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COMPLETELY REGULAR CODES  
IN THE  $n$ -DIMENSIONAL RECTANGULAR GRID

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**ABSTRACT.** It is proved that two sequences of the intersection array of an arbitrary completely regular code in the  $n$ -dimensional rectangular grid are monotonic. It is shown that the minimal distance of an arbitrary completely regular code is at most 4 and the covering radius of an irreducible completely regular code in the grid is at most  $2n$ .

**Keywords:**  $n$ -dimensional rectangular grid, completely regular code, intersection array, covering radius, perfect coloring

## 1. INTRODUCTION

A vertex coloring of a graph is perfect if any two vertices of the same color “see” the same number of vertices of any fixed color. If in addition the vertices are colored by distance from some initial set of vertices then the coloring is distance regular and this set is a completely regular code. These notions are closely related with distance regular graphs. In fact, in a distance regular graph a distance coloring with respect to an arbitrary vertex is perfect and parameters of the corresponding distance regular coloring do not depend on the choice of the vertex. But this property does not hold for the graph of the  $n$ -dimensional rectangular grid in case  $n > 1$ . Completely regular codes in distance regular graphs are extensively investigated.

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The ordered pair  $(b_0, \dots, b_{r-2}; c_1, \dots, c_{r-1})$  of sequences of upper and lower degrees is called as the *intersection array* of the completely regular code  $D$ .

The  $n$ -dimensional rectangular grid is the graph  $G_n$  with the vertex set

$$\mathbb{Z}^n = \{\mathbf{x} = (x_1, \dots, x_n) : x_i \in \mathbb{Z}, i = 1, \dots, n\}$$

and with the edge set

$$\{(\mathbf{x}, \mathbf{y}) : \sum_{i=1}^n |x_i - y_i| = 1\}.$$

Then the graph distance between vertices  $\mathbf{x}$  and  $\mathbf{y}$  can be written as  $\rho(\mathbf{x}, \mathbf{y}) = \sum_{i=1}^n |x_i - y_i|$ . Let  $\mathbf{e}^i \in \mathbb{Z}^n$ ,  $i = 1, \dots, n$ , be the  $(0, 1)$ -vector with the unique one at the  $i$ -th position.

Further in the paper  $D$  denotes a completely regular code in the  $n$ -dimensional rectangular grid,  $\varphi = \varphi_D$  denotes the distance coloring

$$\varphi(\mathbf{x}) = \rho(D, \mathbf{x}), \quad \mathbf{x} \in \mathbb{Z}^n,$$

with respect to  $D$  and

$$\Phi = (\Phi_0 = D, \Phi_1, \dots, \Phi_{r-1}), \quad \Phi_i = \{\mathbf{x} \in \mathbb{Z}^n : \varphi(\mathbf{x}) = i\}, \quad i = 0, \dots, r-1,$$

denotes the corresponding partition of  $\mathbb{Z}^n$  by colors. For an arbitrary vertex  $\mathbf{x} \in \mathbb{Z}^n$  let us introduce the following sets of unit vectors:

$$\begin{aligned} A_\varphi(\mathbf{x}) &= \{\mathbf{y} - \mathbf{x} : \varphi(\mathbf{y}) = \varphi(\mathbf{x}), \rho(\mathbf{x}, \mathbf{y}) = 1\}, \\ B_\varphi(\mathbf{x}) &= \{\mathbf{y} - \mathbf{x} : \varphi(\mathbf{y}) = \varphi(\mathbf{x}) + 1, \rho(\mathbf{x}, \mathbf{y}) = 1\}, \\ C_\varphi(\mathbf{x}) &= \{\mathbf{y} - \mathbf{x} : \varphi(\mathbf{y}) = \varphi(\mathbf{x}) - 1, \rho(\mathbf{x}, \mathbf{y}) = 1\}. \end{aligned}$$

