
&|(\phi(t)-\varepsilon\phi'(t))(a+b\int_\Omega \frac{1}{p(x)}|\nabla u|^{p(x)}dx)\int_\Omega |\nabla u|^{p(x)-1}\nabla\omega(x)dx| \\
&\qquad\le \frac{p^+-1}{p^+}\int_\Omega |\nabla u|^{p(x)}dx + \frac{2}{(p^-)^2}\Big(\int_\Omega |\nabla u|^{p(x)}dx\Big)^2 \\
&\qquad\quad +\frac{1}{4}\Big(\int_\Omega \frac{|\phi(t)-\varepsilon\phi'(t)|^{p(x)}}{p(x)}(p(x)-1)^{p(x)-1}|\nabla\omega(x)|^{p(x)}dx\Big)^2. \qquad (58)
\end{aligned}$$

$$\begin{aligned}
&|\int_\Omega |u_t|^{m(x)-1}|\phi(t)-\varepsilon\phi'(t)|\omega(x)dx| \\
&\qquad\le \int_\Omega \frac{m(x)-1}{m(x)}|u_t|^{m(x)}dx + \int_\Omega \frac{|\phi(t)-\varepsilon\phi'(t)|^{m(x)}}{m(x)}|\omega(x)|^{m(x)}dx \\
&\qquad\le \frac{m^+-1}{m^+}\int_\Omega |u_t|^{m(x)}dx + \int_\Omega \frac{|\phi(t)-\varepsilon\phi'(t)|^{m(x)}}{m(x)}|\omega(x)|^{m(x)}dx. \qquad (59)
\end{aligned}$$

$$\begin{aligned}
&|\int_0^t g(t-\tau)\int_\Omega (\Delta u(\tau)-\Delta u)|\phi(t)-\varepsilon\phi'(t)|\Delta\omega(x)dxd\tau| \\
&\qquad\le \frac{3(1-l)}{2l}(g\odot\Delta u)(t) + \frac{1}{6}|\phi(t)-\varepsilon\phi'(t)|^2\|\Delta\omega\|^2. \qquad (60)
\end{aligned}$$

$$\begin{aligned}
&|\int_0^t g(t-\tau)\int_\Omega \Delta u|\phi(t)-\varepsilon\phi'(t)|\Delta\omega(x)dxd\tau| \\
&\qquad\le \frac{l(m^--2)}{12}\|\Delta u\|^2 + \frac{3(1-l)^2|\phi(t)-\varepsilon\phi'(t)|^2}{l(m^--2)}\|\Delta\omega\|^2. \qquad (61)
\end{aligned}$$

$$\begin{aligned}
&|\alpha\int_\Omega |u|^{q(x)-1}|\phi(t)-\varepsilon\phi'(t)|\omega(x)dx| \\
&\qquad\le \int_\Omega \frac{q(x)-1}{q(x)}|u|^{q(x)}dx + \int_\Omega \frac{(\alpha|\phi(t)-\varepsilon\phi'(t)|)^{q(x)}}{q(x)}|\omega(x)|^{q(x)}dx \\
&\qquad\le \frac{(q^+-1)}{q^+}\int_\Omega |u|^{q(x)}dx + \int_\Omega \frac{(\alpha|\phi(t)-\varepsilon\phi'(t)|)^{q(x)}}{q(x)}|\omega(x)|^{q(x)}dx. \qquad (62)
\end{aligned}$$

Finally, utilizing (54) and (58)-(62) into (15) and using conditions of Theorem 3 about $\phi''(t), \phi'(t)$ and $\phi(t)$, we get inequality (53) and proof of Lemma 5 is completed. □

