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QUASIVARIETIES GENERATED BY SMALL SUBORDER
LATTICES. I. EQUATIONAL BASES

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ABSTRACT. For each cardinal $\kappa > 0$, the quasivariety generated by the suborder lattice of M_κ is a finitely based variety. An equational basis for this variety is found.

Keywords: lattice, quasivariety, variety, poset.

1. INTRODUCTION

Suborder lattices were studied by several authors; we refer to D. Bredikhin and B. Schein [3] and to B. Sivák [15] as well as to [12, 13, 2]. Suborder lattices were used as a convenient tool in establishing some deep results for subsemigroup lattices which are presented in the papers of V. B. Repnitskiĭ [10, 11]; see also [14].

For a positive integer n , let \mathbf{SO}_n denote the class of lattices embeddable into suborder lattices of partial orders of length at most n . It was established in [13] that \mathbf{SO}_n is a finitely based variety and a particular finite equational basis was found for this variety in [13].

There are still some unsolved problems which concern suborder lattices. For example, Question 2 in [13] asks if the quasivariety generated by a finite suborder lattice is a variety. A positive answer to this question was given in [2] for the suborder lattice $\mathbf{O}(M_1)$. Moreover, it was established in [2] that the quasivariety generated by $\mathbf{O}(M_1)$ is a variety and a particular finite equational basis was found for this variety.

In this paper, we extend the results from [2] to a more general case and consider lattices $\mathbf{O}(M_\kappa)$ for an arbitrary cardinal $\kappa > 0$, see Figure 1. Specifically, we

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prove that the quasivariety $\mathbf{Q}(\mathbf{O}(M_\kappa))$ generated by the suborder lattice $\mathbf{O}(M_\kappa)$ is a finitely based variety and find a finite basis for this variety, see Theorem 10, Theorem 13, and Corollaries 14 and 15. In a subsequent article, the results of this paper will be used for establishing categorical dualities for the quasivarieties $\mathbf{Q}(\mathbf{O}(M_\kappa))$ where $1 < \kappa \leq \omega$.

2. BASIC CONCEPTS

For all the notions which are not defined in this section, we refer to A. I. Maltsev [8] and V. A. Gorbunov [6].

2.1. Lattices. Most of the following definitions concerning join covers are in accordance with R. Freese, J. Ježek, and J. B. Nation [5].

Let L be a lattice. For arbitrary two sets $A, B \subseteq L$, we say that A *refines* B and write $A \ll B$ if for each $a \in A$, there is $b \in B$ such that $a \leq b$. If $x \in L$, then A is a *join cover* of x if $\bigvee A$ exists and $x \leq \bigvee A$; we also call $x \leq \bigvee A$ a *join cover* in this case. A join cover $x \leq \bigvee A$ is *nontrivial* if $x \not\leq a$ for all $a \in A$; $x \leq \bigvee A$ is *finite* if the set A is finite. A join cover $x \leq \bigvee A$ is *irredundant* if $x \not\leq \bigvee B$ for any proper subset $B \subset A$. A join cover $x \leq \bigvee A$ is *minimal* if $A \subseteq B$ for each join cover $x \leq \bigvee B$ such that $B \ll A$. The lattice L has the *complete minimal join cover refinement property* $(\text{CR})_X$ for a set $X \subseteq L$ if each nontrivial join cover of each element from X can be refined to a minimal one.

By $J(L)$, we denote the set of all join-irreducible elements of L and by $\text{CJ}(L)$ — the set of all completely join-irreducible elements of L . Similarly, by $\text{P}(L)$, we denote the set of all join-prime elements of L and by $\text{CP}(L)$ — the set of all completely join-prime elements of L .

Definition 1. Let L be a lattice and let $J \subseteq J(L)$. We say that L is a *J-lattice* if L possesses the following properties:

- (i) for each element $a \in L$, there is a subset $J_a \subseteq J$ with $a = \bigvee J_a$;
- (ii) for each element $a \in J$ and each nontrivial join cover $a \leq a_0 \vee \dots \vee a_n$ with $n < \omega$ and $a_0, \dots, a_n \in L$, there is a finite set $F \subseteq J$ such that $a \leq \bigvee F$ is a minimal join cover and $F \ll \{a_0, \dots, a_n\}$.

We say that L is a *CJ-lattice* if L possesses the following properties:

- (i) for each element $a \in L$, there is a subset $J_a \subseteq \text{CJ}(L)$ with $a = \bigvee J_a$;
- (ii) L has the property $(\text{CR})_{\text{CJ}(L)}$.