We will omit the subscript  $\varphi$  if the coloring  $\varphi$  is clear from the context. We will call vectors from the sets  $A_\varphi(\mathbf{x}), B_\varphi(\mathbf{x}), C_\varphi(\mathbf{x})$  as *inner*, *upper* and *lower directions* of the vertex  $\mathbf{x}$  with respect to  $\varphi$ . Obviously,

$$\begin{aligned} |A_\varphi(\mathbf{x})| &= a_i, \quad |B_\varphi(\mathbf{x})| = b_i, \quad |C_\varphi(\mathbf{x})| = c_i, \\ A_\varphi(\mathbf{x}) \cup B_\varphi(\mathbf{x}) \cup C_\varphi(\mathbf{x}) &= \{\pm \mathbf{e}^i : i = 1, \dots, n\}. \end{aligned}$$

For any set  $L$  of directions, we denote  $-L = \{-l : l \in L\}$ .

We say that two colorings  $\varphi$  and  $\psi$  are *equivalent* if  $\psi$  can be obtained from  $\varphi$  by some graph automorphism and some color renumbering. In particular, for a distance regular coloring  $\varphi$ , the coloring  $\psi$  with the inverse order of colors is equivalent and

$$(1) \quad B_\varphi(\mathbf{x}) = C_\psi(\mathbf{x}), \quad C_\varphi(\mathbf{x}) = B_\psi(\mathbf{x}), \quad A_\varphi(\mathbf{x}) = A_\psi(\mathbf{x}), \quad \mathbf{x} \in \mathbb{Z}^n.$$

We say that two codes  $D_1$  and  $D_2$  are *equivalent* if  $D_2$  can be obtained from  $D_1$  by some graph automorphism.

## 2. REDUCIBLE COLORINGS

For an arbitrary  $r$ , there exist only three nonequivalent completely regular codes in the 1-dimensional grid  $G_1$  with the covering radius  $r-1$ :

$$\begin{aligned} &\{2(r-1)t : t \in \mathbb{Z}\}, \\ &\{(2r-1)t : t \in \mathbb{Z}\}, \\ &\{2rt, 2rt-1 : t \in \mathbb{Z}\}. \end{aligned}$$

The corresponding distance colorings are periodical, here their periods are presented as sequences of colors together with the parameter matrices:

$$\begin{aligned} 0, 1, 2, \dots, r-2, r-1, r-2, \dots, 1; & \quad [02|101|101| \dots |101|20], \\ 0, 0, 1, 2, \dots, r-2, r-1, r-2, \dots, 1; & \quad [11|101|101| \dots |101|20], \\ 0, 0, 1, 2, \dots, r-2, r-1, r-1, r-2, \dots, 1; & \quad [11|101|101| \dots |101|11]. \end{aligned}$$

A code  $D$  in the  $n$ -dimensional rectangular grid  $G_n$  is called *reducible* if there exists a completely regular code  $D'$  of  $G_1$  and  $\delta_1, \dots, \delta_n \in \{0, 1, -1\}$  such that

$$(2) \quad D = \{\mathbf{x} = (x_1, x_2, \dots, x_n) : (\delta_1 x_1 + \delta_2 x_2 + \dots + \delta_n x_n) \in D'\}.$$

The parameter matrix is referred to as *reducible* if it admits a reducible coloring  $\varphi_D$ . Obviously, each color of the distance regular coloring with respect to the reducible completely regular code  $D$  is an union of sets

$$\{\mathbf{x} = (x_1, x_2, \dots, x_n) : (\delta_1 x_1 + \delta_2 x_2 + \dots + \delta_n x_n) = const\}.$$

So, the corresponding coloring  $\varphi_D$  of the  $n$ -dimensional grid is also called *reducible* and it means there exists an  $r$ -coloring  $\varphi'$  of  $G_1$  and  $\delta_1, \dots, \delta_n \in \{0, 1, -1\}$  such that for any  $(x_1, x_2, \dots, x_n) \in \mathbb{Z}^n$ ,

$$(3) \quad \varphi(x_1, x_2, \dots, x_n) = \varphi_1(\delta_1 x_1 + \delta_2 x_2 + \dots + \delta_n x_n).$$

