THE DESCRIPTION OF ROTA-BAXTER OPERATORS OF NONZERO WEIGHT ON COMPLEX GENERAL LINEAR LIE ALGEBRA OF ORDER 2.

M. GONCHAROV

Abstract. In the paper, a classification of Rota-Baxter operators of weight 1 on general linear complex Lie algebra of order 2 is given. The description was made up to the action of $Aut(gl_2(\mathbb{C}))$.

Keywords: Lie algebra, Rota—Baxter operator, general linear Lie algebra

1. Introduction

The Rota—Baxter operators first appeared in the work of F. Tricomi [1]. These operators became popular after the work of G. Baxter [2], where these operators were independently defined as a formalism for the study of integral operators. For a long period of time, Rota Baxter operators were studied in combinatorics and probability theory mainly (see [3], [4], [5]). A new impulse the theory of Rota—Baxter operators received in the 80s of the last century, when a deep connection between Rota—Baxter operators of weight 0 and skew-symmetric solutions of the classical Yang—Baxter equation was found [6, 7]. In the recent years, there were found connections between Rota-Baxter operators and various objects of mathematics such as pre- and post- Lie algebras, double Lie algebras, shuffle algebras, associative Yang-Baxter equation (AYBE) etc.

One of the important and interesting areas of research here is the description of the Rota—Baxter operators on the most important classes of algebras. To date, Rota—Baxter operators of zero and nonzero weights on the Lie algebra $sl_2(\mathbb{C})$ were studied in [8, 9, 10, 11], operators on the matrix algebra $M_2(\mathbb{C})$ were studied in [12, 13]. The classification of Rota—Baxter operators of nonzero weight defined on the algebra $sl_3(\mathbb{C})$ was given by V. V. Sokolov [14]. In [15], the classification of nonsplitting Rota—Baxter operators of nonzero weight on the matrix algebra $M_3(F)$, where F is an algebraically closed field of characteristic 0, was given. Note that all the mentioned classes of algebras are simple.

In this article, we classify Rota—Baxter operators on $gl_2(\mathbb{C})$, the general linear Lie algebra of order 2 over the field of complex numbers $\mathbb{C}$. The classification is given up to the action of the group of automorphisms $Aut(gl_2(\mathbb{C}))$. It is worth noting that the algebra $gl_2(\mathbb{C})$ has a non-zero center, so it is not even a semisimple Lie algebra.

Goncharov M.E., The description of Rota-Baxter operators of nonzero weight on general linear complex Lie algebra of order 2.

The work was supported RAS Fundamental Research Program, project FWNF-2022-0002.

2. Preliminary results.

Let A be an algebra over a field F, $\lambda \in F$.

Definition 1. *A linear map $R : A \mapsto A$ is called a Rota-Baxter operator of weight λ, if for all $a, b \in A$:*

$$R(a)R(b) = R(R(a)b + aR(b) + \lambda ab). \tag{1}$$

If R is a Rota-Baxter operator of weight λ on an algebra A and φ is an automorphism or an anti-automorphism of A, then a map $R' = \varphi^{-1} \circ R \circ \varphi$ is also a Rota-Baxter operator of the same weight λ [12]. This remark allows us to define an equivalence relation on the set of Rota—Baxter operators of weight λ and to make the classification up to the action of the group generated by automorphisms and anti-automorphisms of the algebra A. In this case, we will call operators R and R' *similar* for simplicity.

If $0 \neq \alpha \in F$, then a map αR is a Rota-Baxter operator of weight $\alpha\lambda$. Therefore, up to the multiplication by a nonzero scalar, we have only two different cases: $\lambda = 0$ и $\lambda = 1$.

Example. If A is an arbitrary algebra and $0 \neq \lambda \in F$, then maps R_0 and R_1, defined as $R_0(x) = 0$ or $R_1(x) = -\lambda x$ for all $x \in A$, are Rota-Baxter operators of weight λ. Operators of these types are called trivial.

Let $A = gl_2(\mathbb{C}) = (M_2(\mathbb{C}), [\cdot, \cdot])$ be the general linear Lie algebra over the field of complex numbers $\mathbb{C}$ with the Lie product

$$[x, y] = xy - yx.$$

In this work, we describe Rota-Baxter operators of weight 1 on $gl_2(\mathbb{C})$. Note, that if a map φ is an anti-automorphism of a Lie algebra, then a map $-\varphi$ is an automorphism of the same algebra. Thus, in the case of Lie algebras, the classification should be made up to the action of the group of automorphisms.

We will use the following notations: $\mathrm{E} \in gl_2(\mathbb{C})$ is the identity map, e_{ij} is the usual matrix unit, $h = e_{11} - e_{22}$. As a basis of $gl_2(\mathbb{C})$ we will take a set $\{\mathrm{E}, h, e_{12}, e_{21}\}$. We will denote by $sl_2(\mathbb{C})$ an ideal in $gl_2(\mathbb{C})$, that consist of matrices with trace zero.

Note that for any $\theta \neq 0$, a map φ_θ defined as

$$\varphi_\theta(\mathrm{E}) = \theta E, \ \varphi_\theta(z) = z \text{ for all } z \in sl_2(\mathbb{C}), \tag{2}$$

is an automorphism of $gl_2(\mathbb{C})$.

