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EQUATIONALLY NOETHERIAN VARIETIES OF SEMIGROUPS AND B.PLOTKIN'S PROBLEM

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Abstract: We consider systems of semigroup equations with constants. A semigroup S is called equationally Noetherian if any system of equations is equivalent over S to a finite subsystem. In the current paper we describe all semigroup varieties that consist of equationally Noetherian semigroups. Our result solves the problem of B.Plotkin for semigroup varieties.

Keywords: semigroups, varieties, universal algebraic geometry.

1 Introduction

The universal algebraic geometry is the subject on the edge of algebra and model theory and it deals with equations over arbitrary algebraic structures. Obviously, algebraic structures of the same variety have common equational properties. It allows us to develop a uniform algebraic geometry for all algebraic structures of a fixed variety.

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An atomic formula of a semigroup language $\mathcal{L} = \{\cdot\}$ is called an $\mathcal{L}$-equation (examples of $\mathcal{L}$-equations are: $x_1x_2 = x_2x_1$, $x_1^2 = x_1$, $x_1x_2^nx_1 = x_2$). A semigroup S is equationally Noetherian if any system of $\mathcal{L}$-equations in a finite number of variables is equivalent over S to its finite subsystem.

The following general problem for varieties of equationally Noetherian algebraic structures was posed by B. Plotkin in [5].

Problem 1. Describe all varieties $\mathbf{V}$ of algebraic structures such that every $A \in \mathbf{V}$ is equationally Noetherian?

Problem 1 admits positive solution in certain varieties. For example, any abelian variety of groups consists of equationally Noetherian elements (see [2] for more details). In the class of semigroups, the following varieties

- the variety of semigroups with zero multiplication,
- the variety of left (right) zero semigroups

satisfy the statement of Problem 1.

We see above that the language $\mathcal{L}$ defines the class of equations with no constants. One can reformulate Problem 1 for equations with constants.

Problem 2. Describe all varieties $\mathbf{V}$ of $\mathcal{L}$-algebras such that each $A \in \mathbf{V}$ is equationally Noetherian with respect to equations with constants from A (i.e. A is equationally Noetherian in the language $\mathcal{L}(A) = \mathcal{L} \cup \{a \mid a \in A\}$)?

In the current paper we completely solve Problem 2 for semigroup varieties. The description of such varieties $\mathbf{V}$ is given in Theorem 2.

2 Basic notions

Let S be a semigroup, and $E(S) = \{e \in S \mid ee = e\}$ be the set of its *idempotents*. A set $J \subseteq S$ is called an *ideal* if $sJ \subseteq J$ and $Js \subseteq J$ for any $s \in S$. A semigroup S is *simple* if it has a unique ideal $J = S$. By $\langle a \rangle_S$ we denote the ideal of S generated by an element $a \in S$.

An element $s \in S$ is *reducible* if there exist $a, b \in S$ with $s = ab$. The set of all reducible elements of a semigroup S is denoted by $Red(S)$. Obviously, $Red(S)$ is an ideal of S, and $E(S) \subseteq Red(S)$.

An idempotent e is *primitive* if for any $f \in E(S) \setminus \{0\}$ the equalities $fe = ef = f$ imply $f = e$. A simple semigroup S is *completely simple* if it contains a primitive idempotent.

The next classic theorem describes completely simple semigroups (see e.g. [1]).

Theorem 1. *For any completely simple semigroup S there exists a group G and sets I, Λ such that S is isomorphic to the set of triples (λ, g, i), $g \in G$, $\lambda \in \Lambda$, $i \in I$ with the multiplication*

$$(\lambda, g, i)(\mu, h, j) = (\lambda, gp_{i\mu}h, j),$$

where $p_{i\mu} \in G$ is an element of a matrix $\mathbf{P}$ such that

(1) $\mathbf{P}$ *consists of* $|I|$ *rows and* $|\Lambda|$ *columns;*
(2) *the first index of the sets* Λ, I *is denoted by* 1*;*
(3) *the matrix* $\mathbf{P}$ *is* $(1,1)$*-normalised, i.e.*

$$p_{i1} = p_{1\lambda} = 1 \in G \text{ for all } \lambda \in \Lambda,\ i \in I.$$

Following Theorem 1, we denote any completely simple semigroup S by $S = (\Lambda, G, I, \mathbf{P})$. The group G and the matrix $\mathbf{P}$ are called *the structural group* and *sandwich-matrix*, respectively. If the sandwich-matrix $\mathbf{P}$ consists of $1 \in G$ we will use the short denotation (Λ, G, I) instead of $(\Lambda, G, I, \mathbf{P})$.