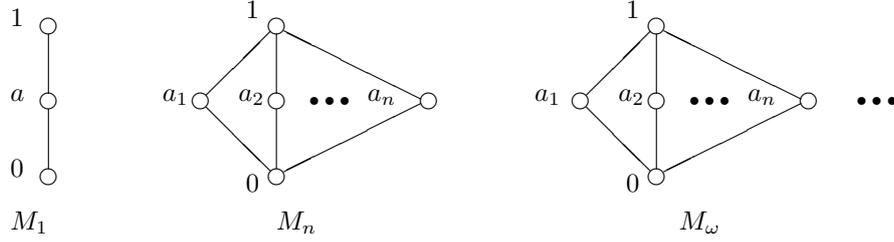
It follows from the definition above that each *CJ-lattice* is a *J-lattice* for $J = \text{CJ}(L)$. *J-lattices* were considered in [1, 4], see also [13].

For a *J-lattice* L and an element $x \in J(L)$, let $\mathfrak{M}(x)$ denote the set of all finite minimal join covers of x .

Remark 1. We note that in an upper continuous lattice L , each minimal join cover of an element $x \in \text{CJ}(L)$ belongs to $\mathfrak{M}(x)$.

Proposition 1. [4] *Let L be a complete dually algebraic lattice. Then the following statements hold.*

- (i) *If L is n -distributive then L is a J -lattice.*
- (ii) *If L is in addition algebraic then L is a CJ -lattice.*

FIGURE 1. Posets M_1 , M_n , and M_ω

2.2. Suborder lattices. Let X be a set and let $R \subseteq X^2$ be a *strict partial order* on X ; that is an irreflexive, antisymmetric, and transitive binary relation. In this case, we also say that $\langle X; R \rangle$ is a *partially ordered set* or a *poset* for short. A subset $R' \subseteq R$ is a (*strict*) *suborder* of R if the structure $\langle X; R' \rangle$ is also a poset. The set $O(X, R)$ of all (strict) suborders of a partial order R on X is a partially ordered set with respect to the relation \subseteq of set-theoretic inclusion. Obviously, \emptyset is a least suborder of R . Thus, \emptyset is a least element in $O(X, R)$. It is also obvious that R is a greatest element in $O(X, R)$. It is straightforward to check that for an arbitrary family $\{R_i \mid i \in I\} \subseteq O(X, R)$, the relation $\bigcap_{i \in I} R_i$ is also a suborder of R ; that is,

$$\bigwedge_{i \in I} R_i = \bigcap_{i \in I} R_i \in O(X, R).$$

Thus, $O(X, R)$ forms a complete lattice, where

$$\bigvee_{i \in I} R_i = \left(\bigcup_{i \in I} R_i \right)^t;$$

here Y^t denotes the transitive closure of a binary relation $Y \subseteq X^2$. It is clear that

$$J(O(X, R)) = \text{CJ}(O(X, R)) = \left\{ \{(a, b)\} \mid (a, b) \in R \right\}.$$

We consider here strict partial orders instead of ordinary partial orders for the sake of simplicity only; the least element of a suborder lattice is in this case \emptyset and not the set $\{(x, x) \mid x \in X\}$.

In this article, we consider suborder lattices of posets M_n , $0 < n \leq \omega$, which all have length 2, see Figure 1.

3. AN EQUATIONAL BASIS FOR $\mathbf{SP}(O(M_n))$

3.1. Identity (D_n) . We consider the identity of n -distributivity, where $0 < n < \omega$, which we denote by (D_n) :

$$x \wedge (y_0 \vee y_1 \vee \dots \vee y_n) = \bigvee_{i \leq n} \left[x \wedge \bigvee_{j \neq i} y_j \right].$$

This identity was introduced by A. Huhn in [7] as a generalization of distributivity—it is clear that (D_1) is just the identity of distributivity. The following lemma is folklore and straightforward to prove, see for example [9].

Lemma 2. *Let L be a lattice, let $J \subseteq J(L)$ be a set such that for each element $a \in L$, there is a subset $J_a \subseteq J$ with $a = \bigvee J_a$. The following conditions are equivalent.*

- (i) (D_n) holds in L .

- (ii) If $a \leq b_0 \vee b_1 \vee \dots \vee b_n$ for some $a \in J$ and some $b_0, b_1, \dots, b_n \in L$, then there is $i \leq n$ such that $a \leq \bigvee_{j \neq i} b_j$.

Corollary 3. Let L be a J -lattice for some set $J \subseteq J(L)$. The following conditions are equivalent.

- (i) (D_n) holds in L .
 (ii) If $a \leq b_0 \vee \dots \vee b_m$ is a minimal nontrivial join cover for some elements a and $b_0, \dots, b_m \in J$ then $0 < m < n$.

3.2. Identity (P). We denote the following identity by (P):

$$x \wedge [(y_0 \wedge (z_0 \vee z_1)) \vee y_1] = [x \wedge y_0 \wedge (z_0 \vee z_1)] \vee [x \wedge y_1] \vee \bigvee_{i < 2} [x \wedge ((y_0 \wedge z_i) \vee y_1)].$$

This identity was introduced in [4] under the name (N_5^1) . It was used in [4] as one of four identities which constitute an equational basis for the [quasi]variety $\mathbf{SP}(N_5)$. It was also used in [2] as one of three identities which form an equational basis of the [quasi]variety $\mathbf{SP}(O(M_1))$, also under the name (N_5^1) . For the next two statements, we refer to [4], see also [2, Lemma 6, Corollary 7].

