

REIDEMEISTER CLASSES IN SOME WREATH PRODUCTS

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ABSTRACT. Among restricted wreath products $G \wr \mathbb{Z}^k$, where G is a finite Abelian group, we find three large classes of groups admitting an automorphism φ with finite Reidemeister number $R(\varphi)$ (number of φ -twisted conjugacy classes). In other words, groups from these classes do not have the R_∞ property.

If a general automorphism φ of $G \wr \mathbb{Z}^k$ has a finite order (this is the case for φ detected in the first part of the paper) and $R(\varphi) < \infty$, we prove that $R(\varphi)$ coincides with the number of equivalence classes of finite-dimensional irreducible unitary representations of $G \wr \mathbb{Z}^k$, which are fixed by the dual map $[\rho] \mapsto [\rho \circ \varphi]$ (i.e. we prove the conjecture about finite twisted Burnside-Frobenius theorem, TBFT_f , for these φ).

1. INTRODUCTION

Suppose, Γ is a group and $\phi : \Gamma \rightarrow \Gamma$ is an endomorphism. Two elements $x, y \in \Gamma$ are ϕ -conjugate or twisted conjugate, if and only if there exists an element $g \in \Gamma$ such that

$$y = gx\phi(g^{-1}).$$

The corresponding classes are called *Reidemeister* or *twisted conjugacy* classes. The number $R(\phi)$ of them is called the *Reidemeister number* of ϕ .

The study of Reidemeister numbers is an important problem related with Topological Dynamics, Number Theory and Representation Theory (see [4]). One of the main problems in the field is to prove or disprove the so-called TBFT (a conjecture about the twisted Burnside-Frobenius theory (or theorem)), which has numerous important consequence for Reidemeister zeta function and for other problems in Topological Dynamics (see a more extended discussion in [14]). Namely the problem is to identify $R(\varphi)$ (when $R(\varphi) < \infty$) in a natural way with the number of fixed points of the induced map $\widehat{\varphi}$ of an appropriate dual object. In the initial formulation of the conjecture [6], the dual object was the unitary dual $\widehat{\Gamma}$ and $\widehat{\varphi} : [\rho] \mapsto [\rho \circ \varphi]$. The conjecture about TBFT was proved in many cases, but failed for an example in [13], which led to the new formulation: TBFT_f , where $\widehat{\Gamma}$ was replaced by its finite-dimensional part, which is evidently invariant under $\widehat{\varphi}$. This is the version, which we will study in this paper for a class of groups. In [14] an example of a group that has neither TBFT nor TBFT_f was presented. The most general proved cases of TBFT_f are the case of polycyclic-by-finite groups [11] and the case of nilpotent torsion-free groups of finite Prüfer rank [10].

Another important problem in the field is to localize the class of groups, where one can consider the TBFT conjecture, i.e. where automorphisms with $R(\varphi) < \infty$ do exist. The

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opposite case is called the R_∞ property. It has some topological consequences itself (see e.g. [18]). A part of recent results about Reidemeister classes and R_∞ can be found in [7, 23, 1, 3, 12, 27, 26, 21] (see also an overview in [9]).

We consider the following restricted wreath product $G \wr \mathbb{Z}^k = \Sigma \rtimes_\alpha \mathbb{Z}^k$, where G is a finite Abelian group, Σ denotes $\bigoplus_{x \in \mathbb{Z}^k} G_x$, and $\alpha(x)(g_y) = g_{x+y}$. Here g_x is $g \in G \cong G_x$.

The R_∞ property was completely studied for $k = 1$ in [17], for $G = \mathbb{Z}_p$ with a prime p and arbitrary k in [27], for $G = \mathbb{Z}_m$ and arbitrary k in [15]. In all these cases the TBFT $_f$ was proved.

The complexity of the study increases drastically when we move from $k = 1$ to $k > 1$, because \mathbb{Z} has only one non-trivial automorphism in contrast with \mathbb{Z}^k .