Suppose a semigroup S has an ideal J. Then the factor-semigroup $H = S/J$ is called *the Rees factor semigroup*. The semigroup H always contains the zero (the ideal J is mapped into the zero of H).

An infinite direct power S^∞ of a semigroup S is the set of all sequences

$$(a_1, a_2, \dots, a_n, \dots),\ a_i \in S,$$

and with the element-wise multiplication.

Let $\mathcal{L} = \{\cdot\}$ be the semigroup language. An *equation over* $\mathcal{L}$ *(*$\mathcal{L}$*-equation)* is an equality of two $\mathcal{L}$-terms: $\tau(X) = \sigma(X)$. The examples of $\mathcal{L}$-equations are the following: $xy = yx$, $x^2 = x$, $xyx = yxy$.

A system $\mathbf{S}$ of $\mathcal{L}$-equations ($\mathcal{L}$-system for shortness) is an arbitrary set of $\mathcal{L}$-equations. The set of all solutions of $\mathbf{S}$ in variables $X = \{x_1, x_2, \dots, x_n\}$ over a semigroup S is denoted by $\mathrm{V}_S(\mathbf{S}(X)) \subseteq S^n$.

A semigroup S is *equationally Noetherian* if any infinite $\mathcal{L}$-system $\mathbf{S}$ in variables $X = \{x_1, x_2, \dots, x_n\}$ is equivalent to a finite subsystem $\mathbf{S}' \subseteq \mathbf{S}$ over S (the equivalence of $\mathbf{S}, \mathbf{S}'$ means $\mathrm{V}_S(\mathbf{S}') = \mathrm{V}_S(\mathbf{S})$).

By $\mathcal{L}(S)$ we denote the language $\mathcal{L} \cup \{s \mid s \in S\}$ extended by new constant symbols. The language extension allows us to use constants in equations. The examples of equations in the extended language $\mathcal{L}(S)$ are the following: $x^2 = s$, $xs = sx$, $s_1 x = s_2 x$, where $s, s_1, s_2 \in S$. Obviously, the class of $\mathcal{L}(S)$-equations is wider than the class of $\mathcal{L}$-equations, so an equationally Noetherian (in the language $\mathcal{L}$) semigroup S may lose this property in the language $\mathcal{L}(S)$. To avoid the ambiguity, we say that S is S*-equationally Noetherian* if S is equationally Noetherian in the language $\mathcal{L}(S)$.

Let

$$\tau(X) = u_1 \dots u_m$$

be an $\mathcal{L}(S)$-term, $u_i \in S \sqcup X$, $X = \{x_1, \dots, x_n\}$ and $u_{i+1} \notin S$ if $u_i \in S$. Each u_i is said to be a *literal* of the term $\tau(X)$. Moreover, the number m is called the *length* of $\tau(X)$ and denoted by $|\tau|$. For example, if S is the free semigroup generated by elements a, b then the length of the $\mathcal{L}(S)$-term $x_1 abx_2x_3b^3$ is 5.

Let S be a semigroup, and $\tau(X), \sigma(X)$ be $\mathcal{L}$-terms. We say that S *satisfies the identity* $\tau(X) = \sigma(X)$ (and denote $S \models [\tau(X) = \sigma(X)]$) if for any n-tuple $(s_1, s_2, \dots, s_n) \in S^n$ it holds $\tau(s_1, s_2, \dots, s_n) = \sigma(s_1, s_2, \dots, s_n)$. The class of semigroups $\mathbf{V} = \{S \mid S \models [\tau(X) = \sigma(X)] \text{ for all } \tau(X) = \sigma(X) \in \mathcal{I}\}$ is called the *variety* defined by a set of identities $\mathcal{I}$. Indeed, any variety is closed under the taking direct products, homomorphic images and subsemigroups.