Proof of Theorem 3. Multiplying equation (1) by u and integrating over Ω yield

$$\begin{aligned}\frac{d}{dt}\int_\Omega uu_t dx &= \|u_t\|^2 + \int_\Omega uu_{tt} dx \\ &= \|u_t\|^2 - (1-\int_0^t g(s)ds)\|\Delta u\|^2 - a\int_\Omega |\nabla u|^{p(x)}dx \\ &\quad -b(\int_\Omega \frac{1}{p(x)}|\nabla u|^{p(x)}dx)(\int_\Omega |\nabla u|^{p(x)}dx) \\ &\quad +\int_0^t g(t-\tau)\int_\Omega \Delta u(\Delta u(\tau)-\Delta u)dxd\tau \\ &\quad -\beta\int_\Omega u|u_t|^{m(x)-2}u_t dx + \alpha\int_\Omega |u|^{q(x)}dx + f(t)\phi(t). \end{aligned} \tag{63}$$

Similar to (42) and using (8), we have

$$|\int_0^t g(t-\tau)\int_\Omega \Delta u(\Delta u(\tau)-\Delta u)dxd\tau| \le \frac{l(m^- -2)}{12}\|\Delta u\|^2 + \frac{3(1-l)}{l(m^- -2)}(g\odot\Delta u)(t). \tag{64}$$

Utilizing (64) in (63) and using (5), we obtain

$$\begin{aligned}\frac{d}{dt}\int_\Omega uu_t dx \ge& \|u_t\|^2 - [(1-\int_0^t g(s)ds) + \frac{l(m^- -2)}{12}]\|\Delta u\|^2 - a\int_\Omega |\nabla u|^{p(x)}dx \\ &- \frac{b}{p^-}(\int_\Omega |\nabla u|^{p(x)}dx)^2 - \frac{3(1-l)}{l(m^- -2)}(g\odot\Delta u)(t) \\ &- \beta\int_\Omega u|u_t|^{m(x)-2}u_t dx + \alpha\int_\Omega |u|^{q(x)}dx + f(t)\phi(t). \end{aligned} \tag{65}$$

For any $\delta > 0$ and using definition of $E(t)$, we have

$$\begin{aligned}\frac{d}{dt}\int_\Omega uu_t dx \ge& -\delta E(t) + (1+\frac{\delta}{2})\|u_t\|^2 + [(\frac{\delta}{2}-1)(1-\int_0^t g(s)ds) - \frac{l(m^- -2)}{12}]\|\Delta u\|^2 \\ &+ a(\frac{\delta}{p^+}-1)\int_\Omega |\nabla u|^{p(x)}dx + b(\frac{\delta}{2(p^+)^2} - \frac{1}{p^-})(\int_\Omega |\nabla u|^{p(x)}dx)^2 \\ &+ (\frac{\delta}{2} - \frac{3(1-l)}{l(m^- -2)})(g\odot\Delta u)(t) + \alpha(1-\frac{\delta}{q^-})\int_\Omega |u|^{q(x)}dx \\ &- \beta\int_\Omega u|u_t|^{m(x)-2}u_t dx + f(t)\phi(t), \end{aligned} \tag{66}$$

where condition (5) has been used.
Also, for any $\varepsilon > 0$ and using (50), we have

$$
\begin{aligned}
&\frac{d}{dt}\Big(\int_\Omega uu_t dx - \varepsilon E(t)\Big) \geq -\delta E(t) + \varepsilon\beta\int_\Omega |u_t|^{m(x)}dx + (1+\frac{\delta}{2})\|u_t\|^2 \\
&+[(\frac{\delta}{2}-1)(1-\int_0^t g(s)ds) - \frac{l(m^- -2)}{12}]\|\Delta u\|^2 \\
&+a(\frac{\delta}{p^+}-1)\int_\Omega |\nabla u|^{p(x)}dx + b(\frac{\delta}{2(p^+)^2} - \frac{1}{p^-})(\int_\Omega |\nabla u|^{p(x)}dx)^2 \\
&+(\frac{\delta}{2} - \frac{3(1-l)}{l(m^- -2)})(g\odot\Delta u)(t) + \alpha(1-\frac{\delta}{q^-})\int_\Omega |u|^{q(x)}dx \\
&-\beta\int_\Omega u|u_t|^{m(x)-2}u_t dx + f(t)|\phi(t) - \varepsilon\phi'(t)|. \qquad (67)
\end{aligned}
$$