The groups under consideration can be viewed as generalized lamplighter groups. For a generalization of the lamplighter group in other directions, the twisted conjugacy was considered in [25], [24], and other papers.

In the present paper, we prove (Theorem 3.1) that the groups under consideration do not have the R_∞ property in the following three cases:

- 1) all prime-power components of G for 2 and 3 have multiplicity at least 2;
- 2) there is no prime-power components for 2 and k is even;
- 3) all prime-power components of G for 2 have multiplicity at least 2 and $k = 4s$ for some s .

To prove this, we construct corresponding examples, and all of them have a finite order. This motivates us to prove the TBFT $_f$ for all groups of the form $G \wr \mathbb{Z}^k$ and their automorphisms of finite order (Corollary 4.2). The proof is based on a description of Reidemeister classes of φ as cylindrical sets (Theorem 4.1).

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2. PRELIMINARIES

We start from some general statements about Reidemeister classes of extensions. Suppose, a normal subgroup H of G is φ -invariant under an automorphism $\varphi : G \rightarrow G$ and $p : G \rightarrow G/H$ is the natural projection. Then φ induces automorphisms $\varphi' : H \rightarrow H$ and $\tilde{\varphi} : G/H \rightarrow G/H$.

Definition 2.1. Denote $C(\varphi) := \{g \in G : \varphi(g) = g\}$, i.e. $C(\varphi)$ is the subgroup of G , formed by φ -fixed elements.

We will use the notation $\tau_g(x) = gxg^{-1}$ for an inner automorphism as well as for its restriction on a normal subgroup.

The following important properties were obtained in [6, 16], see also [11, 18].

Theorem 2.2. *For G , H , φ , φ' , and $\tilde{\varphi}$ as above, we have the following.*

1. Epimorphity: *the projection $G \rightarrow G/H$ maps Reidemeister classes of φ onto Reidemeister classes of $\tilde{\varphi}$, in particular $R(\tilde{\varphi}) \leq R(\varphi)$;*
2. Estimation by fixed elements: *if $|C(\tilde{\varphi})| = n$, then $R(\varphi') \leq R(\varphi) \cdot n$;*
3. Fixed elements-free case: *if $C(\tilde{\varphi}) = \{e\}$, then each Reidemeister class of φ' is an intersection of the appropriate Reidemeister class of φ and H ;*
4. Summation: *if $C(\tilde{\varphi}) = \{e\}$, then $R(\varphi) = \sum_{j=1}^R R(\tau_{g_j} \circ \varphi')$, where g_1, \dots, g_R are some elements of G such that $p(g_1), \dots, p(g_R)$ are representatives of all Reidemeister classes of $\tilde{\varphi}$, $R = R(\tilde{\varphi})$;*

Also we will need the following statement [19] (Lemma 4 and the step (2) in the proof of Theorem A’):

Lemma 2.3. *Suppose, Γ is a residually finite group and $\varphi : \Gamma \rightarrow \Gamma$ is an automorphism with $R(\varphi) < \infty$. Then $|C(\varphi)| < \infty$.*

One can find in [19] an estimation for $|C(\varphi)|$, but we will not use it.

Passing to a semidirect product $\Sigma \rtimes_{\alpha} \mathbb{Z}^k$, we have by [2] that a couple of automorphisms $\varphi' : \Sigma \rightarrow \Sigma$ and $\bar{\varphi} : \Sigma \rtimes_{\alpha} \mathbb{Z}^k / \Sigma \cong \mathbb{Z}^k \rightarrow \mathbb{Z}^k \cong \Sigma \rtimes_{\alpha} \mathbb{Z}^k / \Sigma$ define an automorphism φ of $\Sigma \rtimes_{\alpha} \mathbb{Z}^k$ (not unique) if and only if

$$(1) \quad \varphi'(\alpha(m)(h)) = \alpha(\bar{\varphi}(g))(\varphi'(h)), \quad h \in \Sigma, \quad m \in \mathbb{Z}^k.$$