Lemma 4. [4] Let L be a lattice, let $J \subseteq J(L)$ be a set such that for each element $a \in L$, there is a subset $J_a \subseteq J$ with $a = \bigvee J_a$. The following conditions are equivalent.

- (i) (P) holds in L .
 (ii) If $a \leq a_0 \vee a_1$ is a nontrivial join cover and $a_0 \leq b_0 \vee b_1$ for some $a \in J$ and some $a_0, a_1, b_0, b_1 \in L$, then $a \leq (a_0 \wedge b_i) \vee a_1$ for some $i < 2$.

Corollary 5. [4] Let L be a 2-distributive J -lattice for some set $J \subseteq J(L)$. The following conditions are equivalent.

- (i) (P) holds in L .
 (ii) If $a \leq a_0 \vee a_1$ is a minimal join cover for some $a, a_0, a_1 \in J$, then a_0 and a_1 are join-prime elements.

3.3. Identity (C_n) . We denote the following identity by (C_n) :

$$\begin{aligned} x \wedge \bigwedge_{i \leq n} (y_i \vee z_i) &= \bigvee_{i \leq n} \left[x \wedge y_i \wedge \bigwedge_{j \neq i} (y_j \vee z_j) \right] \vee \bigvee_{i \leq n} \left[x \wedge z_i \wedge \bigwedge_{j \neq i} (y_j \vee z_j) \right] \vee \\ &\vee \bigvee_{i < j \leq n} \left[x \wedge ((y_i \wedge y_j) \vee (z_i \wedge z_j)) \wedge \bigwedge_{k \notin \{i, j\}} (y_k \vee z_k) \right] \vee \\ &\vee \bigvee_{i < j \leq n} \left[x \wedge ((y_i \wedge z_j) \vee (y_j \wedge z_i)) \wedge \bigwedge_{k \notin \{i, j\}} (y_k \vee z_k) \right]. \end{aligned}$$

The identity (C_1) was introduced in [4] and used there, under the name (C), as a member of an equational basis of the [quasi]variety $\mathbf{SP}(N_5)$. It was also used in [2] as one of three identities which form an equational basis of the [quasi]variety $\mathbf{SP}(O(M_1))$, also under the name (C). For the next two statements, we refer to [4], see also [2, Lemma 4, Corollary 5].

Lemma 6. Let L be a lattice, let $J \subseteq J(L)$ be a set such that for each element $a \in L$, there is a subset $J_a \subseteq J$ with $a = \bigvee J_a$. The following conditions are equivalent.

- (i) (C_n) holds in L .

- (ii) If $a \leq a_0 \vee b_0, \dots, a \leq a_n \vee b_n$ are nontrivial join covers for some $a \in J$ and some $a_0, \dots, a_n, b_0, \dots, b_n \in L$, then there are $c, d \in L$ such that $a \leq c \vee d$ and $\{c, d\} \ll \{a_i, b_i\}, \{c, d\} \ll \{a_j, b_j\}$ for some $i < j \leq n$.

Proof. We prove first that (i) implies (ii). Indeed, let the assumptions of (ii) hold. Since (C_n) holds in L , we have

$$\begin{aligned} a &= a \wedge \bigwedge_{i \leq n} (a_i \vee b_i) = \bigvee_{i \leq n} \left[a \wedge a_i \wedge \bigwedge_{j \neq i} (a_j \vee b_j) \right] \vee \bigvee_{i \leq n} \left[a \wedge b_i \wedge \bigwedge_{j \neq i} (a_j \vee b_j) \right] \vee \\ &\vee \bigvee_{i < j \leq n} \left[a \wedge ((a_i \wedge a_j) \vee (b_i \wedge b_j)) \wedge \bigwedge_{k \notin \{i, j\}} (a_k \vee b_k) \right] \vee \\ &\vee \bigvee_{i < j \leq n} \left[a \wedge ((a_i \wedge b_j) \vee (a_j \wedge b_i)) \wedge \bigwedge_{k \notin \{i, j\}} (a_k \vee b_k) \right]. \end{aligned}$$

As a is a join-irreducible element, a equals one of the joinands on the right-hand side of the equality above. Therefore, the following cases are possible.

Case 1: $a = a \wedge a_i \wedge \bigwedge_{j \neq i} (a_j \vee b_j)$. In this case, $a \leq a_i$ which contradicts the assumption that $a \leq a_i \vee b_i$ is a nontrivial join cover. Therefore, this case is impossible.

Case 2: $a = a \wedge b_i \wedge \bigwedge_{j \neq i} (a_j \vee b_j)$. In this case, $a \leq b_i$ which again contradicts the assumption that $a \leq a_i \vee b_i$ is a nontrivial join cover. Therefore, this case is also impossible.