Let $\mathrm{var}(S)$ denote the variety generated by a semigroup S, i.e. $\mathrm{var}(S)$ consists of all semigroups which satisfy all identities $\tau(X)=\sigma(X)$ such that $S\models[\tau(X)=\sigma(X)]$. The variety $\mathrm{var}(\mathbf{K})$ generated by a class of semigroups $\mathbf{K}$ consists of all semigroups S such that $S\models[\tau(X)=\sigma(X)]$ for any identity $\tau(X)=\sigma(X)$ with $T\models\tau(X)=\sigma(X)$ for each $T\in\mathbf{K}$.

3 Main result: structure

Since any variety is closed under direct powers, one should study semigroups that define equationally Noetherian direct powers.

Lemma 1. *Let S be a semigroup such that the direct power S^∞ is S^∞-equationally Noetherian. Then the solution set $Y=\mathrm{V}_S(xa=xb)$ (respectively, $Y=\mathrm{V}_S(ax=bx)$) is either $\emptyset$ or S.*

Proof. Let us consider an equation $xa=xb$, $a,b\in S$. If $Y\notin\{\emptyset,S\}$, there exists c,d such that $ca=cb$, $da\neq db$. Let us consider the following system of $\mathcal{L}(S^\infty)$-equations

$$\mathbf{S}=\begin{cases}\mathbf{x}(a,a,a,\ldots)=\mathbf{x}(a,a,a,\ldots),\\ \mathbf{x}(a,a,a,\ldots)=\mathbf{x}(b,a,a,\ldots),\\ \mathbf{x}(a,a,a,\ldots)=\mathbf{x}(b,b,a,\ldots),\\ \mathbf{x}(a,a,a,\ldots)=\mathbf{x}(b,b,b,\ldots),\\ \ldots\end{cases}$$

An element $\mathbf{x}=(x_1,x_2,x_3,\ldots)$ from the solution set of $\mathbf{S}$ has $x_i\neq d$ for all i. However, the solution set of the first n-equations of $\mathbf{S}$ contains the point

$$(\underbrace{c,c,\ldots,c}_{n-1\text{ times}},d,d,d,\ldots).$$

Thus, $\mathbf{S}$ is not equivalent to its finite subsystems, and S^∞ is not S^∞-equationally Noetherian. □

Corollary 1. *Let S be a semigroup with a zero 0 such that S^∞ is S^∞-equationally Noetherian. Then S has a zero multiplication.*

Proof. Let $a,b\in S$ and consider an equation $ax=0x$. Since the given equation is consistent over S ($0\in\mathrm{V}_S(ax=0x)$), Lemma 1 provides that $b\in\mathrm{V}_S(ax=0x)$. Thus, $ab=0b=0$. □

One can consider any group G in the semigroup language $\mathcal{L}$. For direct powers of groups we have the following result.

Lemma 2. *For any non-abelian group G the direct power G^∞ is not G^∞-equationally Noetherian.*

Proof. Let us consider the following system of $\mathcal{L}(G^\infty)$-equations

$$\mathbf{S} = \begin{cases} \mathbf{x}(1,1,1,\dots) = (1,1,1,\dots)\mathbf{x}, \\ \mathbf{x}(a,1,1,\dots) = (a,1,1,\dots)\mathbf{x}, \\ \mathbf{x}(a,a,1,\dots) = (a,a,1,\dots)\mathbf{x}, \\ \mathbf{x}(a,a,a,\dots) = (a,a,a,\dots)\mathbf{x}, \\ \dots \end{cases}$$

where an element $a \in G$ does not commute with some $b \in G$.

An element $\mathbf{x} = (x_1, x_2, x_3, \dots)$ from the solution set of $\mathbf{S}$ has $x_i \neq b$ for all i. However, the solution set of the first n-equations of $\mathbf{S}$ contains the point

$$(\underbrace{1,1,\dots,1}_{n-1 \text{ times}}, b, b, b, \dots).$$

Thus, G^∞ is not G^∞-equationally Noetherian. □

On the other hand, every abelian group G is G-equationally Noetherian (see [2]).