Let R be a Rota-Baxter operator on $gl_2(\mathbb{C})$. Note, that if $R(\mathrm{E})_J$ is the Jordan form of the matrix $R(\mathrm{E})$ and T is an invertible matrix such that

$$R(\mathrm{E})_J = T^{-1}R(\mathrm{E})T,$$

then a map $\varphi_T : \ gl_2(\mathbb{C}) \mapsto gl_2(\mathbb{C})$, defined as

$$\varphi_T(a) = T^{-1}aT,$$

for all $a \in gl_2(\mathbb{C})$, is an automorphism of $gl_2(\mathbb{C})$. Moreover,

$$\varphi_T \circ R \circ \varphi_{T^{-1}}(\mathrm{E}) = T^{-1}R(T\mathrm{E}T^{-1})T = T^{-1}R(\mathrm{E})T = R(\mathrm{E})_J.$$

Thus, up to the action of $Aut(gl_2(\mathbb{C}))$, we can assume that $R(\mathrm{E})$ is a Jordan matrix. We have the following possibilities for $R(\mathrm{E})_J$:

1. $R(\mathrm{E}) = \lambda E + e_{12}$, $\lambda \in \mathbb{C}$, a Jordan 2×2 block.

2. $R(\mathrm{E}) = \lambda_1 e_{11} + \lambda_2 e_{22}$, $\lambda_1 \neq \lambda_2 \in \mathbb{C}$, a diagonal matrix with different eigenvalues.
3. $R(\mathrm{E}) = \lambda \mathrm{E}$, $\lambda \in \mathbb{C}$, a scalar matrix.

We will consider each of these cases separately.

3. The main part.

The aim of this section is to give a classification of Rota-Baxter operators of weight 1 on $gl_2(\mathbb{C})$. We will break the classification into cases, depending on the Jordan form of the matrix $R(\mathrm{E})$.

Lemma 1. *Let $R : gl_2(\mathbb{C}) \mapsto gl_2(\mathbb{C})$ be a Rota-Baxter operator of weight 1 such that $R(\mathrm{E}) = \lambda\mathrm{E} + e_{12}$. Then R is one of the following:*

$$R(\mathrm{E}) = \lambda\mathrm{E} + e_{12},\ R(h) = R(e_{12}) = R(e_{21}) = 0,\ \lambda \in \mathbb{C}, \tag{3}$$

$$R(\mathrm{E}) = \lambda\mathrm{E} + e_{12},\ R(h) = -h,\ R(e_{12}) = -e_{12},\ R(e_{21}) = -e_{21}\ \lambda \in \mathbb{C}. \tag{4}$$

Proof. Consider

$$[R(h), e_{12}] = [R(h), R(\mathrm{E})] = R([R(h), \mathrm{E}] + [h, R(\mathrm{E})] + [h, \mathrm{E}]) = 2R(e_{12}). \tag{5}$$

Therefore, $R(e_{12}) \in [gl_2(\mathbb{C}), e_{12}]$, that is $R(e_{12}) = \alpha_1 h + \alpha_2 e_{12}$ for some $\alpha_1, \alpha_2 \in \mathbb{C}$.

From

$$[R(e_{12}), e_{12}] = [R(e_{12}), R(\mathrm{E})] = R([R(e_{12}), \mathrm{E}] + [e_{12}, R(\mathrm{E})] + [e_{12}, \mathrm{E}]) = 0 \tag{6}$$

we get that $\alpha_1 = 0$ and $R(e_{12}) = \alpha e_{12}$ for some $\alpha \in \mathbb{C}$.

Similarly,

$$[e_{12}, R(e_{21})] = [R(\mathrm{E}), R(e_{21})] = R([R(\mathrm{E}), e_{21}]) = R(h). \tag{7}$$

This means that

$$R(h) = \beta_1 h + \beta_2 e_{12} \tag{8}$$

for some $\beta_1, \beta_2 \in \mathbb{C}$.

Suppose that $\alpha = 0$. Then $R(e_{12}) = 0$. In this case, from (5) and (8) it follows that $R(h) = \beta_2 e_{12}$. From (7) it follows that $R(e_{21})$ lies in a subspace spanned by e_{12}, h, E. Consider

$$[R(e_{21}), R(h)] = [R(e_{21}), \beta_2 e_{12}] = \gamma e_{12},\ \gamma \in \mathbb{C}. \tag{9}$$

On the other hand, since $R([R(e_{21}), h]) = \delta R(e_{12}) = 0$ for some $\delta \in \mathbb{C}$ and $R([e_{21}, R(h)] = \beta_2 R(h) = \beta_2^2 e_{12}$, we have that

$$R([R(e_{21}), h] + [e_{21}, R(h)] + [e_{21}, h]) = 2R(e_{21}) + \beta_2^2 e_{12}. \tag{10}$$

From the last two equalities, we get that

$$R(e_{21}) = \theta e_{12},\ \theta \in \mathbb{C}.$$

Consider

$$0 = [R(e_{12}), R(e_{21})] = R([R(e_{12}), e_{21}] + [e_{12}, R(e_{21})] + [e_{12}, e_{21}]) = R(h).$$

Hence, $R(h) = 0$. From (9) and (10) we finally get that $R(e_{21}) = 0$ and we obtain the operator (3).

Suppose that $\alpha \neq 0$. From (5) and (8) it follows that $\beta_1 = \alpha$, that is $R(h) = \alpha h + \beta_2 e_{12}$.

From

$$
\begin{aligned}
2\alpha^2 e_{12} &= [R(h), R(e_{12})] \\
&= R([R(h), e_{12}] + [h, R(e_{12})] + [h, e_{12}]) = (4\alpha + 2)R(e_{12}) = (4\alpha + 2)\alpha e_{12}
\end{aligned}
$$

we get that $\alpha = -1$. Using (7), we obtain:

$$
\begin{aligned}
-R(h) &= -[e_{12}, R(e_{21})] \\
&= [R(e_{12}), R(e_{21})] = R([R(e_{12}), e_{21}] + [e_{12}, R(e_{21})] + [e_{12}, e_{21}]) = R(R(h)).
\end{aligned}
$$

That is, $R(R(h)) = -R(h)$. Since $R(h) = -h + \beta_2 e_{12}$ and $R(e_{12}) = -e_{12}$, we obtain $\beta_2 = 0$ and $R(h) = -h$. From (7) it follows that $R(e_{21}) = -e_{21} + \alpha_1 e_{12} + \alpha_2 \mathrm{E}$ for some $\alpha_1, \alpha_2 \in \mathbb{C}$.