If  $t$  denotes the number of nonzero coefficients  $\delta_k, 1 \leq k \leq n$ , then an arbitrary vertex of the color  $i, 1 \leq i \leq r-2$ , "sees" precisely  $t$  vertices of the color  $i-1$  and precisely  $t$  vertices of the color  $i+1$ . As a result, we obtain

**Lemma 1.** *Let  $P$  be an arbitrary reducible matrix of a distance regular coloring of  $G_n$ . Then there exists  $t \in \{0, 1, \dots, n\}$  such that*

$$P = [2n - \varepsilon_1 q, \varepsilon_1 q \mid q, 2n - 2q, q \mid \dots \mid q, n - 2q, q \mid \varepsilon_2 q, 2n - \varepsilon_2 q],$$

where  $\varepsilon_1, \varepsilon_2 \in \{1, 2\}$  (the colorings with  $(\varepsilon_1, \varepsilon_2) = (1, 2)$  and  $(2, 1)$  are equivalent).

### 3. UPPER AND LOWER DEGREES

Throughout this section  $r \geq 3$  and a fixed distance regular  $r$ -coloring of  $G_n$  is denoted by  $\varphi$ . We are going to prove the monotonicity of the upper degrees (and the lower degrees) of the coloring  $\varphi$ .

**Lemma 2.** *Let  $\mathbf{x}$  and  $\mathbf{y}$  be two adjacent vertices and  $\varphi(\mathbf{y}) = \varphi(\mathbf{x}) + 1$ . Then*

$$C(\mathbf{x}) \subseteq C(\mathbf{y}), \quad B(\mathbf{x}) \supseteq B(\mathbf{y}).$$

*Proof.* If  $\varphi(\mathbf{x}) = 0$  then  $\emptyset = C(\mathbf{x}) \subseteq C(\mathbf{y})$ .

Now  $i = \varphi(\mathbf{x}) \geq 1$  and then  $C(\mathbf{x}) \neq \emptyset$ . Let us take  $d \in C(\mathbf{x})$ . If  $d = \mathbf{x} - \mathbf{y}$  then obviously  $d \in C(\mathbf{y})$ . Then consider the case  $d \neq \mathbf{x} - \mathbf{y}$ . The color of the vertex  $\mathbf{x} + d$  equals  $i-1$ . The color of the vertex  $\mathbf{y} + d$  is equal to  $i$ , because  $\varphi(\mathbf{y}) = i+1$  and the vertex  $\mathbf{x} + d = (\mathbf{y} + d) + (\mathbf{x} - \mathbf{y})$  has the color  $i-1$ . Then  $d \in C(\mathbf{y})$ .

Finally,  $C(\mathbf{x}) \subseteq C(\mathbf{y})$ . Further, (1) gives  $B(\mathbf{x}) \supseteq B(\mathbf{y})$ . □

Lemma 2 immediately gives us the monotonicity of the lower degrees and the upper degrees of an arbitrary distance regular  $r$ -coloring of  $G_n$ :

**Theorem 1.** *Let  $D$  be an arbitrary completely regular code in the  $n$ -dimensional rectangular grid with the intersection array  $(b_0, \dots, b_{r-2}; c_1, \dots, c_{r-1})$ . Then*

$$c_1 \leq \dots \leq c_{r-2} \leq c_{r-1},$$

$$b_0 \geq b_1 \geq \dots \geq b_{r-2}.$$

It follows from Theorem 1 that there exist two colors  $I = I(\varphi)$  and  $J = J(\varphi)$  such that

$$(4) \quad I(\varphi) = \max\{i : c_{i+1} \geq b_{i+1}\},$$

$$(5) \quad J(\varphi) = \min\{i : c_{i-1} \leq b_{i-1}\}.$$

Thus, all colors of a distance regular coloring  $\varphi$  are partitioned into three segments:  $\{0, \dots, I(\varphi)\} \neq \emptyset$ , here  $c_i < b_i$  for any  $i, 0 < i \leq I(\varphi)$ ;  $\{I(\varphi) + 1, \dots, J(\varphi) - 1\}$ , here  $c_i = b_i$  for any  $i, I(\varphi) \leq i \leq J(\varphi)$ ;  $\{J(\varphi), \dots, r - 1\} \neq \emptyset$ , here  $c_i > b_i$  for any  $i, J(\varphi) \leq i < r - 1$ .