In Lemmas 3–11 we assume that a semigroup variety $\mathbf{V}$ satisfies the following statement:

"every $S \in \mathbf{V}$ is S-equationally Noetherian".

Lemma 3. *Let I be an ideal of $S \in \mathbf{V}$. Then $Red(S) \subseteq I$.*

Proof. The Rees factor semigroup S/I belongs to $\mathbf{V}$ and contains a zero. By Corollary 1, S/I has zero multiplication. Hence for all $a, b \in S$ it holds $ab \in I$. Thus, $Red(S) \subseteq I$. □

Lemma 4. *The set of all reducible elements $Red(S)$ of a semigroup $S \in \mathbf{V}$ is a simple semigroup.*

Proof. Assume there exists an element $x \in R = Red(S)$ such that $\langle x \rangle_R \subset R$. Consider an ideal

$$I = \langle x^5 \rangle_S = \{axxxxxb \mid a, b \in S^1\}$$

of the semigroup S. By Lemma 3, we obtain the contradiction:

$$\begin{aligned} R &\subseteq I = \{axxxxxb \mid a, b \in S^1\} \\ &= \{r_1 x r_2 \mid r_1 \in S^1 xx \subseteq R, r_2 \in xxS^1 \subseteq R\} \\ &\subseteq \{r_1 x r_2 \mid r_1, r_2 \in R\} \\ &= \langle x \rangle_R. \end{aligned}$$

□

Lemma 5. *Any element of a semigroup $S \in \mathbf{V}$ has finite order.*

Proof. Assume that $s \in S$ generates an infinite cyclic semigroup C_∞. Further in this lemma we consider the semigroup C_∞ in the additive form with the natural linear order $\leq$.

Then $C_\infty \oplus C_\infty \in \mathbf{V}$. Let us show that the two-element idempotent semigroup $L_2 = \{0,1\}$ is a factor of a subsemigroup of $C_\infty \times C_\infty$. Indeed, one can decompose the subsemigroup $\{(a,b) \mid a \leq b\} \subseteq C_\infty \times C_\infty$ into the union of the equivalence classes $A_1 = \{(a,a) \mid a \in C_\infty\}$, $A_0 = \{(a,b) \mid a < b, a, b \in C_\infty\}$.

The multiplication over A_1, A_0 is given by $A_iA_i = A_i$, $A_0A_1 = A_1A_0 = A_0$, and therefore $\{A_1, A_0\}$ is isomorphic to L_2. Thus, $L_2 \in \mathbf{V}$ and $L_2^\infty \in \mathbf{V}$, but L_2^∞ is not L_2^∞-equationally Noetherian (Corollary 1). □

Let us give some facts about the bicyclic semigroup $B = \langle a, b \mid ab = 1\rangle$

Proposition 1. *([1], Theorem 2.54) The bicyclic semigroup B is embedded into a simple but not completely simple semigroup S if S has at least one idempotent.*

Proposition 2. *([3], Theorem 4.1) The bicyclic semigroup B is not B-equationally Noetherian.*

Lemma 6. *Let $S \in \mathbf{V}$ then $R = Red(S)$ is a completely simple semigroup.*

Proof. By Lemma 4, R is simple. By Lemma 5, any element of $R \in \mathbf{V}$ has a finite order. Hence, R contains idempotents.

If R is not completely simple, it contains the bicyclic semigroup B (Proposition 1), and therefore $B \in \mathbf{V}$. Since B is not equationally Noetherian (Proposition 2), we obtain a contradiction with the choice of $\mathbf{V}$. □

Lemma 6 provides that $Red(S)$ has the form $(\Lambda, G, I, \mathbf{P})$.

Lemma 7. *For a variety $\mathbf{V}$ there exists a number $n \geq 1$ such that any group $G \in \mathbf{V}$ is abelian and of the period n (i.e. $g^n = 1$ for all $g \in G$).*

Proof. By Lemma 2, any group G is abelian. Assume that for any number n there exists a group $G \in \mathbf{V}$ and $g \in G$ with $g^n \neq 1$.