Case 3: there are $i < j \leq n$ such that $a = a \wedge ((a_i \wedge a_j) \vee (b_i \wedge b_j)) \wedge \bigwedge_{k \notin \{i, j\}} (a_k \vee b_k)$. In this case, $a \leq c \vee d$, where $c = a_i \wedge a_j$ and $d = b_i \wedge b_j$. Moreover, $\{c, d\} \ll \{a_i, b_i\}$ and $\{c, d\} \ll \{a_j, b_j\}$ whence we get the desired conclusion.

Case 4: there are $i < j \leq n$ such that $a = a \wedge ((a_i \wedge b_j) \vee (a_j \wedge b_i)) \wedge \bigwedge_{k \notin \{i, j\}} (a_k \vee b_k)$. In this case, we put $c = a_i \wedge b_j$ and $d = b_i \wedge a_j$ and obtain the desired conclusion as above in *Case 3*.

We prove now that (ii) implies (i). Let u denote the value of the left-hand side and v denote the value of the right-hand side of the identity (C_n) under interpretation γ , where

$$\gamma(x) = a, \quad \gamma(y_i) = a_i, \quad \gamma(z_i) = b_i, \quad i \leq n.$$

As inequality $v \leq u$ holds in each lattice, in order to prove that (C_n) holds in L , we have to prove that $u \leq v$. According to our assumption about L , it suffices to show that for each element $a' \in J$, the inequality $a' \leq u$ implies that $a' \leq v$. Indeed, $a' \leq u$ means that $a' \leq a$ and $a' \leq a_i \vee b_i$ for all $i \leq n$. If $a' \leq a_i$ for some $i \leq n$ then $a' \leq u \wedge a_i \leq v$. If $a' \leq b_i$ for some $i \leq n$ then $a' \leq u \wedge b_i \leq v$. Assume therefore that $a' \leq a_i \vee b_i$ is a nontrivial join cover for all $i \leq n$. Applying (ii), we obtain that there are elements $c, d \in L$ such that $a' \leq c \vee d$ and $\{c, d\} \ll \{a_i, b_i\}, \{c, d\} \ll \{a_j, b_j\}$ for some $i < j \leq n$. As $a' \leq a_i \vee b_i$ and $a' \leq a_j \vee b_j$ are nontrivial join covers, we conclude that $a' \leq c \vee d$ is also a nontrivial join cover. Therefore, the following cases are possible.

Case 1: $c \leq a_i \wedge a_j$ and $d \leq b_i \wedge b_j$ or $d \leq a_i \wedge a_j$ and $c \leq b_i \wedge b_j$. In this case, $a' \leq u \wedge ((a_i \wedge a_j) \vee (b_i \wedge b_j)) \leq v$.

Case 2: $c \leq a_i \wedge q$ and $d \leq a_i \wedge p$ for some $p, q \in \{a_j, b_j\}$. In this case, $a' \leq c \vee d \leq a_i$ which is impossible by our assumption as the join cover $a' \leq a_i \vee b_i$ is nontrivial.

Case 3: $c \leq a_j \wedge q$ and $d \leq a_j \wedge p$ for some $p, q \in \{a_i, b_i\}$. In this case, $a' \leq c \vee d \leq a_j$ which is impossible as the join cover $a' \leq a_j \vee b_j$ is nontrivial.

Case 4: $c \leq a_i \wedge b_j$ and $d \leq b_i \wedge a_j$ or $d \leq a_i \wedge b_j$ and $c \leq b_i \wedge a_j$. In this case, $a' \leq u \wedge ((a_i \wedge b_j) \vee (a_j \wedge b_i)) \leq v$.

Case 5: $c \leq b_i \wedge q$ and $d \leq b_i \wedge p$ for some $p, q \in \{a_j, b_j\}$. In this case, $a' \leq c \vee d \leq b_i$ which is impossible by our assumption as the join cover $a' \leq a_i \vee b_i$ is nontrivial.

Case 6: $c \leq b_j \wedge q$ and $d \leq b_j \wedge p$ for some $p, q \in \{a_i, b_i\}$. In this case, $a' \leq c \vee d \leq b_j$ which is impossible as the join cover $a' \leq a_j \vee b_j$ is nontrivial.