Since Σ is abelian, by [2, p. 207] the mapping φ_1 defined as φ' on Σ and by $\bar{\varphi}$ on $\mathbb{Z}^k \subset \Sigma \rtimes \mathbb{Z}^k$ is still an automorphism. Moreover, from the following commutative diagrams

$$\begin{array}{ccccccc} 0 & \longrightarrow & \Sigma & \longrightarrow & \Sigma \rtimes \mathbb{Z}^k & \longrightarrow & \mathbb{Z}^k \longrightarrow 0 \\ & & \varphi' \downarrow & & \varphi \downarrow \varphi_1 & & \downarrow \bar{\varphi} \\ 0 & \longrightarrow & \Sigma & \longrightarrow & \Sigma \rtimes \mathbb{Z}^k & \longrightarrow & \mathbb{Z}^k \longrightarrow 0 \end{array}$$

we have $R(\varphi) = R(\varphi_1)$. Indeed, if $R(\bar{\varphi}) = \infty$ then $R(\varphi) = R(\varphi_1) = \infty$. If $R(\bar{\varphi}) < \infty$ then $C_{\bar{\varphi}} = \{0\}$ and by Theorem 2.2

$$R(\varphi) = \sum_{\text{representatives } m \in \mathbb{Z}^k \text{ of Reidemeister classes of } \bar{\varphi}} R(\tau_m \circ \varphi') = R(\varphi_1).$$

So, without loss of generality in the R_{∞} questions (not in Section 4) we will assume

$$(2) \quad \mathbb{Z}^k \subset A \wr \mathbb{Z}^k \text{ is } \varphi\text{-invariant and } \varphi|_{\mathbb{Z}^k} = \bar{\varphi}.$$

This was discussed briefly in [17, Lemma 3.5] in a particular case.

Lemma 2.4. *An automorphism $\varphi : G \wr \mathbb{Z}^k \rightarrow G \wr \mathbb{Z}^k$ has $R(\varphi) < \infty$ if and only if $R(\bar{\varphi}) < \infty$ and $R(\tau_m \circ \varphi') < \infty$ for any $m \in \mathbb{Z}^k$ (in fact, it is sufficient to verify this for representatives of Reidemeister classes of $\bar{\varphi}$).*

Proof. Suppose, $R(\varphi) < \infty$. By Theorem 2.2, we have $R(\bar{\varphi}) < \infty$. Then by Lemma 2.3, we obtain $|C(\bar{\varphi})| < \infty$ (in fact, $|C(\bar{\varphi})| = 1$, because an automorphism of \mathbb{Z}^k can not have finitely many fixed elements except of 0). So, by Theorem 2.2, $R(\varphi') < \infty$. Considering $\tau_z \circ \varphi$, which has $R(\tau_z \circ \varphi) = R(\varphi) < \infty$, instead of φ , we obtain in the same way that $R(\tau_z \circ \varphi') < \infty$.

Conversely, having $|C(\bar{\varphi})| = 1$, one can apply the summation formula from Theorem 2.2. \square

Lemma 2.5. *Suppose, $\bar{\varphi} : \mathbb{Z}^k \rightarrow \mathbb{Z}^k$ is an automorphism and $F : G \rightarrow G$ is an automorphism. Then φ' defined by*

$$(3) \quad \varphi'(a_0) = (Fa)_0, \quad \varphi'(a_x) = (Fa)_{\bar{\varphi}(x)}$$

satisfies (1) and so defines an automorphism of $G \wr \mathbb{Z}^k$.

Evidently the subgroups $\oplus G_x$, where x runs over an orbit of $\bar{\varphi}$, are φ' -invariant summands of Σ .

