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QUASIVARIETIES GENERATED BY SMALL SUBORDER LATTICES. I. EQUATIONAL BASES

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Abstract. For each cardinal $\kappa > 0$, the quasivariety generated by the suborder lattice of M_κ is a finitely based variety. An equational basis for this variety is found.

Keywords: lattice, quasivariety, variety, poset.

1. Introduction

Suborder lattices were studied by several authors; we refer to D. Bredikhin and B. Schein [3] and to B. Sivák [15] as well as to [12, 13, 2]. Suborder lattices were used as a convenient tool in establishing some deep results for subsemigroup lattices which are presented in the papers of V. B. Repnitskiĭ [10, 11]; see also [14].

For a positive integer n, let $\mathbf{SO}_n$ denote the class of lattices embeddable into suborder lattices of partial orders of length at most n. It was established in [13] that $\mathbf{SO}_n$ is a finitely based variety and a particular finite equational basis was found for this variety in [13].

There are still some unsolved problems which concern suborder lattices. For example, Question 2 in [13] asks if the quasivariety generated by a finite suborder lattice is a variety. A positive answer to this question was given in [2] for the suborder lattice $\mathrm{O}(M_1)$. Moreover, it was established in [2] that the quasivariety generated by $\mathrm{O}(M_1)$ is a variety and a particular finite equational basis was found for this variety.

In this paper, we extend the results from [2] to a more general case and consider lattices $\mathrm{O}(M_\kappa)$ for an arbitrary cardinal $\kappa > 0$, see Figure 1. Specifically, we prove that the quasivariety $\mathbf{Q}\big(\mathrm{O}(M_\kappa)\big)$ generated by the suborder lattice $\mathrm{O}(M_\kappa)$ is

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a finitely based variety and find a finite basis for this variety, see Theorem 10, Theorem 13, and Corollaries 14 and 15. In a subsequent article, the results of this paper will be used for establishing categorical dualities for the quasivarieties $\mathbf{Q}\big(\mathrm{O}(M_\kappa)\big)$ where $1 < \kappa \leqslant \omega$.

2. Basic concepts

For all the notions which are not defined in this section, we refer to A. I. Maltsev [8] and V. A. Gorbunov [6].

2.1. **Lattices.** Most of the following definitions concerning join covers are in accordance with R. Freese, J. Ježek, and J. B. Nation [5].

Let L be a lattice. For arbitrary two sets $A, B \subseteq L$, we say that A *refines* B and write $A \ll B$ if for each $a \in A$, there is $b \in B$ such that $a \leq b$. If $x \in L$, then A is a *join cover* of x if $\bigvee A$ exists and $x \leq \bigvee A$; we also call $x \leq \bigvee A$ a *join cover* in this case. A join cover $x \leq \bigvee A$ is *nontrivial* if $x \nleq a$ for all $a \in A$; $x \leq \bigvee A$ is *finite* if the set A is finite. A join cover $x \leq \bigvee A$ is *irredundant* if $x \nleq \bigvee B$ for any proper subset $B \subset A$. A join cover $x \leq \bigvee A$ is *minimal* if $A \subseteq B$ for each join cover $x \leq \bigvee B$ such that $B \ll A$. The lattice L has the *complete minimal join cover refinement property* $(\mathrm{CR})_X$ for a set $X \subseteq L$ if each nontrivial join cover of each element from X can be refined to a minimal one.

By $\mathrm{J}(L)$, we denote the set of all join-irreducible elements of L and by $\mathrm{CJ}(L)$ — the set of all completely join-irreducible elements of L. Similarly, by $\mathrm{P}(L)$, we denote the set of all join-prime elements of L and by $\mathrm{CP}(L)$ — the set of all completely join-prime elements of L.

Definition 1. Let L be a lattice and let $J \subseteq \mathrm{J}(L)$. We say that L is a *J-lattice* if L possesses the following properties:

(i) for each element $a \in L$, there is a subset $J_a \subseteq J$ with $a = \bigvee J_a$;
(ii) for each element $a \in J$ and each nontrivial join cover $a \leq a_0 \vee \ldots \vee a_n$ with $n < \omega$ and $a_0, \ldots, a_n \in L$, there is a finite set $F \subseteq J$ such that $a \leq \bigvee F$ is a minimal join cover and $F \ll \{a_0, \ldots, a_n\}$.

We say that L is a *CJ-lattice* if L possesses the following properties:

(i) for each element $a \in L$, there is a subset $J_a \subseteq \mathrm{CJ}(L)$ with $a = \bigvee J_a$;
(ii) L has the property $(\mathrm{CR})_{\mathrm{CJ}(L)}$.

It follows from the definition above that each CJ-lattice is a J-lattice for $J = \mathrm{CJ}(L)$. J-lattices were considered in [1, 4], see also [13].

For a J-lattice L and an element $x \in \mathrm{J}(L)$, let $\mathfrak{M}(x)$ denote the set of all finite minimal join covers of x.

Remark 1. We note that in an upper continuous lattice L, each minimal join cover of an element $x \in \mathrm{CJ}(L)$ belongs to $\mathfrak{M}(x)$.