Therefore, $a' \leq v$ in any case and the desired conclusion follows. \square

Corollary 7. *Let L be a 2-distributive J -lattice for some set $J \subseteq J(L)$. The following conditions are equivalent.*

- (i) (C_n) holds in L .
- (ii) *If $a \leq a_0 \vee b_0, \dots, a \leq a_m \vee b_m$ are distinct minimal join covers for some $a, a_0, \dots, a_m, b_0, \dots, b_m \in J$, then $m < n$.*

Proof. We prove that (i) implies (ii). Indeed, suppose that $m \geq n$. Then, applying Lemma 6, we obtain that there are $i < j \leq n$ and elements $c, d \in L$ such that $\{c, d\} \ll \{a_i, b_i\}$, $\{c, d\} \ll \{a_j, b_j\}$. As $a \leq a_i \vee b_i$ and $a \leq a_j \vee b_j$ are minimal join covers, we conclude that $a \leq c \vee d$ is a nontrivial join cover and $\{a_i, b_i\} = \{c, d\} = \{a_j, b_j\}$ which contradicts our assumptions. Therefore, $m < n$.

To prove that (ii) implies (i), we show that statement (ii) of Lemma 6 holds. So let $a \leq a_0 \vee b_0, \dots, a \leq a_n \vee b_n$ be nontrivial join covers for some $a \in J$ and some $a_0, \dots, a_n, b_0, \dots, b_n \in L$. As L is a J -lattice for some set $J \subseteq J(L)$, there are finite minimal join covers $a \leq \bigvee F_0, \dots, a \leq \bigvee F_n$ such that $F_i \ll \{a_i, b_i\}$ for all $i \leq n$. As L is 2-distributive, we apply Lemma 2 and obtain that $|F_i| = 2$ for all $i \leq n$. Applying our assumption (ii) to finite minimal join covers $a \leq \bigvee F_0, \dots, a \leq \bigvee F_n$, we obtain that $F_i = F_j = \{c, d\}$ for some $i < j \leq n$ and some $c, d \in L$. This means that $a \leq c \vee d$ and $\{c, d\} \ll \{a_i, b_i\}$, $\{c, d\} \ll \{a_j, b_j\}$ which is our desired conclusion. \square

3.4. An equational basis. For $0 < n < \omega$, we put $\Sigma_n = \{(C_n), (D_2), (P)\}$ and $\mathbf{S}_n = \text{Mod } \Sigma_n$.

Proposition 8. *Let L be a dually algebraic lattice such that $L \models \Sigma_n$, where $0 < n < \omega$. Then for all elements $x \in J(L)$, each element of the set $\mathfrak{M}(x)$ is of the form $\{a, b\}$, where $\{a, b\} \subseteq P(L)$ is an antichain. Moreover, $|\mathfrak{M}(x)| \leq n$.*

In particular, $L \in \mathbf{SP}(\mathbf{O}(M_n))$.

Proof. It follows from Proposition 1(i) that L is a J -lattice. Corollary 3 implies that each minimal nontrivial join cover of an element $x \in J(L)$ contains exactly two elements. Corollary 5 implies that each minimal nontrivial join cover of x consists of join-prime elements. Moreover, $|\mathfrak{M}(x)| \leq n$ by Corollary 7. Thus, the first statement follows.

To prove the second statement, we use the method developed in [12, 13]. We fix an element $x \in J(L) \setminus P(L)$. According to the first statement,

$$\mathfrak{M}(x) = \{\{b_1(x), c_1(x)\}, \dots, \{b_{n(x)}(x), c_{n(x)}(x)\}\}$$

for some natural number $n(x)$ such that $0 < n(x) \leq n$ and some join-prime elements $b_1(x), \dots, b_{n(x)}(x), c_1(x), \dots, c_{n(x)}(x)$. We denote by P_x an isomorphic copy of $M_{n(x)}$. We denote the elements of P_x by $0(x), a_1(x), \dots, a_{n(x)}(x), 1(x)$ respectively, see Figure

1. As $n(x) \leq n$, P_x is a subset of M_n . We define a mapping $\psi_x: J(L) \rightarrow O(P_x)$ as follows:

$$\begin{aligned} \psi_x: x &\mapsto \{(0(x), 1(x))\}; \\ \psi_x: y &\mapsto \{(0(x), a_i(x)) \mid y = b_i(x) \text{ for some } i \in \{1, \dots, n\}\} \cup \\ &\quad \cup \{(a_i(x), 1(x)) \mid y = c_i(x) \text{ for some } i \in \{1, \dots, n\}\}, \quad \text{for all } y \in \bigcup \mathfrak{M}(x); \\ \psi_x: y &\mapsto \emptyset \quad \text{for all } y \notin \{x\} \cup \bigcup \mathfrak{M}(x). \end{aligned}$$

Let $P'(L)$ denote the set of all join-prime elements of L which do not belong to any minimal nontrivial join cover of any element $x \in J(L) \setminus P(L)$. For each element $x \in P'(L)$, we put $P_x = \{0(x), 1(x)\}$, where $0(x) < 1(x)$ and consider the mapping

$$\begin{aligned} \psi_x: J(L) &\rightarrow O(P_x); \\ \psi_x: x &\mapsto \{(0(x), 1(x))\}; \\ \psi_x: y &\mapsto \emptyset \quad \text{for all } y \neq x. \end{aligned}$$

Finally, let $I = (J(L) \setminus P(L)) \cup P'(L)$. We consider the following mapping:

$$\begin{aligned} \psi: L &\rightarrow \prod_{x \in I} O(P_x); \\ \pi_x \psi(a) &= \bigcup \{\psi_x(y) \mid y \in J(L), y \leq a\} \quad \text{for all } a \in L \text{ and all } x \in I. \end{aligned}$$

Claim 1. ψ is well-defined.