Proof. It is sufficient to prove (1) on generating elements of the form a_x . Then for any $z \in \mathbb{Z}^k$,

$$\varphi'(\alpha(z)a_x) = \varphi'(a_{x+z}) = (Fa)_{\overline{\varphi}(x+z)} = \alpha(\overline{\varphi}(z))(Fa)_{\overline{\varphi}(z)} = \alpha(\overline{\varphi}'(z))\varphi'(a_x)$$

and (1) is fulfilled. The first equality in (3) is in fact a particular case of the second one. \square

It is not difficult to prove (see [5]) that, for $\overline{\varphi} : \mathbb{Z}^k \rightarrow \mathbb{Z}^k$ defined by a matrix M , one has

$$(4) \quad R(\overline{\varphi}) = \# \text{Coker}(\text{Id} - \overline{\varphi}) = |\det(E - M)|,$$

if $R(\overline{\varphi}) < \infty$, and $|\det(E - M)| = 0$ otherwise.

3. SOME CLASSES OF WREATH PRODUCTS WITHOUT R_∞ PROPERTY

Theorem 3.1. *Suppose, the prime-power decomposition of G is $\bigoplus_i (\mathbb{Z}_{(p_i)^{r_i}})^{d_i}$. Then under each of the following conditions the corresponding wreath products $G \wr \mathbb{Z}^k$ admit an automorphism φ with $R(\varphi) < \infty$, i.e. do not have the property R_∞ :*

Case 1): for all $p_i = 2$ and $p_i = 3$, we have $d_i \geq 2$ (and is arbitrary for primes > 3);

Case 2): there is no $p_i = 2$ and k is even;

Case 3): for all $p_i = 2$, we have $d_i \geq 2$ and $k = 4s$ for some s .

Proof. In each of these cases we will take an automorphism $\overline{\varphi} : \mathbb{Z}^k \rightarrow \mathbb{Z}^k$ with $R(\overline{\varphi}) < \infty$ (in fact, of finite order) and define $\varphi' : \Sigma \rightarrow \Sigma$ with appropriate properties in accordance with Lemmas 2.4 and 2.5.

Case 1). In this case we can take $\overline{\varphi} = -\text{Id} : \mathbb{Z}^k \rightarrow \mathbb{Z}^k$ and construct φ similarly to [17]. More specifically, note that $R(\overline{\varphi}) = 2^k$ and define $\varphi' : \Sigma \rightarrow \Sigma$ in the following way. The subgroups $G_x \oplus G_{-x}$ will be invariant subgroups of φ' and we define

$$\varphi' : G_x \oplus G_{-x} \rightarrow G_x \oplus G_{-x} \text{ as } \begin{pmatrix} 0 & \Psi \\ \Psi & 0 \end{pmatrix},$$

where $\Psi : G \rightarrow G$ is defined as a direct sum of blocks of the following types:

$$(5) \quad F_2 = \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix} : (\mathbb{Z}_q)^2 \rightarrow (\mathbb{Z}_q)^2, \quad F_3 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix} : (\mathbb{Z}_q)^3 \rightarrow (\mathbb{Z}_q)^3,$$

where q are some $(p_i)^{r_i}$ and for each summand $(\mathbb{Z}_{(p_i)^{r_i}})^{d_i}$ of G ($d_i \geq 2$, $p_i = 2$ or $p_i = 3$) we have s summands F_2 , if $d_i = 2s$, or $s - 1$ summands F_2 and one summand F_3 , if $d_i = 2s + 1$. For the remaining summands (i.e. for $p_i > 3$) we do not need to group summands in the above way and we can consider $F_1 : \mathbb{Z}_q \rightarrow \mathbb{Z}_q$, $1 \mapsto m(q)$ where $q = (p_i)^{r_i}$. This $m = m(q)$ should be taken in such a way that

$$(6) \quad m^2 \text{ and } 1 - m^2 \text{ are invertible in } \mathbb{Z}_q.$$

This can be done for $p_i > 3$: one can take $m = 2$ (and impossible for $p_i = 2$ or 3).

By Lemma 2.5, we defined an automorphism φ of $G \wr \mathbb{Z}^k$ in this way (one may assume (2) to have a unique φ).