Finally, from

$$
\begin{aligned}
-[R(e_{21}), h] &= [R(e_{21}), R(h)] \\
&= R([R(e_{21}), h] + [e_{21}, R(h)] + [e_{21}, h]) = R([R(e_{21}), h]),
\end{aligned}
$$

it follows that $R(e_{21}) = -e_{21}$, and we get that R is equal to the operator (4). □

Lemma 2. *Let $R : gl_2(\mathbb{C}) \mapsto gl_2(\mathbb{C})$ be a Rota-Baxter operator of weight 1 such that $R(\mathrm{E}) = \lambda_1 e_{11} + \lambda_2 e_{22}$, where $\lambda_1 \neq \lambda_2$. Then R is similar to one of the following operators:*

$$R(\mathrm{E}) = \lambda \mathrm{E} + h,\ R(h) = 0,\ R(e_{12}) = R(e_{21}) = 0,\ \lambda \in \mathbb{C}; \tag{11}$$

$$R(\mathrm{E}) = \lambda \mathrm{E} + h,\ R(h) = -h,\ R(e_{12}) = -e_{12},\ R(e_{21}) = -e_{21},\ \lambda \in \mathbb{C}; \tag{12}$$

$$R(\mathrm{E}) = \lambda \mathrm{E} + h,\ R(h) = \alpha_1 \mathrm{E} + \alpha_2 h,\ R(e_{12}) = -e_{12},\ R(e_{21}) = 0,\ \lambda, \alpha_1, \alpha_2 \in \mathbb{C}. \tag{13}$$

Proof. Let $R(\mathrm{E}) = \lambda_1 e_{11} + \lambda_2 e_{22}$, $\lambda_1 \neq \lambda_2$. Then $R(\mathrm{E}) = \alpha_1 E + \alpha_2 h$, where $\alpha_1, \alpha_2 \in \mathbb{C}$. Since $\lambda_1 \neq \lambda_2$, then $\alpha_2 \neq 0$. Define $\theta = \frac{1}{\alpha_2}$. We have

$$\varphi_\theta^{-1} \circ R \circ \varphi_\theta(\mathrm{E}) = \theta \varphi_\theta^{-1}(\alpha_1 \mathrm{E} + \alpha_2 h) = \alpha_1 \mathrm{E} + \frac{\alpha_2}{\alpha_2} h = \alpha_1 \mathrm{E} + h.$$

Therefore, up to the action of $Aut(gl_2(\mathbb{C}))$, we can assume that $R(\mathrm{E}) = \lambda E + h$.

Let $R(e_{12}) = \begin{pmatrix} \beta_{11} & \beta_{12} \\ \beta_{21} & \beta_{22} \end{pmatrix}$. Then we have

$$[R(e_{12}), R(\mathrm{E})] = \left[\begin{pmatrix} \beta_{11} & \beta_{12} \\ \beta_{21} & \beta_{22} \end{pmatrix}, \lambda \mathrm{E} + h \right] = -2\beta_{12} e_{12} + 2\beta_{21} e_{21}.$$

On the other hand,

$$R([R(e_{12}), \mathrm{E}] + [e_{12}, R(\mathrm{E})] + [e_{12}, \mathrm{E}]) = R([e_{12}, \lambda \mathrm{E} + h]) = -2R(e_{12}).$$

From the last two equalities, we can conclude that $\beta_{11} = \beta_{22} = 0$. Moreover, since the characteristic of the ground field in not equal to 2, then $\beta_{21} = 0$. Thus,

$$R(e_{12}) = \beta_{12} e_{12}.$$

Similarly, $R(e_{21}) = \beta_{21} e_{21}$ for some $\beta_{21} \in \mathbb{C}$.

Consider $R(h)$. Let us note that

$$[R(h), \mathrm{E}] + [h, R(\mathrm{E})] + [h, \mathrm{E}] = 0.$$

Hence, $[R(h), R(\mathrm{E})] = 0$ and $R(h) = \begin{pmatrix} \delta_1 & 0 \\ 0 & \delta_2 \end{pmatrix}$, $\delta_i \in \mathbb{C}$.

Let $e_{ij} \in \{e_{12}, e_{21}\}$. Then

$$(\delta_i - \delta_j)\beta_{ij}e_{ij} = [R(h), R(e_{ij})]$$
$$= R([R(h), e_{ij}] + [h, R(e_{ij})] + [h, e_{ij}]) = (\delta_i - \delta_j + (-1)^j(2\beta_{ij} + 2))\beta_{ij}e_{ij},$$

where β_{ij} are defined from conditions $R(e_{ij}) = \beta_{ij}e_{ij}$, $i \neq j$. Thus, we have two options: $\beta_{ij} = 0$ or $\beta_{ij} = -1$.

If $\beta_{12} = \beta_{21} = 0$, then from

$$0 = [R(e_{12}), R(e_{21})] = R([R(e_{12}), e_{21}] + [e_{12}, R(e_{21})] + [e_{12}, e_{21}]) = R(h)$$

it follows that $R(h) = 0$ and we obtain the operator (11). Similarly, if $\beta_{12} = \beta_{21} = -1$, then we get that $R(h) = -h$ and R is equal to the operator (12).