Let $\mathbf{G} \subseteq \mathbf{V}$ be all groups from the variety $\mathbf{V}$. The class $\mathbf{G}$ is not empty, since any semigroup from $\mathbf{V}$ contains a subgroup (Lemma 6).

Let us consider the semigroup variety $\mathrm{var}(\mathbf{G})$ and take an arbitrary identity $\tau(X) = \sigma(X)$ which holds in $\mathrm{var}(\mathbf{G})$.

First, we prove that the identity $\tau(X) = \sigma(X)$ is *balanced*, i.e. for any letter $x_i \in X$ the number of occurrences of x_i in $\tau(X)$ coincides with the number of occurrences in $\sigma(X)$.

Assume the converse: the number of occurrences of x_i in $\tau(X)$ (respectively, $\sigma(X)$) equals k (respectively, l) and $k \neq l$. If we assign $x_j = 1$ for all $j \neq i$ we obtain the identity $x_i^k = x_i^l$. Equivalently, the identity $x_i^{k-l} = 1$ holds for any group $G \in \mathrm{var}(\mathbf{G})$. It follows that any group from $\mathbf{V}$ has the period $k - l$.

Thus, all identities of $\mathrm{var}(\mathbf{G})$ are balanced. Hence, the infinite cyclic group Z belongs to this variety. The last fact contradicts Lemma 5, since all nontrivial elements of Z have infinite order. □

Lemma 8. *Let $S \in \mathbf{V}$ then $Red(S)$ is isomorphic to $(\Lambda, G, I, \mathbf{P})$ with $p_{i\lambda} = 1$ for all $i \in I$, $\lambda \in \Lambda$.*

Proof. Recall that the sandwich-matrix $\mathbf{P}$ is $(1,1)$-normalized. Let us consider an equation:

$$x(\lambda, 1, 1) = x(1, 1, 1).$$

Obviously, this equation satisfies the point $x = (1, 1, 1)$. According to Lemma 1, this equation should be satisfied by any $x \in S$. In particular, we have

$$(1, p_{i\lambda}, 1) = (1, 1, i)(\lambda, 1, 1) = (1, 1, i)(1, 1, 1) = (1, 1, 1).$$

Thus, $p_{i\lambda} = 1$. ∎

Lemma 9. *Let $S \in \mathbf{V}$ and $R = Red(S) = (\Lambda, G, I)$. Then for the any $a \in S$ there exist $g_a \in G$, $\lambda_a \in \Lambda$, $i_a \in I$ such that the multiplication in S is defined as follows:*

$$a(\mu, g, i) = (\lambda_a, g_a g, i),\ (\mu, g, i)a = (\mu, g g_a, i_a) \tag{1}$$

for all $\mu \in \Lambda$, $g \in G$, $i \in I$.

Proof. If $a = (\lambda_a, g_a, i_a) \in R$, we immediately obtain (1). Assume below $a \in S \setminus R$.

Since $a(1, 1, 1) \in R$, there exists λ_a, g_a, i'_a such that $a(1, 1, 1) = (\lambda_a, g_a, i'_a)$. We have:

$$(\lambda_a, g_a, 1) = (\lambda_a, g_a, i'_a)(1, 1, 1) = a(1, 1, 1)(1, 1, 1) = a(1, 1, 1) = (\lambda_a, g_a, i'_a),$$

hence $i'_a = 1$.