We claim that $R(\tau_{z_i} \circ \varphi') = R(\alpha(z_i) \circ \varphi') = 1$, $i = 1, \dots, 2^k$. Consequently, by Theorem 2.2, $R(\varphi) = R(\overline{\varphi}) = 2^k < \infty$. So we need to prove that $\text{Id}_\Sigma - \alpha(z_i) \circ \varphi'$ is an epimorphism, because, for Abelian groups, this is evidently the same as $R(\alpha(z_i) \circ \varphi') = 1$. This homomorphism has a decomposition of Σ into invariant subgroups $G_x \oplus G_{-x+z_i}$, because $\alpha(z_i) : G_{-x} \rightarrow G_{-x+z_i}$, $\varphi' : G_{-x+z_i} \rightarrow G_{x-z_i}$ and $\alpha(z_i) : G_{x-z_i} \rightarrow G_x$. Note that the subgroups G_x and G_{-x+z_i}

coincide if $z_i = 2x$ (this corresponds to the case of G_0 for φ'). Thus it is sufficient to verify the epimorphy for each $G_x \oplus G_{-x+z_i}$ and for the exceptional case. Passing to summands of G , it is sufficient to verify the epimorphy of

$$\begin{pmatrix} -E & F_2 \\ F_2 & -E \end{pmatrix}, \quad \begin{pmatrix} -E & F_3 \\ F_3 & -E \end{pmatrix} \text{ and } \begin{pmatrix} -E & F_1 \\ F_1 & -E \end{pmatrix} = \begin{pmatrix} -1 & m \\ m & -1 \end{pmatrix}.$$

The first two are isomorphisms with the explicit inverses

$$\begin{pmatrix} -1 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 1 & 0 & -1 & 1 \\ 0 & 1 & 1 & 0 \end{pmatrix}, \quad \begin{pmatrix} -2 & 0 & 1 & 1 & 1 & -1 \\ 0 & -1 & 1 & 1 & 0 & 0 \\ 1 & 1 & -1 & -1 & 0 & 1 \\ 1 & 1 & -1 & -2 & 0 & 1 \\ 1 & 0 & 0 & 0 & -1 & 1 \\ -1 & 0 & 1 & 1 & 1 & -1 \end{pmatrix}.$$

For the third one the invertibility follows from (6). For the exceptional case we formally do not need to verify the epimorphy, because it can add only a finite number to $R(\varphi')$, but we wish to prove our (more strong) claim (this will be helpful for TBFT). So we have to prove, that

$$F_2 - E, \quad F_3 - E, \quad m - 1$$

are epimorphisms. This can be done immediately: $\det(F_2 - E) = 1 \pmod{2}$, $\det(F_3 - E) = 1 \pmod{2}$, and $1 - m^2 = (1 - m)(1 + m)$.

Case 2). Now consider the case of even $k = 2t$ and G without 2-subgroup. In this case the construction starts as in [27]: we take $\bar{\varphi} : \mathbb{Z}^{2t} \rightarrow \mathbb{Z}^{2t}$ to be the direct sum of t copies of

$$\mathbb{Z}^2 \rightarrow \mathbb{Z}^2, \quad \begin{pmatrix} u \\ v \end{pmatrix} \mapsto M \begin{pmatrix} u \\ v \end{pmatrix}, \quad M = \begin{pmatrix} 0 & 1 \\ -1 & -1 \end{pmatrix}.$$

Then M generates a subgroup of $GL(2, \mathbb{Z})$, which is isomorphic to \mathbb{Z}_3 (see, [22, p. 179]). All orbits of M have length 3 (except of the trivial one) and the corresponding Reidemeister number $= \det(E - M) = 3$. Similarly for $\bar{\varphi}$: the length of any orbit is 3 (except of the zero) and $R(\bar{\varphi}) = 3^t$. Also

$$(7) \quad M^2 + M + E = \begin{pmatrix} -1 & -1 \\ 1 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 1 \\ -1 & -1 \end{pmatrix} + \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}.$$

Now define φ' as a direct sum of actions for \mathbb{Z}_q , $q = (p_i)^{r_i}$, $p_i \geq 3$.