Proof of Claim. We have to prove that $\pi_x \psi(a)$ is a suborder in P_x for all $x \in I$ and all $a \in L$. As $P_x \cong M_{n(x)}$, it suffices to show that if $(0(x), a_i(x)), (a_i(x), 1(x)) \in \pi_x \psi(a)$ for some $i \in \{1, \dots, n\}$ then $(0(x), 1(x)) \in \pi_x \psi(a)$. Indeed, suppose that $(0(x), a_i(x)), (a_i(x), 1(x)) \in \pi_x \psi(a)$ for some $i \in \{1, \dots, n\}$. In other words, $\psi_x(b_i) \cup \psi_x(c_i) \subseteq \pi_x \psi(a)$ where $x \leq b_i \vee c_i$ is a minimal nontrivial join cover. This means that $b_i, c_i \leq a$ whence $x \leq b_i \vee c_i \leq a$. By our definition of ψ_x this implies that $\{(0(x), 1(x))\} = \psi_x(x) \subseteq \pi_x \psi(a)$ which is our desired conclusion. \square

Claim 2. ψ is a $(0, 1)$ -lattice homomorphism.

Proof of Claim. In order to prove the desired claim, it suffices to show that $\pi_x \psi$ is a $(0, 1)$ -lattice homomorphism for each $x \in I$. Indeed, we fix an element $x \in I$ and elements $u, v \in L$. If u is a least element of L then $y \leq u$ for no element $y \in J(L)$. Therefore, $\pi_x \psi(u) = \emptyset$. If u is a greatest element of L then $y \leq u$ for each element $y \in J(L)$. Therefore, $\pi_x \psi(u)$ is obviously the greatest element of $O(P_x)$. Therefore, $\pi_x \psi$ preserves the bounds.

If $u \leq v$ then $y \leq u$ implies $y \leq v$ for all $y \in J(L)$. Therefore, $\pi_x \psi$ is monotone. We prove that $\pi_x \psi$ preserves meets and joins.

Since $\pi_x \psi$ is monotone, $\pi_x \psi(u) \vee \pi_x \psi(v) \subseteq \pi_x \psi(u \vee v)$. We have to establish that $\pi_x \psi(u \vee v) \subseteq \pi_x \psi(u) \vee \pi_x \psi(v)$. So suppose that $(z_0, z_1) \in \pi_x \psi(u \vee v)$. This means that $(z_0, z_1) \in \psi_x(y) \neq \emptyset$ for some $y \in J(L)$ such that $y \leq u \vee v$. If $y \leq u$ or $y \leq v$ then $(z_0, z_1) \in \pi_x \psi(u) \cup \pi_x \psi(v)$. Otherwise, $y \leq u \vee v$ is a nontrivial join cover. As L is a J -lattice, we can refine this join cover to a minimal one. This implies that $y \in J(L) \setminus P(L)$. As $\psi_x(y) \neq \emptyset$, we conclude by the definition of ψ_x that $y = x$. Moreover, there is i such that $1 \leq i \leq n(x)$ and $y = x \leq b_i \vee c_i$ is a minimal nontrivial join cover with $\{b_i, c_i\} \ll \{u, v\}$. Inclusion $(z_0, z_1) \in \psi_x(y) = \psi_x(x)$ implies that

$z_0 = 0(x)$ and $z_1 = 1(x)$. Furthermore, $(0(x), a_i(x)) \in \psi_x(b_i) \subseteq \pi_x\psi(u) \cup \pi_x\psi(v)$ and $(a_i(x), 1(x)) \in \psi_x(c_i) \subseteq \pi_x\psi(u) \cup \pi_x\psi(v)$ as $\{b_i, c_i\} \ll \{u, v\}$. Hence, $(z_0, z_1) \in \psi_x(b_i) \vee \psi_x(c_i) \subseteq \pi_x\psi(u) \cup \pi_x\psi(v)$. This proves that $\pi_x\psi$ preserves joins.

Since $\pi_x\psi$ is monotone, $\pi_x\psi(u \wedge v) \subseteq \pi_x\psi(u) \cap \pi_x\psi(v)$. We have to establish that $\pi_x\psi(u) \cap \pi_x\psi(v) \subseteq \pi_x\psi(u \wedge v)$. Indeed, let $(z_0, z_1) \in \pi_x\psi(u) \cap \pi_x\psi(v)$. This means that $(z_0, z_1) \in \psi_x(y) \cap \psi_x(y') \neq \emptyset$ for some $y, y' \in J(L)$ such that $y \leq u$ and $y' \leq v$. If $y \neq y'$ then $\psi_x(y) \cap \psi_x(y') = \emptyset$ by the definition of ψ_x , a contradiction. Therefore, $y = y' \leq u \wedge v$ and $(z_0, z_1) \in \psi_x(y) \subseteq \pi_x\psi(u \wedge v)$. This proves that $\pi_x\psi$ preserves meets. \square

Claim 3. ψ is an embedding.