Since the equation $x(\mu, 1, 1) = x(1, 1, 1)$ has a solution $(1, 1, 1)$, Lemma 1 provides

$$a(\mu, 1, 1) = a(1, 1, 1) = (\lambda_a, g_a, 1). \tag{2}$$

Therefore,

$$a(\mu, g, i) = a(\mu, 1, 1)(1, g, i) = (\lambda_a, g_a, 1)(1, g, i) = (\lambda_a, g_a g, i)$$

for all $\mu \in \Lambda$, $g \in G$, $i \in I$. Similarly, one can prove

$$(\mu, g, i)a = (\mu, g g'_a, i_a)$$

for all μ, g, i. Let us prove $g'_a = g_a$. Namely,

$$(1, g'_a, 1) = (1, g'_a, i_a)(1, 1, 1) = (1, 1, 1)a(1, 1, 1) = (1, 1, 1)(\lambda_a, g_a, 1) = (1, g_a, 1).$$

Thus, we proved (1). ∎

Lemma 10. *Let $S \in \mathbf{V}$, and $Red(S) = (\Lambda, G, I)$, then*

$$ab = (\lambda_a, g_a g_b, i_b) \tag{3}$$

for any $a, b \in S$.

Proof. The product ab belongs to $R = Red(S)$, so $ab = (\mu, g, i)$ for some $\mu \in \Lambda$, $g \in G$, $i \in I$.

Using (1), we have

$$(1, g_a g_b, i_b) = (1, g_a, i_a)b = (1, 1, 1)ab = (1, 1, 1)(\mu, g, i) = (1, g, i),$$

hence $g = g_a g_b$, $i = i_b$. On the other hand,

$$(\mu, g_a g_b, 1) = (\mu, g_a g_b, i_b)(1, 1, 1) = ab(1, 1, 1) = a(\lambda_b, g_b, 1) = (\lambda_a, g_a g_b, 1),$$

Thus, $\mu = \lambda_a$, and (3) holds. □

Lemma 11. *Any $S \in \mathbf{V}$ satisfies*

$$aeb = afb \text{ for all } a, b \in S,\ e, f \in E(S). \tag{4}$$

Proof. Since any idempotent belongs to $Red(S)$, and any entry of the sandwich-matrix of $Red(S)$ equals 1 (Lemma 8), arbitrary idempotents of S have the form: $e = (\lambda_e, 1, i_e)$, $f = (\lambda_f, 1, i_f)$ for some $\lambda_e, \lambda_f \in \Lambda$, $i_e, i_f \in I$. Using (3), we have

$$aeb = (\lambda_a, g_a, i_e)b = (\lambda_a, g_a g_b, i_b) = (\lambda_a, g_a, i_f)b = afb.$$

□

Lemma 12. *Suppose a semigroup S satisfies (4), and $Red(S) = (\Lambda, G, I, \mathbf{P})$ is completely simple. Then*

(1) *any entry of P equals* 1*,*
(2) *for any $a, b \in S$ the equality (3) holds.*

Proof. Let $e, f \in E(S) \subseteq Red(S)$. The equality (4) gives $fef = fff = f$. Let $f = (1, 1, 1)$ and e has the indexes $\lambda \in \Lambda$, $i \in I$. According to the multiplication if $Red(S)$, e is the triple $e = (\lambda, p_{i\lambda}^{-1}, i)$. The equality $fef = f$ gives

$$(1, 1, 1)(\lambda, p_{i\lambda}^{-1}, i)(1, 1, 1) = (1, 1, 1) \Rightarrow (1, p_{i\lambda}^{-1}, 1) = (1, 1, 1) \Rightarrow p_{i\lambda} = 1.$$

Let us prove (3). If $e = (\lambda, 1, i)$, $f = (\mu, 1, j)$, $b = (1, 1, 1)$ then (4) gives

$$a(\lambda, 1, 1) = a(\mu, 1, 1).$$

Hence, the exists λ_a, g_a, i'_a with $a(\lambda, 1, 1) = (\lambda_a, g_a, i'_a)$ for any $\lambda \in \Lambda$. The further proof coincides with Lemmas 9 and 10. □

Suppose a semigroup satisfies (3) and $Red(S)$ is completely simple. One can define a mapping:

$$\phi\colon S \to Red(S),\ \phi(s) = (\lambda_s, g_s, i_s).$$

Moreover, the equality (3) implies that ϕ is a homomorphism of semigroups.

Let $P = (p_1, \dots, p_n) \in S^n$, then $\phi(P)$ denotes the point $(\phi(p_1), \dots, \phi(p_n)) \in Red(S)^n$.