Proposition 1. [4] *Let L be a complete dually algebraic lattice. Then the following statements hold.*

(i) *If L is n-distributive then L is a J-lattice.*
(ii) *If L is in addition algebraic then L is a CJ-lattice.*

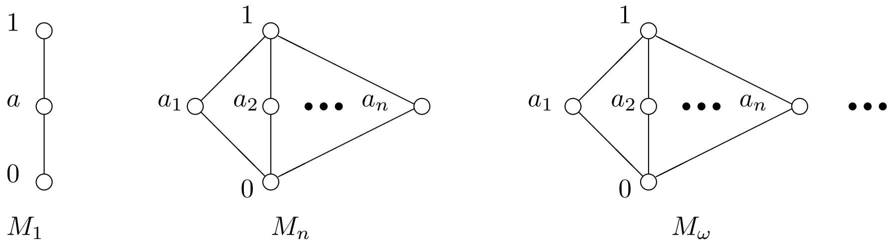

Рис. 1. Posets M_1, M_n, and M_ω

2.2. **Suborder lattices.** Let X be a set and let $R \subseteq X^2$ be a *strict* partial order on X; that is an antireflexive, antisymmetric, and transitive binary relation. In this case, we also say that $\langle X; R\rangle$ is a *partially ordered set* or a *poset* for short. A subset $R' \subseteq R$ is a (*strict*) *suborder* of R if the structure $\langle X; R'\rangle$ is also a poset. The set $\mathrm{O}(X,R)$ of all (strict) suborders of a partial order R on X is a partially ordered set with respect to the relation $\subseteq$ of set-theoretic inclusion. Obviously, $\varnothing$ is a least suborder of R. Thus, $\varnothing$ is a least element in $\mathrm{O}(X,R)$. It is also obvious that R is a greatest element in $\mathrm{O}(X,R)$. It is straightforward to check that for an arbitrary family $\{R_i \mid i\in I\} \subseteq \mathrm{O}(X,R)$, the relation $\bigcap_{i\in I} R_i$ is also a suborder of R; that is,

$$\bigwedge_{i\in I} R_i = \bigcap_{i\in I} R_i \in \mathrm{O}(X,R).$$

Thus, $\mathrm{O}(X,R)$ forms a complete lattice, where

$$\bigvee_{i\in I} R_i = \big(\bigcup_{i\in I} R_i\big)^t;$$

here Y^t denotes the transitive closure of a binary relation $Y\subseteq X^2$. It is clear that

$$\mathrm{J}\big(\mathrm{O}(X,R)\big) = \mathrm{CJ}\big(\mathrm{O}(X,R)\big) = \Big\{\big\{(a,b)\big\} \mid (a,b)\in R\Big\}.$$

In this article, we consider, in particular, suborder lattices of posets M_n, $0<n\leqslant\omega$, see Figure 1.

3. An equational basis for $\mathbf{SP}\big(\mathrm{O}(M_n)\big)$

3.1. **Identity** (D_n)**.** We consider the identity of n-distributivity, where $0<n<\omega$, which we denote by (D_n):

$$x\wedge(y_0\vee y_1\vee\ldots\vee y_n) = \bigvee_{i\leqslant n}\big[x\wedge\bigvee_{j\neq i} y_j\big].$$

This identity was introduced by A. Huhn in [7] as a generalization of distributivity—it is clear that (D_1) is just the identity of distributivity. The following lemma is folklore and straightforward to prove, see for example [9].

Lemma 2. *Let L be a lattice, let $J\subseteq \mathrm{J}(L)$ be a set such that for each element $a\in L$, there is a subset $J_a\subseteq J$ with $a=\bigvee J_a$. The following conditions are equivalent.*

(i) (D_n) *holds in L.*

(ii) *If $a\leq b_0\vee b_1\vee\ldots\vee b_n$ for some $a\in J$ and some $b_0,b_1,\ldots,b_n\in L$, then there is $i\leqslant n$ such that $a\leq\bigvee_{j\neq i} b_j$.*

Corollary 3. *Let L be a J-lattice for some set $J \subseteq \mathrm{J}(L)$. The following conditions are equivalent.*

(i) (D_n) *holds in L.*

(ii) *If $a \leq b_0 \vee \ldots \vee b_m$ is a minimal nontrivial join cover for some elements a and $b_0, \ldots, b_m \in J$ then $0 < m < n$.*

3.2. **Identity** (P)**.** We denote the following identity by (P):

$$x \wedge \big[\big(y_0 \wedge (z_0 \vee z_1)\big) \vee y_1\big] = \big[x \wedge y_0 \wedge (z_0 \vee z_1)\big] \vee \big[x \wedge y_1\big] \vee \bigvee_{i<2} \big[x \wedge \big((y_0 \wedge z_i) \vee y_1\big)\big].$$

This identity was introduced in [4] under the name (N_5^1). It was used in [4] as one of four identities which constitute an equational basis for the [quasi]variety $\mathbf{SP}(N_5)$. It was also used in [2] as one of three identities which form an equational basis of the [quasi]variety $\mathbf{SP}\big(\mathrm{O}(M_1)\big)$, also under the name (N_5^1). For the next two statements, we refer to [4], see also [2, Lemma 6, Corollary 7].