For $p_i \geq 3$ choose $m = m_i$ such that

$$(8) \quad m^3 \text{ and } 1 - m^3 \text{ are invertible in } \mathbb{Z}_q.$$

This can be done for $p_i \geq 3$: one can take $m = 3$ for $p_i = 7$ and $m = 2$ in the remaining cases (and impossible for $p_i = 2$). Define $\varphi'(a_0) = (ma)_0$ and $\varphi'(a_x) = (ma)_{\bar{\varphi}(g)}$, where $a \in \mathbb{Z}_q \subset G$. So, the corresponding subgroup $\bigoplus_{g \in \mathbb{Z}^k} (\mathbb{Z}_q)_g \subset \Sigma$ is φ' -invariant and decomposed into infinitely many invariant summands $(\mathbb{Z}_q)_g \oplus (\mathbb{Z}_q)_{\bar{\varphi}(g)} \oplus (\mathbb{Z}_q)_{\bar{\varphi}^2(g)}$ isomorphic to $(\mathbb{Z}_q)^3$ (over generic orbits of $\bar{\varphi}$) and one summand $(\mathbb{Z}_q)_0$ (over the trivial orbit). Then the corresponding restrictions of φ' and $1 - \varphi'$ can be written as multiplication by

$$\begin{pmatrix} 0 & 0 & m \\ m & 0 & 0 \\ 0 & m & 0 \end{pmatrix}, \quad \begin{pmatrix} 1 & 0 & -m \\ -m & 1 & 0 \\ 0 & -m & 1 \end{pmatrix}, \quad \text{and } m, \quad 1 - m,$$

respectively. The three-dimensional mappings are isomorphisms by (8). Since an element ℓ is not invertible in $\mathbb{Z}_{(p_i)^{r_i}}$ if and only if $\ell = u \cdot p_i$, the invertibility of one-dimensional mappings follows from (8) and the factorization $1 - m^3 = (1 - m)(1 + m + m^2)$. (This construction gives a more explicit presentation of a part of proof of [27, Theorem 4.1])

For $\tau_z \circ \varphi'$ we have

$$\begin{aligned} (\tau_z \circ \varphi')(g_x) &= (mg)_{\overline{\varphi}(x)+z}, & (\tau_z \circ \varphi')(g_{\overline{\varphi}(x)+z}) &= (mg)_{\overline{\varphi}^2(x)+\overline{\varphi}z+z}, \\ (\tau_z \circ \varphi')g_{\overline{\varphi}^2(x)+\overline{\varphi}z+z} &= (mg)_{\overline{\varphi}^3(x)+\overline{\varphi}^2z+\overline{\varphi}z+z} = (mg)_x, \end{aligned}$$

because $\overline{\varphi}^3(x) = x$ and $\overline{\varphi}^2z + \overline{\varphi}z + z = 0$ by (7). So $\tau_z \circ \varphi'$ has the same matrices as φ' , but on new invariant summands $(\mathbb{Z}_q)_x \oplus (\mathbb{Z}_q)_{\overline{\varphi}(x)+z} \oplus (\mathbb{Z}_q)_{\overline{\varphi}^2(x)+\overline{\varphi}z+z}$. Similarly for the exceptional orbit. This completes the proof of this case.

Case 3): when $d_i > 1$ for $p_i = 2$ and $k = 4s$. Using the cyclotomic polynomial we can define (similarly to the above M) an element of order 5 in $GL(4, \mathbb{Z})$

$$M_4 = \begin{pmatrix} 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & -1 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & -1 \end{pmatrix}$$

(see e.g. [20] for an elementary introduction). For any $k = 4s$, let $M \in GL(k, \mathbb{Z})$ be the direct sum of s copies of M_4 . Let $\overline{\varphi} : \mathbb{Z}^k \rightarrow \mathbb{Z}^k$ be defined by M . One can calculate

$$\det(M_4 - E) = 5, \quad \det(M - E) = 5^s.$$

Hence, by (4), $R(\overline{\varphi}) = 5^s < \infty$. The length of any non-trivial orbit is 5, hence an *odd* number.