Let $\tau(X) = u_1 \dots u_m$ be an $\mathcal{L}(S)$-term with literals $u_i \in S \sqcup X$. One can define the $\mathcal{L}(R)$-term $\tau_\phi(X) = (u_1)_\phi \dots (u_m)_\phi$ as follows:

$$(u_i)_\phi = \begin{cases} \phi(u_i) \text{ if } u_i \in S, \\ u_i \text{ if } u_i \in X \end{cases}$$

For example, if $\tau(X) = x_1 a x_2 b$ then $\tau_\phi(X) = x_1(\lambda_a, g_a, i_a)x_2(\lambda_b, g_b, i_b)$.

Lemma 13. *Let $\tau(X)$ be an $\mathcal{L}(S)$-term with $|\tau| \geq 2$ and a semigroup S satisfies (3). Then for any $P \in S^n$ we have the equalities:*

$$\tau(P) = \tau(\phi(P)) = \tau_\phi(P) = \tau_\phi(\phi(P)). \tag{5}$$

Proof. It is sufficient to prove the equalities for any term of length 2 (the proof for longer terms can be done by the induction). Thus, we have the following types of $\mathcal{L}(S)$-terms of length 2 (here $x, y \in X$, $s \in S$):

(1) $\tau(X) = xy$,
(2) $\tau(X) = xs$,
(3) $\tau(X) = sx$.

Let us consider the first case. For a point $P = (a, b)$ we have $\tau_\phi(X) = xy$, $\phi(P) = ((\lambda_a, g_a, i_a), (\lambda_b, g_b, i_b))$. One can directly prove that

$$\tau(P) = \tau(\phi(P)) = \tau_\phi(P) = \tau_\phi(\phi(P)) = (\lambda_a, g_a g_b, i_b),$$

and (5) holds.

For the second case we have (here $P = a$): $\phi(P) = (\lambda_a, g_a, i_a)$, $\tau_\phi(X) = x\sigma_\phi = x(\lambda_s, g_s, i_s)$, and

$$\tau(P) = \tau(\phi(P)) = \tau_\phi(P) = \tau_\phi(\phi(P)) = (\lambda_a, g_a g_s, i_s).$$

The third case is similar to the previous one. □

In [4] it was proved the following criterion of the Noetherian property for completely simple semigroups.

Proposition 3. ([4], Theorem 8) *Let $S = (\Lambda, G, I, \mathbf{P})$ be a completely simple semigroup. The semigroup is S-equationally Noetherian iff G is G-equationally Noetherian.*

In particular, if G is abelian, any semigroup $S = (\Lambda, G, I, \mathbf{P})$ is S-equationally Noetherian.

Theorem 2. *A variety $\mathbf{V}$ completely consists of S-equationally Noetherian semigroups iff there exists a number n such that every $S \in \mathbf{V}$ satisfies the following conditions:*

(1) *the set of reducible elements $R = Red(S)$ is isomorphic to a completely simple semigroup $(\Lambda, G, I, \mathbf{P})$;*
(2) *the group G is abelian and of the period n;*
(3) *S satisfies (4).*

Proof. The "only if" part of the theorem directly follows from Lemmas 6, 7, 11.

Let us prove the "if" part of the theorem. By Lemma 12, we obtain that any entry of the sandwich-matrix $\mathbf{P}$ equals 1 and (3) holds.

Let us consider a system of $\mathcal{L}(S)$-equations $\mathbf{S} = \{t_j(X) = s_j(X) \mid j \in J\}$ over a semigroup S in variables $X = \{x_1, \dots, x_n\}$.

Applying the map $\phi\colon S \to R$, we obtain the system $\phi(\mathbf{S}) = \{(t_j)_\phi(X) = (s_j)_\phi(X) \mid j \in J\}$ of $\mathcal{L}(R)$-equations. By Proposition 3, the system $\phi(\mathbf{S})$ is equivalent to a finite subsystem $\mathbf{S}_\phi \subseteq \phi(\mathbf{S})$ over the semigroup R.