Lemma 4. [4] *Let L be a lattice, let $J \subseteq \mathrm{J}(L)$ be a set such that for each element $a \in L$, there is a subset $J_a \subseteq J$ with $a = \bigvee J_a$. The following conditions are equivalent.*

(i) (P) *holds in L.*

(ii) *If $a \leq a_0 \vee a_1$ is a nontrivial join cover and $a_0 \leq b_0 \vee b_1$ for some $a \in J$ and some $a_0, a_1, b_0, b_1 \in L$, then $a \leq (a_0 \wedge b_i) \vee a_1$ for some $i < 2$.*

Corollary 5. [4] *Let L be a 2-distributive J-lattice for some set $J \subseteq \mathrm{J}(L)$. The following conditions are equivalent.*

(i) (P) *holds in L.*

(ii) *If $a \leq a_0 \vee a_1$ is a minimal join cover for some $a, a_0, a_1 \in J$, then a_0 and a_1 are join-prime elements.*

3.3. **Identity** (C_n)**.** We denote the following identity by (C_n):

$$\begin{aligned} x \wedge \bigwedge_{i \leqslant n} (y_i \vee z_i) = {} & \bigvee_{i \leqslant n} \Big[x \wedge y_i \wedge \bigwedge_{j \neq i} (y_j \vee z_j)\Big] \;\vee\; \bigvee_{i \leqslant n} \Big[x \wedge z_i \wedge \bigwedge_{j \neq i} (y_j \vee z_j)\Big] \;\vee \\ \vee \; & \bigvee_{i<j \leqslant n} \Big[x \wedge \big((y_i \wedge y_j) \vee (z_i \wedge z_j)\big) \wedge \bigwedge_{k \notin \{i,j\}} (y_k \vee z_k)\Big] \;\vee \\ \vee \; & \bigvee_{i<j \leqslant n} \Big[x \wedge \big((y_i \wedge z_j) \vee (y_j \wedge z_i)\big) \wedge \bigwedge_{k \notin \{i,j\}} (y_k \vee z_k)\Big]. \end{aligned}$$

The identity (C_1) was introduced in [4] and used there, under the name (C), as a member of an equational basis of the [quasi]variety $\mathbf{SP}(N_5)$. It was also used in [2] as one of three identities which form an equational basis of the [quasi]variety $\mathbf{SP}\big(\mathrm{O}(M_1)\big)$, also under the name (C). For the next two statements, we refer to [4], see also [2, Lemma 4, Corollary 5].

Lemma 6. *Let L be a lattice, let $J \subseteq \mathrm{J}(L)$ be a set such that for each element $a \in L$, there is a subset $J_a \subseteq J$ with $a = \bigvee J_a$. The following conditions are equivalent.*

(i) (C_n) *holds in L.*

(ii) *If $a \leq a_0 \vee b_0$, …, $a \leq a_n \vee b_n$ are nontrivial join covers for some $a \in J$ and some $a_0, \ldots, a_n$, $b_0, \ldots, b_n \in L$, then there are $c, d \in L$ such that $a \leq c \vee d$ and $\{c, d\} \ll \{a_i, b_i\}$, $\{c, d\} \ll \{a_j, b_j\}$ for some $i < j \leqslant n$.*

Proof. We prove first that (i) implies (ii). Indeed, let the assumptions of (ii) hold. Since (C_n) holds in L, we have

$$\begin{aligned} a = a \wedge \bigwedge_{i\leqslant n}(a_i \vee b_i) = \bigvee_{i\leqslant n}\Big[a\wedge a_i \wedge \bigwedge_{j\neq i}(a_j\vee b_j)\Big] \;\vee\; \bigvee_{i\leqslant n}\Big[a\wedge b_i \wedge \bigwedge_{j\neq i}(a_j\vee b_j)\Big] \vee \\ \vee \bigvee_{i<j\leqslant n}\Big[a\wedge \big((a_i\wedge a_j)\vee(b_i\wedge b_j)\big)\wedge \bigwedge_{k\notin\{i,j\}}(a_k\vee b_k)\Big] \vee \\ \vee \bigvee_{i<j\leqslant n}\Big[a\wedge \big((a_i\wedge b_j)\vee(a_j\wedge b_i)\big)\wedge \bigwedge_{k\notin\{i,j\}}(a_k\vee b_k)\Big]. \end{aligned}$$

As a is a join-irreducible element, a equals one of the joinands on the right-hand side of the equality above. Therefore, the following cases are possible.

Case 1: $a = a\wedge a_i\wedge\bigwedge_{j\neq i}(a_j\vee b_j)$. In this case, $a\leq a_i$ which contradicts the assumption that $a\leq a_i\vee b_i$ is a nontrivial join cover. Therefore, this case is impossible.

Case 2: $a = a\wedge b_i\wedge\bigwedge_{j\neq i}(a_j\vee b_j)$. In this case, $a\leq b_i$ which again contradicts the assumption that $a\leq a_i\vee b_i$ is a nontrivial join cover. Therefore, this case is also impossible.