Finally, if $\beta_{12} \neq \beta_{21}$, then up to the action of an automorphism $\varphi : z \mapsto -z^\tau$ (where τ is the transpose) for all $z \in gl_2(\mathbb{C})$, we can assume that $\beta_{12} = 0$, $\beta_{21} = -1$. Thus, in this case we obtain the operator (13). □

Lemma 3. *Let $R : gl_2(\mathbb{C}) \mapsto gl_2(\mathbb{C})$ be a Rota-Baxter operator of weight 1 such that $R(\mathrm{E}) = \lambda\mathrm{E}$. Then R is similar to one of the following operators:*

$$R(\mathrm{E}) = \lambda\mathrm{E},\ R(h) = 0,\ R(e_{12}) = -e_{12} + th;\ R(e_{21}) = 0,\ t \in \{0, 1\} \tag{14}$$

$$R(\mathrm{E}) = \lambda\mathrm{E},\ R(h) = 0,\ R(e_{12}) = -e_{12} + th + E;\ R(e_{21}) = 0,\ t \in \{0, 1\} \tag{15}$$

$$R(\mathrm{E}) = \lambda\mathrm{E},\ R(h) = E,\ R(e_{12}) = -e_{12} + h + \alpha\mathrm{E};\ R(e_{21}) = 0,\ \alpha \in \mathbb{C} \tag{16}$$

$$R(\mathrm{E}) = \lambda\mathrm{E},\ R(h) = E,\ R(e_{12}) = -e_{12} + \mathrm{E};\ R(e_{21}) = 0 \tag{17}$$

$$R(\mathrm{E}) = \lambda\mathrm{E},\ R(h) = th,\ R(e_{12}) = -e_{12},\ R(e_{21}) = 0,\ t \in \mathbb{C},\ t \neq 0 \tag{18}$$

$$R(\mathrm{E}) = \lambda\mathrm{E},\ R(h) = th + \mathrm{E},\ R(e_{12}) = -e_{12},\ R(e_{21}) = 0,\ t \in \mathbb{C},\ t \neq 0 \tag{19}$$

$$R(\mathrm{E}) = \lambda\mathrm{E},\ R(h) = -h + \alpha\mathrm{E},\ R(e_{12}) = -e_{12},\ R(e_{21}) = \mathrm{E},\ \alpha \in \mathbb{C} \tag{20}$$

$$R(\mathrm{E}) = \lambda\mathrm{E}, R(h) = th,\ R(e_{12}) = te_{12},\ R(e_{21}) = te_{21},\ t \in \{0, -1\}. \tag{21}$$

Proof. In this case, the center of the algebra $gl_2(\mathbb{C})$ is R-invariant. Hence, R induces a Rota-Baxter operator $\bar{R}$ on the quotient algebra $g_1 = gl_2(\mathbb{C})/Z(gl_2(\mathbb{C})) \cong sl_2(\mathbb{C})$. Given an element $a \in gl_2(\mathbb{C})$, $\bar{a}$ denotes the coset generated by a.

The classification of Rota-Baxter operators of nonzero weight 1 on $sl_2(\mathbb{C})$ was already made in [10]. Unfortunately, we can't use this classification since it was made without taking into account the action of the group $Aut(sl_2(\mathbb{C}))$. The description of Rota-Baxter operators up to the action of $Aut(sl_2(\mathbb{C}))$ was made in a preprint of V. Gubarev and R. Kozlov. Beside the trivial ones, there are two nonsimilar operators on $sl_2(\mathbb{C})$:

I. $R(h) = R(e_{21}) = 0,\ R(e_{12}) = -e_{12} + th,\ t \in \{0, 1\}$,

II. $R(h) = th,\ R(e_{12}) = -e_{12},\ R(e_{21}) = 0, t \in \mathbb{C},\ t \neq 0$.

Let $\bar{\varphi}$ be an automorphism of g_1. For a coset $\bar{a} \in g_1$ we can uniquely choose a generator $a \in gl_2(\mathbb{C})$ such that $tr(a) = 0$. Define a map $\varphi : gl_2(\mathbb{C}) \mapsto gl_2(\mathbb{C})$ as $\varphi(\mathrm{E}) = \mathrm{E}$ and $\varphi(p) = q$ if $tr(p) = tr(q) = 0$ and $\bar{\varphi}(\bar{p}) = \bar{q}$. It is easy to see that the map φ is an automorphism of the algebra $gl_2(\mathbb{C})$. Moreover, for all $x \in gl_2(\mathbb{C})$ we have $\bar{\varphi}(\bar{x}) = \overline{\varphi(x)}$. Note that if R is an arbitrary Rota-Baxter operator (of any weight) on $gl_2(\mathbb{C})$ satisfying $R(\mathrm{E}) = \theta\mathrm{E}$, $\bar{R} : g_1 \mapsto g_1$ is the corresponding

Rota-Baxter operator on g_1 and $\bar{\varphi}$ is an arbitrary automorphism of g_1, then the composition $R_1 = \varphi^{-1}\circ R\circ\varphi$ is a Rota-Baxter operator of the same weight satisfying $R_1(\mathrm{E}) = \theta\mathrm{E}$ and $\overline{R_1} = \overline{\varphi}^{-1} \circ \bar{R} \circ \overline{\varphi}$. This allows us to consider that $\bar{R}$ is a trivial Rota-Baxter operator or one of the operators I or II defined above. We will consider these situations one by one.

Case 1. Let $\bar{R}(\bar{h}) = \bar{R}(\overline{e_{21}}) = 0,\ \bar{R}(\overline{e_{12}}) = -\overline{e_{12}} + t\bar{h},\ t \in \{0,1\}$.