Lower degrees in the first segment and upper degrees in the last segment are considered in the following

**Lemma 3.** *a) If  $i \leq I(\varphi)$  then  $c_i \neq c_{i+1}$ . b) If  $i \geq J(\varphi)$  then  $b_i \neq b_{i-1}$ .*

*Proof.* a) Let us suppose  $i \leq I(\varphi)$  and  $c_i = c_{i+1}$ . We take an arbitrary vertex  $\mathbf{x}$  of the color  $i$  and an arbitrary direction  $d \in B(\mathbf{x})$ . It means that  $-d \in C(\mathbf{x} + d)$ . By assumption,  $c_i = c_{i+1}$ , then it follows from Lemma 2 that  $-d \in C(\mathbf{x})$ . Hence,  $-B(\mathbf{x}) \subseteq C(\mathbf{x})$  and  $b_i \leq c_i$  in contrary to the choice of  $i$ . Therefore a) is true. Now (1) gives b).  $\square$

#### 4. COLORS WITH THE SAME DEGREE TRIPLE

Lemma 3 establishes that only the degree triple of form  $(t, 2n - 2t, t)$  can be repeated in the intersection matrix of a completely regular code. Everywhere in this section  $\varphi$  denotes an arbitrary distance regular coloring, moreover, we suppose  $r \geq 4$  and  $J(\varphi) > I(\varphi) + 2$ ; i.e., the different colors  $I(\varphi) + 1$  and  $J(\varphi) - 1$  have the same degree triple.

**Lemma 4.** *Let the colors  $i$  and  $i + 1$  have the same degree triple. Then for any two adjacent vertices  $\mathbf{x}$  and  $\mathbf{y}$  of colors  $i$  and  $i + 1$ , respectively, we have*

$$(6) \quad C(\mathbf{x}) = C(\mathbf{y}) = -B(\mathbf{x}) = -B(\mathbf{y}),$$

$$(7) \quad A(\mathbf{x}) = A(\mathbf{y}) = -A(\mathbf{x}) = -A(\mathbf{y}).$$

*Proof.* Equalities  $C(\mathbf{x}) = C(\mathbf{y})$  and  $B(\mathbf{x}) = B(\mathbf{y})$  follow from Lemma 2. Further, we put the direction  $d \in B(\mathbf{x})$ . It means that  $-d \in C(\mathbf{x} + d)$ . Then  $-d \in C(\mathbf{x})$  by Lemma 2 provided  $c_i = c_{i+1}$ . Hence  $-B(\mathbf{x}) \subseteq C(\mathbf{x})$ . But we have  $b_i = c_i$ , so  $-B(\mathbf{x}) = C(\mathbf{x})$ . Finally, (7) follows from (6).  $\square$

We emphasize that according to Lemma 4, two opposite directions,  $d$  and  $-d$ , belong or do not belong to the set  $A(\mathbf{x})$  of inner directions of a vertex  $\mathbf{x}$  of a color  $i$ ,  $i = I(\varphi) + 1, \dots, J(\varphi) - 1$ , simultaneously.

For any set  $V \subseteq \mathbb{Z}^n$  of vertices  $G(V)$  denotes the subgraph of the  $n$ -dimensional rectangular grid generated by  $V$ . Let  $V_i^{i+1}$  be the vertex set of an arbitrary connected component of the graph  $G(\Phi_i \cup \Phi_{i+1})$ .