Case 3: there are $i<j\leqslant n$ such that $a = a\wedge\big((a_i\wedge a_j)\vee(b_i\wedge b_j)\big)\wedge\bigwedge_{k\notin\{i,j\}}(a_k\vee b_k)$. In this case, $a\leq c\vee d$, where $c = a_i\wedge a_j$ and $d = b_i\wedge b_j$. Moreover, $\{c,d\}\ll\{a_i,b_i\}$ and $\{c,d\}\ll\{a_j,b_j\}$ whence we get the desired conclusion.

Case 4: there are $i<j\leqslant n$ such that $a = a\wedge\big((a_i\wedge b_j)\vee(a_j\wedge b_i)\big)\wedge\bigwedge_{k\notin\{i,j\}}(a_k\vee b_k)$. In this case, we put $c = a_i\wedge b_j$ and $d = b_i\wedge a_j$ and obtain the desired conclusion as above in *Case 3*.

We prove now that (ii) implies (i). Let u denote the value of the left-hand side and v denote the value of the right-hand side of the identity (C_n) under interpretation γ, where

$$\gamma(x) = a, \quad \gamma(y_i) = a_i, \quad \gamma(z_i) = b_i, \quad i\leqslant n.$$

As inequality $v\leq u$ holds in each lattice, in order to prove that (C_n) holds in L, we have to prove that $u\leq v$. According to our assumption about L, it suffices to show that for each element $a'\in J$, the inequality $a'\leq u$ implies that $a'\leq v$. Indeed, $a'\leq u$ means that $a'\leq a$ and $a'\leq a_i\vee b_i$ for all $i\leqslant n$. If $a'\leq a_i$ for some $i\leqslant n$ then $a'\leq u\wedge a_i\leq v$. If $a'\leq b_i$ for some $i\leqslant n$ then $a'\leq u\wedge b_i\leq v$. Assume therefore that $a'\leq a_i\vee b_i$ is a nontrivial join cover for all $i\leqslant n$. Applying (ii), we obtain that there are elements $c,d\in L$ such that $a'\leq c\vee d$ and $\{c,d\}\ll\{a_i,b_i\}$, $\{c,d\}\ll\{a_j,b_j\}$ for some $i<j\leqslant n$. As $a'\leq a_i\vee b_i$ and $a'\leq a_j\vee b_j$ are nontrivial join covers, we conclude that $a'\leq c\vee d$ is also a nontrivial join cover. Therefore, the following cases are possible.

Case 1: $c\leq a_i\wedge a_j$ and $d\leq b_i\wedge b_j$ or $d\leq a_i\wedge a_j$ and $c\leq b_i\wedge b_j$. In this case, $a'\leq u\wedge\big((a_i\wedge a_j)\vee(b_i\wedge b_j)\big)\leq v$.

Case 2: $c\leq a_i\wedge q$ and $d\leq a_i\wedge p$ for some $p,q\in\{a_j,b_j\}$. In this case, $a'\leq c\vee d\leq a_i$ which is impossible by our assumption as the join cover $a'\leq a_i\vee b_i$ is nontrivial.

Case 3: $c\leq a_j\wedge q$ and $d\leq a_j\wedge p$ for some $p,q\in\{a_i,b_i\}$. In this case, $a'\leq c\vee d\leq a_j$ which is impossible as the join cover $a'\leq a_j\vee b_j$ is nontrivial.

Case 4: $c\leq a_i\wedge b_j$ and $d\leq b_i\wedge a_j$ or $d\leq a_i\wedge b_j$ and $c\leq b_i\wedge a_j$. In this case, $a'\leq u\wedge\big((a_i\wedge b_j)\vee(a_j\wedge b_i)\big)\leq v$.

Case 5: $c \leq b_i \wedge q$ and $d \leq b_i \wedge p$ for some $p, q \in \{a_j, b_j\}$. In this case, $a' \leq c \vee d \leq b_i$ which is impossible by our assumption as the join cover $a' \leq a_i \vee b_i$ is nontrivial.

Case 6: $c \leq b_j \wedge q$ and $d \leq b_j \wedge p$ for some $p, q \in \{a_i, b_i\}$. In this case, $a' \leq c \vee d \leq b_j$ which is impossible as the join cover $a' \leq a_j \vee b_j$ is nontrivial.

Therefore, $a' \leq v$ in any case and the desired conclusion follows. □

Corollary 7. *Let L be a 2-distributive J-lattice for some set $J \subseteq \mathrm{J}(L)$. The following conditions are equivalent.*

(i) (C_n) *holds in L.*

(ii) *If $a \leq a_0 \vee b_0, \ldots, a \leq a_m \vee b_m$ are distinct minimal join covers for some $a, a_0, \ldots, a_m, b_0, \ldots, b_m \in J$, then $m < n$.*

Proof. We prove that (i) implies (ii). Indeed, suppose that $m \geqslant n$. Then, applying Lemma 6, we obtain that there are $i < j \leqslant n$ and elements $c, d \in L$ such that $\{c, d\} \ll \{a_i, b_i\}$, $\{c, d\} \ll \{a_j, b_j\}$. As $a \leq a_i \vee b_i$ and $a \leq a_j \vee b_j$ are minimal join covers, we conclude that $a \leq c \vee d$ is a nontrivial join cover and $\{a_i, b_i\} = \{c, d\} = \{a_j, b_j\}$ which contradicts our assumptions. Therefore, $m < n$.