Going back to the operator R, we can say that

$$R(\mathrm{E})=\lambda\mathrm{E},\ R(h)=\alpha_1\mathrm{E},\ R(e_{21})=\alpha_2\mathrm{E},\ R(e_{12})=-e_{12}+th+\alpha_3\mathrm{E},\ t \in \{0,1\},\ \alpha_i \in \mathbb{C}.$$

From

$$0 = [R(e_{21}), R(h)] = R([R(e_{21}), h] + [e_{21}, R(h)] + [e_{21}, h]) = 2R(e_{21})$$

it follows that $\alpha_2 = 0$.

If $\alpha_1 = \alpha_3 = 0$, then we obtain (14). Suppose that $\alpha_1 = 0, \alpha_3 \neq 0$. Let $R_1 = \varphi^{-1}_{\alpha_3^{-1}} \circ R \circ \varphi_{\alpha_3^{-1}}$. Then direct computations show that

$$R_1(\mathrm{E}) = \lambda\mathrm{E},\ R_1(e_{12}) = -e_{12} + th + \mathrm{E},\ R_1(e_{21}) = 0,\ R_1(h) = 0.$$

We obtain the operator (15).

Suppose that $\alpha_1 \neq 0$. Using the conjugation with the automorphism $\varphi_{\alpha_1^{-1}}$, we obtain an operator

$$R(\mathrm{E}) = \lambda\mathrm{E},\ R(h) = E,\ R(e_{12}) = -e_{12} + th + \frac{\alpha_3}{\alpha_1}\mathrm{E},\ R(e_{21}) = 0. \tag{22}$$

If $t = 1$, then we get (16). Let $t = 0$. Consider an automorphism $\psi_{\alpha_3^{-1}}$. If $R_1 = \varphi^{-1}_{\alpha_3^{-1}} \circ R \circ \varphi_{\alpha_3^{-1}}$, then direct computations show that $R_1(\mathrm{E}) = \lambda\mathrm{E}$, $R_1(h) = \mathrm{E}$, $R_1(e_{21}) = 0$. Moreover,

$$R_1(e_{12}) = \varphi^{-1}_{\alpha_3^{-1}}(R(\alpha_3^{-1}e_{12})) = \alpha_3^{-1}\varphi^{-1}_{\alpha_3^{-1}}(-e_{12} + \alpha_3\mathrm{E}) = -e_{12} + \mathrm{E}.$$

Hence, we obtain (17).

Case 2. In this case, we have that

$$R(\mathrm{E}) = \lambda\mathrm{E},\ R(h) = th+\alpha_1\mathrm{E},\ R(e_{21}) = \alpha_2\mathrm{E},\ R(e_{12}) = -e_{12}+\alpha_3\mathrm{E},\ t, \alpha_i \in \mathbb{C},\ t \neq 0.$$

From

$$0 = [R(e_{21}), R(h)] = R([R(e_{21}), h] + [e_{21}, R(h)] + [e_{21}, h]) = 2(t+1)R(e_{21})$$

it follows that a condition $t \neq -1$ implies $\alpha_2 = 0$. At the same time, if $t = -1$, then α_2 is arbitrary.

Note that an equality

$$[R(e_{12}), R(e_{21})] = R([R(e_{12}), e_{21}] + [e_{12}, R(e_{21})] + [e_{12}, e_{21}])$$

holds and gives no restrictions for parameters α_i. Consider

$$[R(e_{12}), R(h)] = [-e_{12} + \alpha_3\mathrm{E}, th + \alpha_1\mathrm{E}] = 2te_{12},$$

$$R([R(e_{12}), h]+[e_{12}, R(h)]+[e_{12}, h]) = R(-[e_{12}, h]+t[e_{12}, h]+[e_{12}, h])=2te_{12} + 2t\alpha_3\mathrm{E}.$$

Since $t \neq 0$, then $\alpha_3 = 0$.

Let $\alpha_2 = 0$. If, at the same time, $\alpha_1 = 0$, then we obtain (18). If $\alpha_1 \neq 0$, then the conjugation with $\varphi_{\alpha_1^{-1}}$ gives us the operator (19).

If $t = -1$ and $\alpha_2 \neq 0$, then the conjugation with $\varphi_{\alpha_2^{-1}}$ gives us (20).

Case 3. It remains to consider the cases of trivial operator $\overline{R}$, that is $\overline{R}(\overline{v}) = t\overline{v}$, $\overline{v} \in gl_2(\mathbb{C})/Z(gl_2(\mathbb{C}))$, $t \in \{0, -1\}$. Note that in these cases for all $a, b \in sl_2(\mathbb{C})$:

$$t^2[a, b] = [R(a), R(b)] = R([R(a), b] + [a, R(b)] + [a, b]) = (-1)^t R([a, b]).$$

Therefore, $R(sl_2(\mathbb{C})) \subset sl_2(\mathbb{C})$ and R is equal to (21). □

Remark 1. In Lemmas 1-3, in order to obtain conditions for operators (3)-(4) and (11)-(21), we've checked that these operators satisfy the condition (1) for the case when a and b are vectors from the basis of $gl_2(\mathbb{C})$. Thus, these operators are Rota-Baxter operators of weight 1.

4. The similarity check.

In this section, we will prove that operators from lemmas 1-3 are not similar.