**Lemma 5.** *Let the colors  $i$  and  $i + 1$  have the same degree triples. Then for any two vertices  $\mathbf{x}, \mathbf{y} \in V_i^{i+1}$  the equalities (6) and (7) hold.*

*Proof.* It is sufficient to prove (6) and (7) for two adjacent vertices of the same color  $i$  or  $i + 1$ , without loss of generality of the color  $i$ . Let  $\mathbf{x}, \mathbf{y} \in \Phi_i$  be two adjacent vertices. Then  $\mathbf{x} - \mathbf{y} \in A(\mathbf{y})$  and by Lemma 4 also  $\mathbf{y} - \mathbf{x} \in A(\mathbf{y})$ , whence we have

$$(8) \quad 2\mathbf{y} - \mathbf{x} = \mathbf{y} + (\mathbf{y} - \mathbf{x}) \in \Phi_i.$$

Suppose (7) is not true for the vertices  $\mathbf{x}$  and  $\mathbf{y}$ . Then there exists  $d \in A(\mathbf{x}) \setminus A(\mathbf{y})$ . In this case  $d \in B(\mathbf{y}) \cup C(\mathbf{y})$ . First let  $d \in B(\mathbf{y})$ . Then  $\mathbf{y} + d \in \Phi_{i+1}$  and  $\mathbf{x} - \mathbf{y} = (\mathbf{x} + d) - (\mathbf{y} + d) \in C(\mathbf{y} + d)$ . By Lemma 4,  $\mathbf{y} - \mathbf{x} \in B(\mathbf{y} + d)$  from which we have

$$2\mathbf{y} - \mathbf{x} + d = (\mathbf{y} + d) + (\mathbf{y} - \mathbf{x}) \in \Phi_{i+2}$$

that contradicts (8). It remains to consider the case  $d \in C(\mathbf{y})$ . Here  $-d \in B(\mathbf{y})$  and analogously we obtain the contradiction. Thus, (7) holds for the vertices  $\mathbf{x}$  and  $\mathbf{y}$ .

The equation (6) follows from (7) and Lemma 4.  $\square$

Now we will recognize that an arbitrary connected component  $V_i^{i+1}$  of the graph  $G(\Phi_i \cup \Phi_{i+1})$ ,  $I(\varphi) < i < J(\varphi)$ , consists of two hyperplanes. Let  $\gamma \in \mathbb{Z}$  and  $\delta = (\delta_1, \delta_2, \dots, \delta_n) \in \{0, 1, -1\}^n$ . Let us denote

$$(9) \quad M(\delta, \gamma) = \{\mathbf{x} = (x_1, x_2, \dots, x_n) \in \mathbb{Z}^n : \delta_1 x_1 + \delta_2 x_2 + \dots + \delta_n x_n = \gamma\}.$$

**Lemma 6.** *Let the colors  $i$  and  $i + 1$  have the same degree triple. Then there exist integer  $\gamma$  and  $(0, 1, -1)$ -valued vector  $\delta = (\delta_1, \dots, \delta_n)$  such that*

$$\Phi_{i+\varepsilon} \cap V_i^{i+1} = M(\delta, \gamma + \varepsilon), \quad \varepsilon \in \{0, 1\}.$$

*Proof.* Let  $0 \leq t \leq n$ , the repeated degree triple be  $(t, 2n - 2t, t)$  and  $\mathbf{v} = (v_1, v_2, \dots, v_n) \in V_i^{i+1}$  be the vertex of color  $i$ . Lemma 5 follows that

$$B(\mathbf{v}) = \{\mathbf{e}^1, \dots, \mathbf{e}^s, -\mathbf{e}^{s+1}, \dots, -\mathbf{e}^t\},$$

$$C(\mathbf{v}) = \{-\mathbf{e}^1, \dots, -\mathbf{e}^s, \mathbf{e}^{s+1}, \dots, \mathbf{e}^t\},$$

$$A(\mathbf{v}) = \{\pm \mathbf{e}^{t+1}, \dots, \pm \mathbf{e}^n\},$$

up to the numbering of unit vectors. Then let us put:

$$\delta_1 = \dots = \delta_s = 1,$$

$$\delta_{s+1} = \dots = \delta_t = -1,$$

$$\delta_{t+1} = \dots = \delta_n = 0,$$

$$\gamma = v_1 + \dots + v_s - v_{s+1} - \dots - v_t.$$

Under this choice of constants,  $\mathbf{v} \in M(\delta, \gamma)$  obviously. For an arbitrary vertex  $\mathbf{x} \in V_i^{i+1}$ , one can easily check by induction on distance between  $\mathbf{x}$  and  $\mathbf{v}$  that  $\mathbf{x} \in M(\delta, \gamma)$  in case  $\mathbf{x} \in \Phi_i$  and  $\mathbf{x} \in M(\delta, \gamma + 1)$  in case  $\mathbf{x} \in \Phi_{i+1}$ .  $\square$

**Theorem 2.** *Let  $\varphi : \mathbb{Z}^n \rightarrow \{0, 1, \dots, r - 1\}$  be an arbitrary distance regular coloring,  $2 \leq i < j \leq r - 2$ , the colors  $i$  and  $j$  have the same degree triple. Then the degree triples coincide for all colors from 1 to  $r - 2$  and the coloring is reducible.*

*Proof.* We will show that any color consists of hyperplanes of form (9). All colors from  $i$  to  $j$ , in particular, the colors  $i$  and  $i + 1$  have the same degree triple. By Lemma 6, there exists  $\gamma \in \mathbb{Z}$  and  $\delta \in \{0, 1, -1\}^n$  such that  $M(\delta, \gamma + \varepsilon) \subseteq \Phi_{i+\varepsilon}$ ,  $\varepsilon \in \{0, 1\}$ . Then for any  $k \in \{0, \dots, r - 1\}$  by induction on  $|k - i|$  one can easily check that  $M(\delta, \gamma + k - i) \subseteq \Phi_k$  because the coloring is distance regular. In particular, for the initial and the last colors it holds  $M(\delta, \gamma - i) \subseteq \Phi_0$  and  $M(\delta, \gamma + r - i - 1) \subseteq \Phi_{r-1}$ . By distance regularity of the coloring,  $M(\delta, \gamma - i - 1) \subseteq \Phi_1$  or  $M(\delta, \gamma - i - 1) \subseteq \Phi_0$  and  $M(\delta, \gamma + r - i) \subseteq \Phi_{r-2}$  or  $M(\delta, \gamma + r - i) \subseteq \Phi_{r-1}$ .

Hence, by distance regularity of the coloring, for each  $\gamma' \in \mathbb{Z}$  all vertices of  $M(\delta, \gamma')$  have the same color which depends on  $\gamma' = \delta_1 x_1 + \delta_2 x_2 + \dots + \delta_n x_n$  only. It means that there exists a distance regular coloring  $\varphi'$  of the graph  $G_1$  such that

$$\varphi(x_1, x_2, \dots, x_n) = \varphi'(\delta_1 x_1 + \delta_2 x_2 + \dots + \delta_n x_n),$$

i.e., the coloring  $\varphi$  of  $G_n$  is distance regular.  $\square$

## 5. MINIMAL DISTANCE AND COVERING RADIUS

The Hamming graph  $H_N$  consists of the following vertex set and edge set:

$$\mathbf{F}^N = \{\alpha = (\alpha_1, \dots, \alpha_N) : \alpha_k \in \{0, 1\}, k = 1, \dots, N\},$$

$$\{(\alpha, \beta) : \alpha, \beta \in \mathbf{F}^N, \sum_{k=1}^N |\alpha_k - \beta_k| = 1\}.$$

The Hamming graph  $H_{2n}$  is covered by the graph  $G_n$  of  $n$ -dimensional rectangular grid and the covering mapping is  $g : \mathbb{Z}^n \rightarrow \mathbf{F}^{2n}$ , such that for every  $\mathbf{x} = (x_1, \dots, x_n) \in \mathbb{Z}^n$

$$g(x_1, \dots, x_n) = (g_0(x_1 \bmod 4), \dots, g_0(x_n \bmod 4)),$$

$$g_0(0) = 00, g_0(1) = 01, g_0(2) = 11, g_0(3) = 10.$$

Actually, this mapping preserves the adjacency: if two vertices  $\mathbf{x}, \mathbf{y}$  of the  $n$ -dimensional rectangular grid differ in exactly one position, then clearly the vertices  $g(\mathbf{x}), g(\mathbf{y})$  of the  $2n$ -dimensional Hamming graph  $H_{2n}$  also differ in exactly one position.