To prove that (ii) implies (i), we show that statement (ii) of Lemma 6 holds. So let $a \leq a_0 \vee b_0, \ldots, a \leq a_n \vee b_n$ be nontrivial join covers for some $a \in J$ and some $a_0, \ldots, a_n, b_0, \ldots, b_n \in L$. As L is a J-lattice for some set $J \subseteq \mathrm{J}(L)$, there are finite minimal join covers $a \leq \bigvee F_0, \ldots, a \leq \bigvee F_n$ such that $F_i \ll \{a_i, b_i\}$ for all $i \leqslant n$. As L is 2-distributive, we apply Lemma 2 and obtain that $|F_i| = 2$ for all $i \leqslant n$. Applying our assumption (ii) to finite minimal join covers $a \leq \bigvee F_0, \ldots,$ $a \leq \bigvee F_n$, we obtain that $F_i = F_j = \{c, d\}$ for some $i < j \leqslant n$ and some $c, d \in L$. This means that $a \leq c \vee d$ and $\{c, d\} \ll \{a_i, b_i\}$, $\{c, d\} \ll \{a_j, b_j\}$ which is our desired conclusion. □

3.4. **An equational basis.** For $0 < n < \omega$, we put $\Sigma_n = \{(\mathrm{C}_n), (\mathrm{D}_2), (\mathrm{P})\}$ and $\mathbf{S}_n = \mathrm{Mod}\, \Sigma_n$.

Proposition 8. *Let L be a dually algebraic lattice such that $L \models \Sigma_n$, where $0 < n < \omega$. Then for each element $x \in \mathrm{J}(L)$, we have*

$$\mathfrak{M}(x) = \big\{\{a, b\} \mid a, b \in \mathrm{P}(L),\ a \neq b\big\} \text{ and } |\mathfrak{M}(x)| \leqslant n.$$

In particular, $L \in \mathbf{SP}\big(\mathrm{O}(M_n)\big)$.

Proof. It follows from Proposition 1(i) that L is a J-lattice. Corollary 3 implies that each minimal nontrivial join cover of an element $x \in \mathrm{J}(L)$ contains exactly two elements. Corollary 5 implies that implies that each minimal nontrivial join cover of x consists of join-prime elements. Moreover, $|\mathfrak{M}(x)| \leqslant n$ by Corollary 7. Thus, the first statement follows.

To prove the second statement, we use the method developed in [12, 13]. We fix an element $x \in \mathrm{J}(L) \backslash \mathrm{P}(L)$. According to the first statement,

$$\mathfrak{M}(x) = \big\{\{b_1(x), c_1(x)\}, \ldots, \{b_{n(x)}(x), c_{n(x)}(x)\}\big\}$$

for some natural number $n(x)$ such that $0 < n(x) \leqslant n$ and some join-prime elements $b_1(x), \ldots, b_{n(x)}, c_1(x), \ldots, c_{n(x)}$. We denote by P_x an isomorphic copy of $M_{n(x)}$. We denote the elements of P_x by $0(x), a_1(x), \ldots, a_{n(x)}(x), 1(x)$ respectively, see Figure 1. As $n(x) \leqslant n$, P_x is a subposet of M_n. We define a mapping $\psi_x \colon \mathrm{J}(L) \to \mathrm{O}(P_x)$

as follows:

$$\psi_x\colon x \mapsto \big\{\big(0(x), 1(x)\big)\big\};$$
$$\psi_x\colon y \mapsto \big\{\big(0(x), a_i(x)\big) \mid y = b_i(x) \text{ for some } i \in \{1, \dots, n\}\big\}\cup$$
$$\cup\big\{\big(a_i(x), 1(x)\big) \mid y = c_i(x) \text{ for some } i \in \{1, \dots, n\}\big\}, \quad \text{for all } y \in \bigcup \mathfrak{M}(x);$$
$$\psi_x\colon y \mapsto \varnothing \quad \text{for all } y \notin \{x\} \cup \bigcup \mathfrak{M}(x).$$

Let $\mathrm{P}'(L)$ denote the set of all join-prime elements of L which do not belong to any minimal nontrivial join cover of any element $x \in \mathrm{J}(L) \backslash \mathrm{P}(L)$. For each element $x \in \mathrm{P}'(L)$, we put $P_x = \big\{0(x), 1(x)\big\}$, where $0(x) < 1(x)$ and consider the mapping

$$\psi_x\colon \mathrm{J}(L) \to \mathrm{O}(P_x);$$
$$\psi_x\colon x \mapsto \big\{\big(0(x), 1(x)\big)\big\};$$
$$\psi_x\colon y \mapsto \varnothing \quad \text{for all } y \neq x.$$

Finally, let $I = \big(\mathrm{J}(L) \backslash \mathrm{P}(L)\big) \cup \mathrm{P}'(L)$. We consider the following mapping:

$$\psi\colon L \to \prod_{x \in I} \mathrm{O}(P_x);$$
$$\pi_x\psi(a) = \bigcup \big\{\psi_x(y) \mid y \in \mathrm{J}(L),\ y \leq a\big\} \quad \text{for all } a \in L \text{ and all } x \in I.$$

Claim 1. ψ is well-defined.