Similarly to M , one can verify that

$$(9) \quad (M_4)^4 + (M_4)^3 + (M_4)^2 + M_4 + E = 0.$$

This can be also deduced from the fact that the characteristic polynomial of the ‘‘companion matrix’’ of a polynomial p is just p .

For p -power components \mathbb{Z}_{p^i} with $p > 2$, we define φ' (as above) by $a_0 \mapsto (p-1)a_0$. Then, for an orbit $u, \overline{\varphi}u, \dots, \overline{\varphi}^\gamma u$, we need to verify (for finiteness of $R(\varphi')$) that $(p-1)^\gamma$ as a homomorphism $\mathbb{Z}_{p^i} \rightarrow \mathbb{Z}_{p^i}$ has no non-trivial fixed elements, i.e. $(p-1)^\gamma \not\equiv 1 \pmod{p}$. This is fulfilled because, for an odd γ , $(p-1)^\gamma - 1 \equiv -2 \not\equiv 0 \pmod{p}$.

For 2-power components $\mathbb{Z}_{2^i} \oplus \mathbb{Z}_{2^i}$, we define φ' by $a_0 \mapsto F_2 a_0$ (as in (5)). Then, for an orbit $u, \overline{\varphi}u, \dots, \overline{\varphi}^\gamma u$, we need to verify that $(F_2)^\gamma$ as a homomorphism $\mathbb{Z}_{2^i} \oplus \mathbb{Z}_{2^i} \rightarrow \mathbb{Z}_{2^i} \oplus \mathbb{Z}_{2^i}$ has no non-trivial fixed elements. Here we need to use not only the fact that γ is odd, but its more specific form: $\gamma = 5$. In particular it can not be divided by 3 = order of $F_2 \pmod{2}$. Hence $(F_2)^\gamma = (F_2)^5 = (F_2)^2 = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix} \pmod{2}$. It has no non-trivial fixed elements mod 2^i for any i .

For 2-power components $\mathbb{Z}_{2^i} \oplus \mathbb{Z}_{2^i} \oplus \mathbb{Z}_{2^i}$, we define φ' by $a_0 \mapsto F_3 a_0$ (as in (5)). Then, for an orbit of $\overline{\varphi}$ of length γ , we need to verify that $(F_3)^\gamma$ as a homomorphism $\mathbb{Z}_{2^i} \oplus \mathbb{Z}_{2^i} \oplus \mathbb{Z}_{2^i} \rightarrow \mathbb{Z}_{2^i} \oplus \mathbb{Z}_{2^i} \oplus \mathbb{Z}_{2^i}$ has no non-trivial fixed elements. One can verify, for $i = 1$, i.e. for $2^i = 2$, that the order of F_3 is relatively prime with 5, namely it is equal to 7. Moreover, $(F_3)^j$, $j = 1, \dots, 6$, has no non-trivial fixed elements. The absence of non-trivial fixed elements is equivalent to $\det((F_3)^j - E) \not\equiv 0 \pmod{2}$. Then $\det((F_3)^j - E) \not\equiv 0 \pmod{2^i}$. Hence, for $i > 1$ these automorphisms still have no non-trivial fixed elements. The elements $(F_3)^{7u}$,

$u = 1, 2, \dots$, typically are not $E \pmod{2^i}$, but in any case $7u \neq 5$, for any u . In fact, we are interested only in properties of $(F_3)^5$.