Thanks to the Lemma 5, we get

$$
\begin{aligned}
\frac{d}{dt}\Big(\int_\Omega uu_t dx - \varepsilon E(t)\Big) \geq &-\delta E(t) + \beta(\varepsilon - \frac{m^+ -1}{m^+})\int_\Omega |u_t|^{m(x)}dx + (1+\frac{\delta}{2})\|u_t\|^2 \\
&+[(\frac{\delta}{2}-1)(1-\int_0^t g(s)ds) - \frac{l(m^- -2)}{4}]\|\Delta u\|^2 \\
&+a(\frac{\delta+1}{p^+}-2)\int_\Omega |\nabla u|^{p(x)}dx \\
&+b(\frac{\delta}{2(p^+)^2} - \frac{1}{p^-} - \frac{2}{(p^-)^2})(\int_\Omega |\nabla u|^{p(x)}dx)^2 \\
&+(\frac{\delta}{2} - \frac{3(1-l)}{l(m^- -2)} - \frac{3(1-l)}{2l})(g\odot\Delta u)(t) \\
&+[\alpha(1-\frac{\delta}{q^-}) - \frac{q^+ -1}{q^+}]\int_\Omega |u|^{q(x)}dx \\
&-\beta\int_\Omega u|u_t|^{m(x)-2}u_t dx - D_3. \qquad (68)
\end{aligned}
$$

Again by using Young's inequality (11), we obtain

$$
\begin{aligned}
|\int_\Omega u|u_t|^{m(x)-1}dx| &\leq \int_\Omega \frac{1}{m(x)}|u|^{m(x)}dx + \int_\Omega \frac{m(x)-1}{m(x)}|u_t|^{m(x)}dx \\
&\leq \frac{1}{m^-}\int_\Omega |u|^{m(x)}dx + \frac{m^+ -1}{m^+}\int_\Omega |u_t|^{m(x)}dx. \qquad (69)
\end{aligned}
$$

On the other hand, let c^* be the best constant of embedding $H_0^2(\Omega) \hookrightarrow L^{m(.)}(\Omega)$. Then we have

$$
\begin{aligned}
\int_\Omega |u|^{m(x)}dx &\leq \max\{\|u\|_{m(x)}^{m^-}, \|u\|_{m(x)}^{m^+}\} \\
&\leq \max\{(c^*)^{m^-}\|\Delta u\|^{m^-}, (c^*)^{m^+}\|\Delta u\|^{m^+}\} \\
&\leq \max\{(c^*)^{m^-}\|\Delta u\|^{m^- -2}, (c^*)^{m^+}\|\Delta u\|^{m^+ -2}\}\|\Delta u\|^2 \\
&\leq \overline{C}\|\Delta u\|^2. \qquad (70)
\end{aligned}
$$

Combining (69) with (70), we get

$$|\int_\Omega u|u_t|^{m(x)-1}dx| \leq \frac{\overline{C}}{m^-}\|\Delta u\|^2 + \frac{m^+-1}{m^+}\int_\Omega |u_t|^{m(x)}dx.$$

Substituting last inequality into (68) and set $\varepsilon := m^+,\ \delta := m^-$, we obtain

$$\begin{aligned}\frac{d}{dt}\Big(\int_\Omega uu_t dx - m^+E(t)\Big) \geq & -m^-E(t) + \beta(m^+ - \frac{2(m^+-1)}{m^+})\int_\Omega |u_t|^{m(x)}dx \\ & + (1+\frac{m^-}{2})\|u_t\|^2 + [\frac{l(m^--2)}{4} - \frac{\beta\overline{C}}{m^-}]\|\Delta u\|^2 \\ & + a(\frac{m^-+1}{p^+} - 2)\int_\Omega |\nabla u|^{p(x)}dx \\ & + b(\frac{m^-}{2(p^+)^2} - \frac{1}{p^-} - \frac{2}{(p^-)^2})(\int_\Omega |\nabla u|^{p(x)}dx)^2 \\ & + (\frac{m^-}{2} - \frac{3(1-l)}{l(m^--2)} - \frac{3(1-l)}{2l})(g\odot\Delta u)(t) \\ & + [\alpha(1-\frac{m^-}{q^-}) - \frac{q^+-1}{q^+}]\int_\Omega |u|^{q(x)}dx - D_3, \end{aligned} \tag{71}$$