Lemma 4. *Let R_i, $(i = 1, 2, 3)$ be Rota-Baxter operators of weight 1 on $gl_2(\mathbb{C})$, such that $R_1(\mathrm{E}) = \lambda_1\mathrm{E} + e_{12}$, $R_2(\mathrm{E}) = \lambda_2\mathrm{E} + h$, $R_3(\mathrm{E}) = \lambda_3\mathrm{E}$. Then R_i are not similar.*

Proof. First we note that R_3 can't be similar to R_1 or R_2. Indeed, if φ is an automorphism of the algebra $gl_2(\mathbb{C})$, then $\varphi(\mathrm{E}) = \theta\mathrm{E}$ for some $\theta \in \mathbb{C}\backslash\{0\}$. Hence,

$$(\varphi^{-1} \circ R_3 \circ \varphi)(\mathrm{E}) = (\varphi^{-1} \circ R)(\theta\mathrm{E}) = \theta\lambda_3\varphi^{-1}(\mathrm{E}) = \lambda_3\mathrm{E}.$$

Consider R_1 and R_2. Let $\varphi \in Aut(gl_2(\mathbb{C}))$. Then

$$(\varphi^{-1} \circ R_1 \circ \varphi)(\mathrm{E}) = (\varphi^{-1} \circ R)(\theta\mathrm{E}) = \theta\varphi^{-1}(\lambda_1\mathrm{E} + e_{12}) = \lambda_1\mathrm{E} + \varphi^{-1}(e_{12}).$$

Since e_{12} is an ad-nilpotent element of $gl_2(\mathbb{C})$, then $\varphi(e_{12})$ can't be semisimple. Therefore, R_1 and R_2 are not similar. □

Remark 2. All operators from lemmas 1-3 depend on the parameter λ. From the proof of Lemma 4, it follows that two Rota-Baxter operators from lemmas 1-3 corresponding to different parameters $\lambda_1 \neq \lambda_2$, can't be similar.

Lemma 5. *Operators* (3) *and* (4) *are not similar.*

Proof. Indeed, the image of the operator (3) is a one-dimensional subspace spanned by $R(\mathrm{E})$ while in the image of (4) there are at least two linear independent vectors e_{12} and e_{21}. □

Lemma 6. *Operators* (11), (12) *and* (13) *are not similar. Moreover, operators of the type* (13) *corresponding to different pairs of parameters* (α_1, α_2) *are not similar as well.*

Proof. The image of an operator of type (11) is one dimensional while the image of an operator from (12) or (13) is at least two. Thus, (11) can't be similar to an operator from (12) or (13).

Let R_1 and R_2 be Rota-Baxter operators of types (12) and (13) respectively. Then $[Im(R_1), Im(R_1)] = sl_2(\mathbb{C})$ while $Im(R_2)$ is a solvable algebra. Hence, operators R_1 and R_2 are not similar.

It remains to prove that different pairs of parameters (α_1,α_2) in (13) gives us non-similar Rota-Baxter operators. Let R be a Rota-Baxter operator of type (13), φ be an automorphism of $gl_2(\mathbb{C})$). Suppose that an operator $R_1 = \varphi^{-1} \circ R \circ \varphi$ is again a Rota-Baxter operator of type (13). Let $\varphi(\mathrm{E}) = \theta\mathrm{E}$, then

$$(\varphi^{-1} \circ R \circ \varphi)(\mathrm{E}) = (\varphi^{-1} \circ R)(\theta \mathrm{E}) = \theta\varphi^{-1}(\lambda_1 \mathrm{E} + h) = \lambda \mathrm{E} + \theta\varphi^{-1}(h).$$

Therefore, $\varphi(h) = \theta h$. Then,

$$(\varphi^{-1} \circ R \circ \varphi)(h) = (\varphi^{-1} \circ R)(\theta h) = \theta\varphi^{-1}(\alpha_1 \mathrm{E} + \alpha_2 h) = \alpha_1 \mathrm{E} + \alpha_2 h.$$

Hence, the conjugation by an automorphism can't transform R into a Rota-Baxter operator of the same type (13) with another pair of parameters (α_1, α_2). □

Lemma 7. *Operators* (14) − (21) *from Lemma 3, including operators of the same type corresponding to different parameters, are not similar.*

Proof. Note that an operator R_1 of a type (14) − (17) and an operator R_2 of a type (18)–(20) can't be similar since they can't be conjugate by an automorphism of type φ_α and the induced operators $\overline{R_i} : gl_2(\mathbb{C})/Z(gl_2(\mathbb{C})) \mapsto gl_2(\mathbb{C})/Z(gl_2(\mathbb{C}))$ are not similar as operators defined on $g/Z(gl_2(\mathbb{C})) = sl_2(\mathbb{C})$. Similar arguments show that different parameters t give us non-similar operators.

An operator of type (21) can't be similar to an operator of another type from Lemma 3 since only in (21) we have that $Im(R) \subset Z(gl_2(\mathbb{C}))$ (if $t = 0$) and $sl_2(\mathbb{C}) \subset Im(R)$ (if $t = -1$). Note that similar arguments show that the case $t = 0$ is not similar to the case when $t = -1$ in (21).

Consider operators (14) − (17). If R is a Rota-Baxter operator of type (14), then for all $x \in sl_2(\mathbb{C})$, $R(x) \in sl_2(\mathbb{C})$. Since $sl_2(\mathbb{C})$ is $Aut(gl_2(\mathbb{C}))$-invariant, then $sl_2(\mathbb{C})$ is $\varphi^{-1} \circ R \circ \varphi$-invariant. Hence, R can't be similar to an operator from (15), (16) and (17). Similar arguments show that an operator from (18) can't be similar to an operator from (19) or (20).