Collecting together these homomorphisms defined on the summands, we obtain as in the first two cases, φ' with the desired properties. It remains only to verify the epimorphity of $\tau_z \circ \varphi'$. This can be done quite similarly to the end of Case 2) with the help of (9). \square

4. TWISTED BURNSIDE-FROBENIUS THEOREM

Theorem 4.1. *Suppose that φ is an automorphism of the restricted wreath product $G \wr \mathbb{Z}^k = \bigoplus_{m \in \mathbb{Z}^k} G_m \rtimes_{\alpha} \mathbb{Z}^k$, where G is a finite abelian group. Suppose that φ is of finite order. Then $R(\varphi')$ is 1 or ∞ .*

Corollary 4.2. *In particular, φ has the $TBFT_f$ property.*

Proof of Corollary. By Lemma 2.4, $R(\varphi) < \infty$ implies $R(\varphi') < \infty$. Hence, by Theorem 4.1, $R(\varphi') = 1$. Considering $\tau_z \circ \varphi$ instead of φ from the very beginning, we see that $R(\tau_z \circ \varphi') = 1$, for any $z \in \mathbb{Z}^k$. Thus, by Theorem 2.2, Reidemeister classes $\{g\}_{\varphi}$ of φ are pull-backs of Reidemeister classes $\{z\}_{\bar{\varphi}}$ of $\bar{\varphi}$ under the natural projection $\pi : G \wr \mathbb{Z}^k \rightarrow \mathbb{Z}^k$, i.e. $\{g\}_{\varphi} = \pi^{-1}(\{\pi(g)\}_{\bar{\varphi}})$. So, if classes of $\bar{\varphi}$ are separated by an epimorphism $f : \mathbb{Z}^k \rightarrow F$ onto a finite group F , then classes of φ are separated by $f \circ \pi$. It remains to use the equivalence between $TBFT_f$ and separability of Reidemeister classes in the case of finite Reidemeister number (see [11] and [8]). \square

Remark 4.3. In particular, this covers all automorphisms, which were considered in [27]. Indeed, it was proved there, that all orbits are finite and their length is equal to the length of orbits of $\bar{\varphi}$. But the structure of \mathbb{Z}^k implies that $\bar{\varphi}$ has finite order (consider generators). Hence, φ' and φ are of finite order.

Proof of Theorem. Suppose, $R(\varphi') > 1$. Then there exists an element $\sigma \in \Sigma$ such that $\sigma \notin \text{Im}(\text{Id} - \varphi')$. Moreover, $\sigma \notin \text{Im}(\text{Id} - \varphi'_{\sigma})$, where φ'_{σ} is the restriction of φ' onto the φ' -invariant subgroup Σ_{σ} generated by σ . In particular, $R(\varphi'_{\sigma}) > 1$. By the supposition Σ_{σ} is a finite group with generators $\sigma, \varphi'(\sigma), \dots, (\varphi')^s(\sigma)$ for some s . Hence, φ'_{σ} has a nontrivial fixed element σ_0 , $\varphi'_{\sigma}(\sigma_0) = \sigma_0$ and $\sigma_0 \neq 0$. For an element $m \in \mathbb{Z}^k$ consider the orbit

$$\alpha(m)\sigma_0, \quad \varphi'(\alpha(m)\sigma_0) = \alpha(\bar{\varphi}(m))\sigma_0, \quad (\varphi')^t(\alpha(m)\sigma_0) = \alpha(\bar{\varphi}^t(m))\sigma_0, \quad \bar{\varphi}^{t+1}(m) = m.$$

Then $(\varphi')^{t+1}(\alpha(m)\sigma_0) = \alpha(m)\sigma_0$. Passing from m to nm , $n \in \mathbb{Z}$, $m \in \mathbb{Z}^k$, if necessary, we can assume that the supports in \mathbb{Z}^k of $\sigma_0, \alpha(\bar{\varphi}^j(nm))\sigma_0, j = 0, \dots, t$, do not intersect. Then $\sum_{j=1}^t \alpha(\bar{\varphi}^j(nm))\sigma_0$ is a fixed element of φ' , which is distinct from 0 and σ_0 . Increasing n “in sufficiently large steps” we obtain infinitely many distinct fixed elements in the same way. Then by Lemma 2.3, $R(\varphi') = \infty$. \square

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