where $1-\int_0^t g(s)ds > 1 - \int_0^\infty g(s)ds = l$ has been used.
Using the conditions of Theorem 3, if α is large enough and β sufficiently small and (51) satisfied, then we have

$$\begin{aligned}&\frac{d}{dt}\Big(\int_\Omega uu_t dx - m^+E(t)\Big) \\ &\geq \quad -m^-E(t) + (1+\frac{m^-}{2})\|u_t\|^2 + [\frac{l(m^--2)}{4} - \frac{\beta\overline{C}}{m^-}]\|\Delta u\|^2 - D_3 \\ &\geq \quad -m^-E(t) + (1+\frac{m^-}{2})\|u_t\|^2 + \frac{1}{B^2}[\frac{l(m^--2)}{4} - \frac{\beta\overline{C}}{m^-}]\|u\|^2 - D_3,\end{aligned} \tag{72}$$

where B is the best constant of embedding $H_0^2(\Omega) \hookrightarrow L^2(\Omega)$.
By virtue of the Young's inequality and condition (51) i.e. $m^+ > \frac{m^-}{\sqrt{2(m^-+2)}}$ and for sufficiently small β, it is easy to see that

$$\begin{aligned}\frac{m^-}{m^+}\int_\Omega uu_t dx \quad &\leq \quad \|u\|^2 + \big(\frac{m^-}{2m^+}\big)^2\|u_t\|^2 \\ &\leq \quad \frac{1}{B^2}[\frac{l(m^--2)}{4} - \frac{\beta\overline{C}}{m^-}]\|u\|^2 + \frac{(m^-+2)}{2}\|u_t\|^2.\end{aligned} \tag{73}$$

Thus using inequality (73) into (72) yields

$$\frac{d}{dt}\Big(\int_\Omega uu_t dx - m^+E(t)\Big) \geq \frac{m^-}{m^+}\Big(\int_\Omega uu_t dx - m^+E(t)\Big) - D_3. \tag{74}$$

Let define

$$H(t) = \int_\Omega uu_t dx - m^+E(t),$$

and therefore

$$H'(t) \geq \frac{m^-}{m^+}H(t) - D_3,$$

integrating over $(0,t)$ to get

$$H(t) \geq e^{\frac{m^- t}{m^+}}(H(0) - \frac{m^+ D_3}{m^-}) + \frac{m^+ D_3}{m^-}, \qquad \forall t > 0 \tag{75}$$

where by the assumption of Theorem 3, $H(0) > \frac{m^+ D_3}{m^-}$.
Finally, inequality (75) shows that $H(t)$ tends to infinity when time goes to infinity and thus the proof of Theorem 3 is completed.

4. Conclusion

In this paper, we studied blow up of solutions for a class of plate viscoelastic $p(x)-$Kirchhoff type inverse source problem with variable-exponent nonlinearities. We obtained blow up of solutions for the inverse problem (1)-(4) in a finite time when $m(x) \equiv 2$. Moreover, if $2 < m^- \leq m(x)$, then we proved blow-up at infinity of solutions for the inverse problem (1)-(4). Therefore, in the case of $2 < m^- \leq m(x)$, blow-up of solutions in a finite time is an open problem for the inverse problem (1)-(4).

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Mohammad Shahrouzi
Department of Mathematics,
Jahrom University, Jahrom,
P.O.Box: 74137-66171, Iran
Email address: mshahrouzi@jahromu.ac.ir

Jorge Ferreira
Department of Exact Sciences,
Federal Fluminense University, Volta Redonda,
27255-125, Brazil
Email address: ferreirajorge2012@gmail.com

Erhan Pişkin
Department of Mathematics,
Dicle University, Diyarbakir,
TR-21280, Turkey
Email address: episkin@dicle.edu.tr

Nouri Boumaza
Department of Mathematics and Computer Science,
Larbi Tebessi University, Tebessa,
Algeria
Email address: nouri.boumaza@univ-tebessa.dz