Proof of Claim. We have to prove that $\pi_x\psi(a)$ is a suborder in P_x for all $x \in I$ and all $a \in L$. As $P_x \cong M_{n(x)}$, it suffices to show that if $\big(0(x), a_i(x)\big), \big(a_i(x), 1(x)\big) \in \pi_x\psi(a)$ for some $i \in \{1, \dots, n\}$ then $\big(0(x), 1(x)\big) \in \pi_x\psi(a)$. Indeed, suppose that $\big(0(x), a_i(x)\big), \big(a_i(x), 1(x)\big) \in \pi_x\psi(a)$ for some $i \in \{1, \dots, n\}$. In other words, $\psi_x(b_i) \cup \psi_x(c_i) \subseteq \pi_x\psi(a)$ where $x \leq b_i \vee c_i$ is a minimal nontrivial join cover. This means that $b_i, c_i \leq a$ whence $x \leq b_i \vee c_i \leq a$. By our definition of ψ_x this implies that $\big\{\big(0(x), 1(x)\big)\big\} = \psi_x(x) \subseteq \pi_x\psi(a)$ which is our desired conclusion. □

Claim 2. ψ is a $(0, 1)$-lattice homomorphism.

Proof of Claim. In order to prove the desired claim, it suffices to show that $\pi_x\psi$ is a $(0, 1)$-lattice homomorphism for each $x \in I$. Indeed, we fix an element $x \in I$ and elements $u, v \in L$. If u is a least element of L then $y \leq u$ for no element $y \in \mathrm{J}(L)$. Therefore, $\pi_x\psi(u) = \varnothing$. If u is a greatest element of L then $y \leq u$ for each element $y \in \mathrm{J}(L)$. Therefore, $\pi_x\psi(u)$ is obviously the greatest element of $\mathrm{O}(P_x)$. Therefore, $\pi_x\psi$ preserves the bounds.

If $u \leq v$ then $y \leq u$ implies $y \leq v$ for all $y \in \mathrm{J}(L)$. Therefore, $\pi_x\psi$ is monotone. We prove that $\pi_x\psi$ preserves meets and joins.

Since $\pi_x\psi$ is monotone, $\pi_x\psi(u) \vee \pi_x\psi(v) \subseteq \pi_x\psi(u \vee v)$. We have to establish that $\pi_x\psi(u \vee v) \subseteq \pi_x\psi(u) \vee \pi_x\psi(v)$. So suppose that $(z_0, z_1) \in \pi_x\psi(u \vee v)$. This means that $(z_0, z_1) \in \psi_x(y) \neq \varnothing$ for some $y \in \mathrm{J}(L)$ such that $y \leq u \vee v$. If $y \leq u$ or $y \leq v$ then $(z_0, z_1) \in \pi_x\psi(u) \cup \pi_x\psi(v)$. Otherwise, $y \leq u \vee v$ is a nontrivial join cover. As L is a J-lattice, we can refine this join cover to a minimal one. This implies that $y \in \mathrm{J}(L) \backslash \mathrm{P}(L)$. As $\psi_x(y) \neq \varnothing$, we conclude by the definition of ψ_x that $y = x$. Moreover, there is i such that $1 \leqslant i \leqslant n(x)$ and $y = x \leq b_i \vee c_i$ is a minimal nontrivial join cover with $\{b_i, c_i\} \ll \{u, v\}$. Inclusion $(z_0, z_1) \in \psi_x(y) = \psi_x(x)$ implies that $z_0 = 0(x)$ and $z_1 = 1(x)$. Furthermore, $\big(0(x), a_i(x)\big) \in \psi_x(b_i) \subseteq \pi_x\psi(u) \cup \pi_x\psi(v)$

and $\big(a_i(x), 1(x)\big) \in \psi_x(c_i) \subseteq \pi_x\psi(u) \cup \pi_x\psi(v)$ as $\{b_i, c_i\} \ll \{u, v\}$. Hence, $(z_0, z_1) \in \psi_x(b_i) \vee \psi_x(c_i) \subseteq \pi_x\psi(u) \cup \pi_x\psi(v)$. This proves that $\pi_x\psi$ preserves joins.

Since $\pi_x\psi$ is monotone, $\pi_x\psi(u \wedge v) \subseteq \pi_x\psi(u) \cap \pi_x\psi(v)$. We have to establish that $\pi_x\psi(u) \cap \pi_x\psi(v) \subseteq \pi_x\psi(u \wedge v)$. Indeed, let $(z_0, z_1) \in \pi_x\psi(u) \cap \pi_x\psi(v)$. This means that $(z_0, z_1) \in \psi_x(y) \cap \psi_x(y') \neq \varnothing$ for some $y, y' \in \mathrm{J}(L)$ such that $y \leq u$ and $y' \leq v$. If $y \neq y'$ then $\psi_x(y) \cap \psi_x(y') = \varnothing$ by the definition of ψ_x, a contradiction. Therefore, $y = y' \leq u \wedge v$ and $(z_0, z_1) \in \psi_x(y) \subseteq \pi_x\psi(u \wedge v)$. This proves that $\pi_x\psi$ preserves meets. □

Claim 3. ψ is an embedding.