Proof of Claim. Suppose that $u \not\leq v$ in L . As L is a J -lattice, there is $y \in J(L)$ such that $y \leq u$ and $y \not\leq v$. By our definition, there is $x \in I$ such that $\psi_x(y) \neq \emptyset$. But then $\emptyset \neq \psi_x(y) \subseteq \pi_x\psi(u)$ and $\psi_x(y) \cap \pi_x\psi(v) = \emptyset$. This implies that $\pi_x\psi(u) \not\subseteq \pi_x\psi(v)$ whence $\psi(u) \not\leq \psi(v)$. \square

It follows from the claims above that

$$L \in \mathbf{SP}(O(P_x) \mid x \in I) \subseteq \mathbf{SPS}(O(M_n)) \subseteq \mathbf{SP}(O(M_n)).$$

The proof of Proposition 8 is complete. \square

Proposition 9. *Let L be a bi-algebraic lattice such that $L \models \Sigma_n$, where $0 < n < \omega$. Then for all elements $x \in \text{CJ}(L)$, each element of the set $\mathfrak{M}(x)$ is of the form $\{a, b\}$, where $\{a, b\} \subseteq P(L)$ is an antichain. Moreover, $|\mathfrak{M}(x)| \leq n$.*

In particular, $L \in \mathbf{SP}(O(M_n))$.

Proof. The argument is similar to the one in the proof of Proposition 8 and uses Proposition 1(ii). \square

Theorem 10. Σ_n forms an equational basis for $\mathbf{SP}(O(M_n))$. In particular, the class $\mathbf{SP}(O(M_n)) = \mathbf{S}_n$ is a lattice variety.

Proof. Let $L \models \Sigma_n$ and let F be the dual filter lattice of L . It is well-known that F is dually algebraic and it follows that $F \models \Sigma_n$. By Proposition 1, F is a J -lattice. By Proposition 8, $F \in \mathbf{SP}(O(M_n))$ whence $L \in \mathbf{SP}(O(M_n))$ as L embeds into F . On the other hand, the lattice $O(M_n)$ has the only minimal join covers:

$$A \leq A_i \vee B_i, \quad 1 \leq i \leq n, \quad \text{where} \\ A = \{(0, 1)\}, \quad A_i = \{(0, a_i)\}, \quad B_i = \{(a_i, 1)\}, \quad 1 \leq i \leq n,$$

see Figure 1. Thus, $O(M_n)$ is 2-distributive by Corollary 3. Moreover, $O(M_n)$ satisfies the condition (ii) of Corollaries 7 and 5. This implies that $O(M_n) \models \Sigma_n$. \square

Let \mathbf{L}_{01} denote the variety of $(0, 1)$ -lattices and let $\mathbf{S}_n^{01} = \mathbf{L}_{01} \cap \text{Mod } \Sigma_n$.

Theorem 11. *The set Σ_n forms an equational basis for $\mathbf{SP}(O(M_n))$ within the variety \mathbf{L}_{01} . In particular, $\mathbf{SP}(O(M_n)) = \mathbf{S}_n^{01}$ is a variety of $(0, 1)$ -lattices.*

Proof. If L is a $(0, 1)$ -lattice then taking in the proof of Theorem 10 the dual lattice of nonempty filters as F , we obtain that L is a $(0, 1)$ -sublattice of F and $F \in \mathbf{SP}(O(M_n))$ by Proposition 8. Therefore, L belongs in this case to the variety of $(0, 1)$ -lattices generated by $O(M_n)$. \square

4. AN EQUATIONAL BASIS FOR $\mathbf{Q}(\mathbf{O}(M_\omega))$

We put $\Sigma = \{(D_2), (P)\}$.

Proposition 12. *Let L be a dually algebraic lattice such that $L \models \Sigma$. The following statements hold.*

- (i) *For all $x \in J(L)$, each element of the set $\mathfrak{M}(x)$ is of the form $\{a, b\}$, where $\{a, b\} \subseteq P(L)$ is an antichain.*
- (ii) *If L is bi-algebraic then for all $x \in CJ(L)$, each element of the set $\mathfrak{M}(x)$ is of the form $\{a, b\}$, where $\{a, b\} \subseteq P(L)$ is an antichain.*

In particular, $L \in \mathbf{SP}(\mathbf{O}(M_\kappa)) \subseteq \mathbf{Q}(\mathbf{O}(M_\omega))$ for some cardinal κ .