Let  $\psi$  be a coloring of the Hamming graph  $H_{2n}$ . Define the coloring  $\varphi_\psi$  of the  $n$ -dimensional rectangular grid  $G_n$  as follows:

$$(10) \quad \varphi_\psi(x_1, \dots, x_n) = \psi(g(x_1, \dots, x_n)), \quad (x_1, \dots, x_n) \in \mathbb{Z}^n.$$

Therefore the colorings  $\psi$  and  $\varphi_\psi$  are perfect (respectively, distance regular) simultaneously and have the same parameter matrix. We take as  $\psi$  the distance coloring of  $H_{2n}$  with respect to the all-zero vertex:

$$(11) \quad \psi(\alpha) = wt(\alpha), \quad \alpha \in \mathbf{F}^{2n},$$

where  $wt(\alpha) = \sum_{i=1}^{2n} \alpha_i$  is the Hamming weight of the vertex  $\alpha$ . In this case  $\psi$  distance regular  $(2n + 1)$ -coloring with the parameter matrix

$$[0, 2n|1, 0, 2n - 1| \dots |i, 0, 2n - i| \dots |2n - 1, 0, 1|2n, 0].$$

Then  $\varphi_\psi$  is also the distance regular  $(2n + 1)$ -coloring with the same parameter matrix, moreover, it is not reducible because its parameter matrix is not reducible.

Finally, we can state the main theorem.

**Theorem 3.** *For an arbitrary irreducible distance regular  $r$ -coloring of  $n$ -dimensional rectangular grid, it holds  $r \leq 2n + 1$ . An irreducible distance regular  $(2n + 1)$ -coloring exists.*

*Proof.* Let  $\varphi$  be an irreducible distance regular coloring of the  $n$ -dimensional rectangular grid. By Theorem 2, every two colors have different degree triples. This means that  $J(\varphi) - I(\varphi) \leq 2$ . By definition (4) of  $I(\varphi)$ , we have  $c_{I(\varphi)} < b_{I(\varphi)}$ , and also we know that  $c_{I(\varphi)} + b_{I(\varphi)} \leq 2n$ . It follows from Lemma 3 that  $\{c_1, \dots, c_{I(\varphi)}\}$  are pairwise different. Then  $I(\varphi) + 1 \leq n$ . Analogously, we obtain that  $J(\varphi) \geq r - n$ . Finally,  $r = (I(\varphi) + 1) + (J(\varphi) - I(\varphi) - 1) + (r - J(\varphi)) \leq n + 1 + n = 2n + 1$ .

The coloring  $\varphi_\phi$  defined by (10), (11) gives us the example of the irreducible  $(2n + 1)$ -coloring. □

Let us rewrite Theorem 3 in terms of completely regular codes:

**Corollary 1.** *The covering radius of an arbitrary completely regular code in the  $n$ -dimensional rectangular grid is at most  $2n$ .*

We also can obtain the upper bound for the minimal distance of the completely regular code in  $n$ -dimensional rectangular grid.

**Theorem 4.** *The minimal distance of an arbitrary completely regular code in  $n$ -dimensional rectangular grid is at most 4.*

*Proof.* Let  $D$  be an arbitrary completely regular code with minimal distance  $d = d(D) \geq 5$  in the graph  $G_n$ , and  $\mathbf{x} \in D$ . Let us consider the corresponding distance regular  $r$ -coloring  $\Phi = (\Phi_0 = D, \dots, \Phi_{r-1})$ . Here  $r - 1$  is equal to the covering radius of the code  $D$ , and  $r - 1 \geq 2$  as far as  $d \geq 5$ . Then any vertex of the sphere  $S_2(\mathbf{x})$  (of radius 2 centered in the vertex  $\mathbf{x}$ ) is of the color 2 and has adjacent vertices of color 1 only in the sphere  $S_1(\mathbf{x})$  (of radius 1 centered in the vertex  $\mathbf{x}$ ).