For every equation from $\mathbf{S}_\phi$ we choose a pre-image in $\mathbf{S}$ to collect a finite subsystem $\mathbf{S}' \subseteq \mathbf{S}$. Let

$$\mathbf{S}^1 = \{\tau(X) = \sigma(X) \mid |\tau(X)| = 1 \text{ or } |\sigma(X)| = 1\} \subseteq \mathbf{S}.$$

Obviously, $\mathbf{S}^1$ is a union

$$\mathbf{S}^1 = \mathbf{S}^{11} \cup \bigcup_{i=1}^{n} \mathbf{S}^1_i,$$

where

(1) $\mathbf{S}^{11} \subseteq \mathbf{S}$ is the set of all equations $\pi(X) = \rho(X) \in \mathbf{S}$ with $|\pi| = |\rho| = 1$;
(2) $\mathbf{S}^1_i \subseteq \mathbf{S}$ is the set of all equations $\pi(X) = \rho(X) \in \mathbf{S} \setminus \mathbf{S}^{11}$ such that either $\pi(X)$ or $\rho(X)$ equals $x_i \in X$.

Let us choose an arbitrary equation $\mathbf{Eq}_i$ from each system $\mathbf{S}^1_i$ and denote

$$\mathbf{S}'' = \mathbf{S}' \cup \bigcup_i \mathbf{Eq}_i \cup \mathbf{S}^{11} \subseteq \mathbf{S}.$$

Let us prove that the finite subsystem $\mathbf{S}''$ is equivalent to $\mathbf{S}$ over S.

Assume there exists a point $P = (p_1, \dots, p_n)$ with $P \in \mathrm{V}_S(\mathbf{S}'') \setminus \mathrm{V}_S(\mathbf{S})$. In other words, there exists an equation $\tau(X) = \sigma(X) \in \mathbf{S} \setminus \mathbf{S}''$ and $\tau(P) \neq \sigma(P)$.

Since ϕ is a homomorphism, we have $\phi(P) \in \mathrm{V}_R(\phi(\mathbf{S}''))$. By the choice of the system $\mathbf{S}'$, we have $\phi(P) \in \mathrm{V}_R(\phi(\mathbf{S}))$, in particular

$$\phi(P) \in \mathrm{V}_R(\tau_\phi(X) = \sigma_\phi(X)). \tag{6}$$

By (5), $P \in \mathrm{V}_S(\tau(X) = \sigma(X))$ if $|\tau|, |\sigma| \geq 2$. Since $\tau(P) \neq \sigma(P)$, at least one length of terms τ, σ equals 1. The case $|\tau| = |\sigma| = 1$ is impossible, since the system $\mathbf{S}^{11}$ is included into $\mathbf{S}''$.

Thus, we may assume that $\tau(X) = x_i$ and $|\sigma| \geq 2$. The choice of the equation $x_i = \sigma(X)$ gives $p_i \neq \sigma(P)$. By (5), (6), $\phi(p_i) = \sigma_\phi(\phi(P)) = \sigma(P)$.

Hence, $p_i \neq \phi(p_i)$ and the definition of ϕ gives $p_i \in S \setminus R$. However, the system $\mathbf{S}''$ contains an equation of the form $x_i = \sigma(X)$, $|\sigma| \geq 2$ with $p_i = \sigma(P)$. Since $|\sigma| \geq 2$, we have $p_i = \sigma(P) \in R$, a contradiction. □

Example 1. *Let* $\mathbf{V}$ *be the variety of rectangular bands, i.e.* $\mathbf{V}$ *is defined by the identities* $xx = x$, $xyz = xz$. *It is well-known fact that any semigroup from* $\mathbf{V}$ *is isomorphic to* $(\{1\}, \Lambda, I)$ *for some index sets* Λ, I.

Let us check the conditions of Theorem 2 for $\mathbf{V}$*. Namely, for* $S \in \mathbf{V}$ *we have* $S = Red(S) = (\Lambda, \{1\}, I)$*, and* $n = 1$*. The property (4) follows from the identity* $xyz = xz$*.*

Thus, $\mathbf{V}$ *completely consists of equationally Noetherian semigroups.*

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