Let R be a Rota-Baxter operator of type (15). Then $R(h) = 0$. Let φ be an automorphism of $gl_2(\mathbb{C})$ such that $\varphi^{-1} \circ R \circ \varphi(h) = \mathrm{E}$. Then $R(\varphi(h)) = \theta\mathrm{E}$, $\theta \neq 0$ that is not possible since $\mathrm{tr}(\varphi(h)) = 0$. Hence, R can't be similar to an operator from (16) or (17).

Operators (16) and (17) are not similar since the induced operators on the quotient algebra $gl_2(\mathbb{C})/Z(gl_2(\mathbb{C})$ are not similar (they correspond to different values of the parameter t, see the proof of Lemma 3).

Let R be a Rota-Baxter operator from (16) and φ be an automorphism of $gl_2(\mathbb{C})$ such that $R_1 = \varphi^{-1} \circ R \circ \varphi$ is again a Rota-Baxter operator of type (16). Suppose that $\varphi(\mathrm{E}) = \theta\mathrm{E}$. Since $R_1(h) = \mathrm{E}$, then $R(\varphi(h)) = \theta\mathrm{E}$. Note that $\mathrm{tr}(\varphi(h)) = 0$. Hence, $\varphi(h) = \theta h$. Condition $R_1(e_{21}) = 0$ implies that $R(\varphi(e_{12})) = 0$. Therefore, $\varphi(e_{21}) = \delta e_{21}$ which in turn implies that $\varphi(e_{12}) = \gamma e_{12}$. From $[\varphi(h), \varphi(e_{12})] = \varphi([h, e_{12}])$ we get that $\theta = 1$.

Suppose that $R_1(e_{12}) = -e_{12} + h + \alpha_1\mathrm{E}$. Then

$$R(\varphi(e_{12})) = -\varphi(e_{12}) + \theta h + \theta\alpha_1\mathrm{E} = -\gamma e_{12} + h + \alpha_1\mathrm{E}.$$

Thus,

$$\gamma(-e_{12} + h + \alpha\mathrm{E}) = -\gamma e_{12} + h + \alpha_1\mathrm{E}.$$

This means that $\gamma = 1$ and $\alpha = \alpha_1$. In other words, Rota-Baxter operators of type (16), corresponding to different values of the parameter α, are not similar.

It remains to consider operators (18)–(20). We've already noted that an operator from (18) can't be similar to an operator from (19) or (20).

If R is a Rota-Baxter operator of type (19), then $e_{21} \in ker(R)$. At the same time, if R_1 is an operator of type (20), then $ker(R_1) \cap sl_2(\mathbb{C}) = \{0\}$. Note that for every

$\varphi \in Aut(gl_2(\mathbb{C}))$ we have that $\varphi(e_{21}) \in sl_2(\mathbb{C})$. Therefore, $\varphi^{-1} \circ R_1 \circ \varphi(e_{21}) \neq 0$ for every $\varphi \in Aut(gl_2(\mathbb{C}))$. Hence, operators from (19) and (20) lie in different orbits under the action of $Aut(gl_2(\mathbb{C}))$.

What is left is to prove that different values of the parameter α in (20) give non-similar Rota-Baxter operators. Let R be a Rota-Baxter operator of type (20) and φ be an automorphism of $gl_2(\mathbb{C})$ such that $R_1 = \varphi^{-1} \circ R \circ \varphi$ is again a Rota-Baxter operator from (20). Suppose that $\varphi(\mathrm{E}) = \theta\mathrm{E}$. From $R(e_{21}) = R_1(e_{21}) = \mathrm{E}$ we get that $\varphi(e_{21}) = \theta e_{21}$. From $R_1(e_{12}) = -e_{12}$ it follows that $\varphi(e_{12}) = \delta_1 e_{12} + \delta_2 h$ for some $\delta_1, \delta_2 \in \mathbb{C}$. Since $R_1(h) = -h + \alpha_1\mathrm{E}$ for some $\alpha_1 \in \mathbb{C}$, we have

$$\varphi(h) = -R(\varphi(h)) + \alpha_1\theta\mathrm{E}. \tag{23}$$

On the other hand,

$$\varphi(h) = [\varphi(e_{12}), \varphi(e_{21})] \in \mathbb{C}h + \mathbb{C}e_{21}. \tag{24}$$

Note that $e_{21} \notin Im(R)$. Hence, from (23) and (24) it follows that $\varphi(h) = \gamma h$ for some $\gamma \in \mathbb{C}$. Since φ is an automorphism of $gl_2(\mathbb{C})$, then $\gamma = 1$. Then (23) implies that $\theta = 1$. Therefore,

$$R_1(h) = \varphi^{-1} \circ R \circ \varphi(h) = -h + \alpha\mathrm{E}.$$

That completes the proof. □

Lemmas 1-6 imply the following

Theorem 1. *Let R be a Rota-Baxter operator of weight 1 defined on $gl_2(\mathbb{C})$. Then, up to the action of $Aut(gl_2(\mathbb{C}))$, R is one of the following:*

$$1.\ R(\mathrm{E}) = \lambda\mathrm{E} + e_{12},\ R(h) = R(e_{12}) = R(e_{21}) = 0,\ \lambda \in \mathbb{C};$$

$$2. R(\mathrm{E}) = \lambda\mathrm{E} + e_{12},\ R(h) = -h,\ R(e_{12}) = -e_{12},\ R(e_{21}) = -e_{21},\ \lambda \in \mathbb{C};$$

$$3.\ R(\mathrm{E}) = \lambda\mathrm{E} + h,\ R(h) = 0,\ R(e_{12}) = R(e_{21}) = 0,\ \lambda \in \mathbb{C};$$

$$4.\ R(\mathrm{E}) = \lambda\mathrm{E} + h,\ R(h) = -h,\ R(e_{12}) = -e_{12},\ R(e_{21}) = -e_{21},\ \lambda \in \mathbb{C};$$