Proof. Applying the same argument as in the proof of Proposition 8, we obtain that $L \in \mathbf{SP}(\mathbf{O}(M_\kappa))$ for some infinite cardinal $\kappa \geq |L|$. As M_κ embeds into an ultrapower of M_ω , we conclude that $\mathbf{O}(M_\kappa) \in \mathbf{SP}_u(\mathbf{O}(M_\omega))$ and

$$\mathbf{SP}(\mathbf{O}(M_\kappa)) \subseteq \mathbf{SPP}_u(\mathbf{O}(M_\omega)) = \mathbf{Q}(\mathbf{O}(M_\omega)),$$

which is our desired conclusion. \square

Theorem 13. *The following statements hold.*

- (i) *The quasivariety $\mathbf{Q}(\mathbf{O}(M_\omega))$ is a lattice variety and Σ forms an equational basis for this variety.*
- (ii) *The class $\mathbf{Q}(\mathbf{O}(M_\omega))$ of $(0, 1)$ -lattices is a variety of $(0, 1)$ -lattices and Σ forms an equational basis for this variety.*

Proof. (i) If $L \models \Sigma$, then $L \in \mathbf{Q}(\mathbf{O}(M_\omega))$ by Proposition 12. Hence, $\text{Mod } \Sigma \subseteq \mathbf{Q}(\mathbf{O}(M_\omega))$. Conversely, the lattice $\mathbf{O}(M_\omega)$ has the only minimal join covers:

$$A \leq A_i \vee B_i, \quad 1 \leq i < \omega, \quad \text{where}$$

$$A = \{(0, 1)\}, \quad A_i = \{(0, a_i)\}, \quad B_i = \{(a_i, 1)\}, \quad 1 \leq i < \omega,$$

see Figure 1. Thus, $\mathbf{O}(M_n)$ is 2-distributive by Corollary 3. Moreover, $\mathbf{O}(M_n)$ satisfies the condition (ii) of Corollary 5. Therefore, $\mathbf{O}(M_\omega) \models \Sigma$ and

$$\mathbf{Q}(\mathbf{O}(M_\omega)) = \mathbf{SPP}_u(\mathbf{O}(M_\omega)) \models \Sigma$$

as identities are stable with respect to the operators \mathbf{S} , \mathbf{P} , and \mathbf{P}_u . It follows that $\text{Mod } \Sigma = \mathbf{Q}(\mathbf{O}(M_\omega))$.

The proof of (ii) is similar. \square

Corollary 14. *The following equalities hold for an arbitrary infinite cardinal κ :*

$$\mathbf{SO}_2 = \mathbf{Q}(\mathbf{O}(M_n) \mid 0 < n < \omega) = \mathbf{SP}(\mathbf{O}(M_\kappa)).$$

Proof. By [13, Theorem 4.8], Σ forms an equational basis for \mathbf{SO}_2 . Taking into account Theorem 13, we conclude that $\mathbf{SO}_2 = \mathbf{SP}(\mathbf{O}(M_\omega))$. Furthermore, each algebraic structure embeds into an ultraproduct of its finitely generated substructures, see for example [6, Theorem 1.2.8]. Therefore, $M_\kappa \in \mathbf{SP}_u(M_n \mid 0 < n < \omega)$ for each infinite cardinal kappa whence $\mathbf{O}(M_\kappa) \in \mathbf{SP}_u(\mathbf{O}(M_n) \mid 0 < n < \omega)$ and

$$\begin{aligned} \mathbf{SO}_2 = \mathbf{SP}(\mathbf{O}(M_\omega)) &= \mathbf{SP}(\mathbf{O}(M_\kappa)) \subseteq \mathbf{SPP}_u(\mathbf{O}(M_n) \mid 0 < n < \omega) = \\ &= \mathbf{Q}(\mathbf{O}(M_n) \mid 0 < n < \omega) \subseteq \mathbf{SO}_2. \end{aligned}$$

The desired conclusion follows. \square

The following problem was raised in [13].

Problem 1. [13, Question 2] If $\langle P; \leq \rangle$ is a finite poset, is it true that the quasivariety $\mathbf{SP}(\mathbf{O}(P; \leq))$ is a variety?

The next statement solves Problem 1 in the positive for finite posets of length at most two.

Corollary 15. *If $\langle P; \leq \rangle$ is a finite poset of length at most two then $\mathbf{SP}(\mathbf{O}(P; \leq))$ is a finitely based variety.*

Proof. It follows from Corollary 14 and the fact that the poset $\langle P; \leq \rangle$ is finite that $\mathbf{SP}(\mathbf{O}(P; \leq)) = \mathbf{SP}(\mathbf{O}(M_n))$ or $\mathbf{SP}(\mathbf{O}(P; \leq))$ is the variety of distributive lattices. In the first case, $\mathbf{SP}(\mathbf{O}(P; \leq))$ is a finitely based variety by Theorem 10. \square

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