Proof of Claim. Suppose that $u \nleq v$ in L. As L is a J-lattice, there is $y \in \mathrm{J}(L)$ such that $y \leq u$ and $y \nleq v$. By our definition, there is $x \in I$ such that $\psi_x(y) \neq \varnothing$. But then $\varnothing \neq \psi_x(y) \subseteq \pi_x\psi(u)$ and $\psi_x(y) \cap \pi_x\psi(v) = \varnothing$. This implies that $\pi_x\psi(u) \nsubseteq \pi_x\psi(v)$ whence $\psi(u) \nleq \psi(v)$. □

It follows from the claims above that

$$L \in \mathbf{SP}\big(\mathrm{O}(P_x) \mid x \in I\big) \subseteq \mathbf{SPS}\big(\mathrm{O}(M_n)\big) \subseteq \mathbf{SP}\big(\mathrm{O}(M_n)\big).$$

The proof of Proposition 8 is complete. □

Proposition 9. *Let L be a bi-algebraic lattice such that $L \models \Sigma_n$, where $0 < n < \omega$. Then for each element $x \in \mathrm{CJ}(L)$, we have*

$$\mathfrak{M}(x) = \big\{\{a, b\} \mid a, b \in \mathrm{CP}(L),\ \ a \neq b\big\} \text{ and } |\mathfrak{M}(x)| \leqslant n.$$

In particular, $L \in \mathbf{SP}\big(\mathrm{O}(M_n)\big)$.

Proof. The argument is similar to the one in the proof of Proposition 8 and uses Proposition 1(ii). □

Theorem 10. *Σ_n forms an equational basis for $\mathbf{SP}\big(\mathrm{O}(M_n)\big)$. In particular, the class $\mathbf{SP}\big(\mathrm{O}(M_n)\big) = \mathbf{S}_n$ is a lattice variety.*

Proof. Let $L \models \Sigma_n$ and let F be the dual filter lattice of L. It is well-known that F is dually algebraic and it follows that $F \models \Sigma_n$. By Proposition 1, F is a J-lattice. By Proposition 8, $F \in \mathbf{SP}\big(\mathrm{O}(M_n)\big)$ whence $L \in \mathbf{SP}\big(\mathrm{O}(M_n)\big)$ as L embeds into F. On the other hand, the lattice $\mathrm{O}(M_n)$ has the only minimal join covers:

$$\begin{aligned} &A \leq A_i \vee B_i, \quad 1 \leqslant i \leqslant n, \text{ where} \\ &A = \big\{(0,1)\big\},\ A_i = \big\{(0, a_i)\big\},\ B_i = \big\{(a_i, 1)\big\}, \quad 1 \leqslant i \leqslant n, \end{aligned}$$

see Figure 1. Thus, $\mathrm{O}(M_n)$ is 2-distributive by Corollary 3. Moreover, $\mathrm{O}(M_n)$ satisfies the condition (ii) of Corollaries 7 and 5. This implies that $\mathrm{O}(M_n) \models \Sigma_n$. □

Let $\mathbf{L}_{01}$ denote the variety of $(0,1)$-lattices and let $\mathbf{S}_n^{01} = \mathbf{L}_{01} \cap \operatorname{Mod} \Sigma_n$.

Theorem 11. *The set Σ_n forms an equational basis for $\mathbf{SP}\big(\mathrm{O}(M_n)\big)$ within the variety $\mathbf{L}_{01}$. In particular, $\mathbf{SP}\big(\mathrm{O}(M_n)\big) = \mathbf{S}_n^{01}$ is a variety of $(0,1)$-lattices.*

Proof. If L is a $(0,1)$-lattice then taking in the proof of Theorem 10 the dual lattice of nonempty filters as F, we obtain that L is a $(0,1)$-sublattice of F and $F \in \mathbf{SP}\big(\mathrm{O}(M_n)\big)$ by Proposition 8. Therefore, L belongs in this case to the variety of $(0,1)$-lattices generated by $\mathrm{O}(M_n)$. □

4. An equational basis for $\mathbf{Q}\big(\mathrm{O}(M_\omega)\big)$

We put $\Sigma = \{(\mathrm{D}_2), (\mathrm{P})\}$.

Proposition 12. *Let L be a dually algebraic lattice such that $L \models \Sigma$. The following statements hold.*

(i) *For each element $x \in \mathrm{J}(L)$, we have*

$$\mathfrak{M}(x) = \big\{\{a, b\} \mid a, b \in \mathrm{P}(L),\ a \neq b\big\}.$$

(ii) *If L is bi-algebraic then for each element $x \in \mathrm{CJ}(L)$, we have*

$$\mathfrak{M}(x) = \big\{\{a, b\} \mid a, b \in \mathrm{CP}(L),\ a \neq b\big\}.$$

In particular, $L \in \mathbf{SP}\big(\mathrm{O}(M_\kappa)\big) \subseteq \mathbf{Q}\big(\mathrm{O}(M_\omega)\big)$ for some infinite cardinal κ.