We consider the vertices  $\mathbf{x} + 2\mathbf{e}^1$  and  $\mathbf{x} + \mathbf{e}^1 + \mathbf{e}^2$  in the sphere  $S_2(\mathbf{x})$ . Their sets of lower directions are  $C(\mathbf{x} + 2\mathbf{e}^1) = \{\mathbf{e}^1\}$  and  $C(\mathbf{x} + \mathbf{e}^1 + \mathbf{e}^2) = \{\mathbf{e}^1, \mathbf{e}^2\}$  and have different cardinalities. This lead us to the contradiction. □

#### REFERENCES

- [1] *Avgustinovich S.V., Krotov D.S., Vasil'eva A. Yu.* Completely regular codes in the infinite hexagonal grid, Siberian Electronic Mathematical Reports, 2016, V.13, 987–1016. DOI:10.17377/semi.2016.13.079
- [2] *Avgustinovich S.V., Vasil'eva A. Yu., Sergeeva I.V.* Distance regular colorings of an infinite rectangular grid, Journal of Applied and Industrial Mathematics, 2012, V.6, No.3, 280–285. DOI:10.1134/S1990478912030027
- [3] *Avgustinovich S.V., Vasil'eva A. Yu.* Distance regular colorings of  $n$ -dimensional rectangular grid, Proceedings of 13th International Workshop on Algebraic and Combinatorial Coding Theory – Sofia: Inst. of Math. and Informatics, 2012, 35–40. <http://www.moi.math.bas.bg/moiuser/ACCT2012/b6.pdf>
- [4] *Axenovich M.* On multiple coverings of the infinite rectangular grid with balls of constant radius, Discrete Math. 2003. V.268, No.1-3. P. 31–49. DOI:10.1016/S0012-365X(02)00744-6
- [5] *Khoroshilova D.B.* On two-colour perfect colourings of circular graphs, Discretn. Anal. Issled. Oper., 2009. V.16, No.1, 80–92. (in Russian)
- [6] *Khoroshilova D.B.* On parameters of 2-colorings of circulant graphs, Diskretn. Anal. Issled. Oper., 2011, V. 18, No.6, 82–89 (in Russian)
- [7] *Koolen J. H.* On a conjecture of Martin on the parameters of completely regular codes and the classification of the completely regular codes in the Biggs-Smith graph, Linear and Multilinear Algebra, 1995, V.39, No.1-2, 3–17. DOI:10.1080/03081089508818376

- [8] *Krotov D.S.*, Perfect colorings of  $\mathbb{Z}^2$ : Nine colors, E-print 0901.0004, arXiv.org, 2009. <https://arxiv.org/abs/0901.0004>
- [9] *Martin W. J.* Completely Regular Subsets, Ph.D. thesis, Univ. Waterloo, 1992. <https://users.wpi.edu/~martin/RESEARCH/THESIS/t1.pdf>
- [10] *Parshina O. G.* Perfect 2-colorings of infinite circulant graphs with continuous set of distances, Diskretn. Anal. Issled. Oper., 2014, V.21, No.2. 76-83 (in Russian)
- [11] *S.A. Puzyrina* Perfect colorings of vertices of the graph  $G(\mathbb{Z}^2)$  in three colors, Diskretn. Anal. Issled. Oper. Ser. 2, 2005, V.12, No.1, 37-54. (in Russian).
- [12] *Vasil'eva A.Yu.* Completely regular codes in the triangular grid, Proceedings of MIPT, 2022, V.14, No.2(54), 14-42 (in Russian). <https://mipt.ru/upload/medialibrary/4f7/01.pdf>

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