$$5.\ R(\mathrm{E}) = \lambda\mathrm{E} + h,\ R(h) = \alpha_1\mathrm{E} + \alpha_2 h,\ R(e_{12}) = -e_{12},\ R(e_{21}) = 0,\ \lambda, \alpha_1, \alpha_2 \in \mathbb{C};$$

$$6.\ R(\mathrm{E}) = \lambda\mathrm{E},\ R(h) = 0,\ R(e_{12}) = -e_{12} + th;\ R(e_{21}) = 0,\ \lambda \in \mathbb{C},\ t \in \{0,1\};$$

$$7.\ R(\mathrm{E}) = \lambda\mathrm{E},\ R(h) = 0,\ R(e_{12}) = -e_{12} + th + E;\ R(e_{21}) = 0,\ \lambda \in \mathbb{C},\ t \in \{0,1\};$$

$$8.\ R(\mathrm{E}) = \lambda\mathrm{E},\ R(h) = E,\ R(e_{12}) = -e_{12} + h + \alpha\mathrm{E};\ R(e_{21}) = 0,\ \lambda, \alpha \in \mathbb{C};$$

$$9.\ R(\mathrm{E}) = \lambda\mathrm{E},\ R(h) = E,\ R(e_{12}) = -e_{12} + \mathrm{E};\ R(e_{21}) = 0,\ \lambda \in \mathbb{C};$$

$$10.\ R(\mathrm{E}) = \lambda\mathrm{E},\ R(h) = th,\ R(e_{12}) = -e_{12},\ R(e_{21}) = 0,\ \lambda \in \mathbb{C},\ t \in \mathbb{C},\ t \neq 0;$$

$$11.\ R(\mathrm{E}) = \lambda\mathrm{E},\ R(h) = th + \mathrm{E},\ R(e_{12}) = -e_{12},\ R(e_{21}) = 0,\ \lambda \in \mathbb{C},\ t \in \mathbb{C},\ t \neq 0;$$

$$12.\ R(\mathrm{E}) = \lambda\mathrm{E},\ R(h) = -h + \alpha\mathrm{E},\ R(e_{12}) = -e_{12},\ R(e_{21}) = \mathrm{E},\ \lambda, \alpha \in \mathbb{C};$$

$$13.\ R(\mathrm{E}) = \lambda\mathrm{E}, R(h) = th,\ R(e_{12}) = te_{12},\ R(e_{21}) = te_{21},\ \lambda \in \mathbb{C},\ t \in \{0, -1\}.$$

Operators 1-13 (including operators of the same type satisfying different values of parameters) lie in different orbits under the action of the group $Aut(gl_2(\mathbb{C}))$.

References

[1] F.G. Tricomi, *On the finite Hilbert transformation*, Quart. J. Math. 2 (1951), 199–211. MR0043258

[2] G. Baxter, *An analytic problem whose solution follows from a simple algebraic identity*, Pacific J. Math. **10** (1960) 731–742. MR0119224

[3] F.V. Atkinson, *Some aspects of Baxter's functional equation*, J. Math. Anal. Appl. **7** (1963) 1–30. MR0155196

[4] P. Cartier, *On the structure of free Baxter algebras*, Adv. Math. **9** (1972) 253–265. MR0338040

[5] G.-C. Rota, *Baxter algebras and combinatorial identities.* I, Bull. Amer. Math. Soc. **75** (1969) 325–329. MR0244070

[6] A.A. Belavin, V.G. Drinfeld, *Solutions of the classical Yang - Baxter equation for simple Lie algebras*, Funct. Anal. Appl., **16**:3 (1982), 159—180. MR0674005

[7] M.A. Semenov-Tyan-Shanskii, *What is a classical r-matrix?*, Funct. Anal. Appl., 17:4 (1983), 259–272. MR0725413

[8] P.S. Kolesnikov, *Homogeneous averaging operators on simple finite conformal Lie algebras*, J. Math. Phys. **56** (2015), 071702. MR3405939

[9] E.I. Konovalova, *Double Lie algebras*, PhD thesis, Samara (2009), in Russian.

[10] Yu Pan, Q. Liu, C. Bai, L. Guo, *PostLie algebra structures on the Lie algebra* $\mathrm{sl}(2,\mathbb{C})$, Electron. J. Linear Algebra **23** (2012) 180–197. MR2889581

[11] J. Pei, C. Bai, and L. Guo, *Rota-Baxter operators on* $\mathrm{sl}(2,\mathbb{C})$ *and solutions of the classical Yang-Baxter equation*, J. Math. Phys. **55** (2014), 021701, 17 p. MR3202860

[12] P. Benito, V. Gubarev, A. Pozhidaev, *Rota—Baxter operators on quadratic algebras*, Mediterr. J. Math. **15** (2018), 23 p. (N189). MR3843415

[13] X. Tang, Y. Zhang, and Q. Sun, *Rota-Baxter operators on 4-dimensional complex simple associative algebras*, Appl. Math. Comp. **229** (2014) 173–186. MR3159865

[14] V.V. Sokolov, *Classification of constant solutions of the associative Yang—Baxter equation on* Mat_3, Theor. Math. Phys. (3) **176** (2013) 1156–1162. MR3230739

[15] M. Goncharov, V. Gubarev, *Rota–Baxter operators of nonzero weight on the matrix algebra of order three*, Linear and Multilinear Algebra, **70**:6 (2022) 1055-1080. MR4402515

Maxim Goncharov
Sobolev Institute of Mathematics,
Novosibirsk State University, Department of Mechanics and Mathematics, Russia
Email address: goncharov.gme@gmail.com