Proof. Applying the same argument as in the proof of Proposition 8, we obtain that $L \in \mathbf{SP}\big(\mathrm{O}(M_\kappa)\big)$ for some infinite cardinal $\kappa \geqslant |L|$. As M_κ embeds into an ultrapower of M_ω, we conclude that $\mathrm{O}(M_\kappa) \in \mathbf{SP}_u\big(\mathrm{O}(M_\omega)\big)$ and

$$\mathbf{SP}\big(\mathrm{O}(M_\kappa)\big) \subseteq \mathbf{SPP}_u\big(\mathrm{O}(M_\omega)\big) = \mathbf{Q}\big(\mathrm{O}(M_\omega)\big),$$

which is our desired conclusion. □

Theorem 13. *The following statements hold.*

(i) *The quasivariety $\mathbf{Q}\big(\mathrm{O}(M_\omega)\big)$ is a lattice variety and Σ forms an equational basis for this variety.*

(ii) *The class $\mathbf{Q}\big(\mathrm{O}(M_\omega)\big)$ of $(0,1)$-lattices is a variety of $(0,1)$-lattices and Σ forms an equational basis for this variety.*

Proof. (i) If $L \models \Sigma$, then $L \in \mathbf{Q}\big(\mathrm{O}(M_\omega)\big)$ by Proposition 12. Hence, $\operatorname{Mod}\Sigma \subseteq \mathbf{Q}\big(\mathrm{O}(M_\omega)\big)$. Conversely, the lattice $\mathrm{O}(M_\omega)$ has the only minimal join covers:

$$A \leq A_i \vee B_i, \quad 1 \leqslant i < \omega, \text{ where}$$
$$A = \big\{(0,1)\big\},\ A_i = \big\{(0, a_i)\big\},\ B_i = \big\{(a_i, 1)\big\}, \quad 1 \leqslant i < \omega,$$

see Figure 1. Thus, $\mathrm{O}(M_n)$ is 2-distributive by Corollary 3. Moreover, $\mathrm{O}(M_n)$ satisfies the condition (ii) of Corollary 5. Therefore, $\mathrm{O}(M_\omega) \models \Sigma$ and

$$\mathbf{Q}\big(\mathrm{O}(M_\omega)\big) = \mathbf{SPP}_u\big(\mathrm{O}(M_\omega)\big) \models \Sigma$$

as identities are stable with respect to the operators $\mathbf{S}$, $\mathbf{P}$, and $\mathbf{P}_u$. It follows that $\operatorname{Mod}\Sigma = \mathbf{Q}\big(\mathrm{O}(M_\omega)\big)$.

The proof of (ii) is similar. □

Corollary 14. *The following equalities hold for an arbitrary infinite cardinal κ:*

$$\mathbf{SO}_2 = \mathbf{Q}\big(\mathrm{O}(M_n) \mid 0 < n < \omega\big) = \mathbf{SP}\big(\mathrm{O}(M_\kappa)\big).$$

Proof. By [13, Theorem 4.8], Σ forms an equational basis for $\mathbf{SO}_2$. Taking into account Theorem 13, we conclude that $\mathbf{SO}_2 = \mathbf{SP}\big(\mathrm{O}(M_\omega)\big)$. Furthermore, each algebraic structure embeds into an ultraproduct of its finitely generated substructures, see for example [6, Theorem 1.2.8]. Therefore, $M_\kappa \in \mathbf{SP}_u(M_n \mid 0 < n < \omega)$ for each infinite cardinal kappa whence $\mathrm{O}(M_\kappa) \in \mathbf{SP}_u\big(\mathrm{O}(M_n) \mid 0 < n < \omega\big)$ and

$$\begin{aligned}\mathbf{SO}_2 =& \mathbf{SP}\big(\mathrm{O}(M_\omega)\big) = \mathbf{SP}\big(\mathrm{O}(M_\kappa)\big) \subseteq \mathbf{SPP}_u\big(\mathrm{O}(M_n) \mid 0 < n < \omega\big) = \\ =& \mathbf{Q}\big(\mathrm{O}(M_n) \mid 0 < n < \omega\big) \subseteq \mathbf{SO}_2.\end{aligned}$$

The desired conclusion follows. □

The following problem was raised in [13].

Problem 1. [13, Question 2] If $\langle P; \leq\rangle$ is a finite poset, is it true that the quasivariety $\mathbf{SP}\big(\mathrm{O}(P;\leq)\big)$ is a variety?

The next statement solves Problem 1 in the positive for finite posets of length at most two.

Corollary 15. *If $\langle P; \leq\rangle$ is a finite poset of length at most two then $\mathbf{SP}\big(\mathrm{O}(P;\leq)\big)$ is a finitely based variety.*

Proof. It follows from Corollary 14 and the fact that the poset $\langle P; \leq\rangle$ is finite that $\mathbf{SP}\big(\mathrm{O}(P;\leq)\big) = \mathbf{SP}\big(\mathrm{O}(M_n)\big)$ or $\mathbf{SP}\big(\mathrm{O}(P;\leq)\big)$ is the variety of distributive lattices. In the first case, $\mathbf{SP}\big(\mathrm{O}(P;\leq)\big)$ is a finitely based variety by Theorem 10. □

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