On the Shevlyakov's paper "Equationally Noetherian varieties of semigroups and B.Plotkin's problem"

The paper under consideration is devoted to the description of varieties consisting of semigroups satisfying the property to be equationally Noetherian. Prof. B. I. Plotkin mentions that "Notherianity plays a crucial role in many problems related to universal algebraic geometry". He prefers to use the term "geometrically Noetherian algebras" instead of "equationally Noetherian algebras" used by some other authors. These terms are synonyms.

Let $\mathbf{V}$ be a variety of algebras and $W(X)$ denote a free $\mathbf{V}$-algebra over a finite set X. A $\mathbf{V}$-algebra H is called geometrically (equationally) Noetherian if for every system of equations T in $W(X)$ there exists a finite subsystem $T_0 \subset T$ such that T and T_0 have the same set of solutions in H. If this property takes place for every $\mathbf{V}$-algebra the variety $\mathbf{V}$ is called geometrically (equationally) Noetherian. Below I use the term "equationally" following the author of the reviewed paper.

The author indicates that the problem of describing such varieties $\mathbf{V}$ of algebraic structures that every $A \in \mathbf{V}$ is equationally Noetherian was posed by B. Plotkin and he notes that this problem admits positive solutions in certain varieties. The author reformulates this problem in the following way: "Describe all such varieties $\mathbf{V}$ that each $\mathbf{V}$-algebra A is equationally Noetherian with respect to equations with constants that are elements of A". It is worth mentioning that B.I. Plotkin also considered equations with constants but in that case constants were elements of a fixed algebra included in the signature of the variety.

Thus the paper suggests a new variant of Notherianity, namely, an algebra S is called S-equationally Noetherian if S is equationally Noetherian in the language obtained from the language corresponding to the given variety by adding elements of S as constants. It is clear that a variety consisting of such algebras has to satisfy very strong restrictions. The author completely solves the problem formulated above for semigroup varieties. The main result is the following one which I give in a more convenient version to avoid any references to other parts of the paper.

Main Theorem. *A variety $\mathbf{V}$ completely consists of S-equationally Noetherian semigroups iff there exists a number n such that every $S \in \mathbf{V}$ satisfies the following conditions: 1. the set of reducible elements $R = SS$ is isomorphic to a completely simple semigroup (G, P, Λ, I); 2. the group G is abelian and of the period n; 3. $aeb = afb$ for all $a, b \in S$ and all idempotents $e, f \in S$.*

There is no doubt that this result is worthy of publication. However, there are some inaccuracies in the article that force me to refrain from the recommendation to publish the article in its present form.

The main of them one can find in the proof of Lemma 3.6. "Any element of a semigroup $S \in \mathbf{V}$ has finite order". This fact is very important for the main result and maybe it takes place. But the author obtains it from the statement that the two-element idempotent semigroup $L_2 = \{0, 1\}$ belongs to V since L_2

is a factor of $C_\infty \times C_\infty$, where C_∞ is the infinite cyclic semigroup generated by an element $s \in S$. To my mind L_2 cannot be a factor of $C_\infty \times C_\infty$ at all. If the author thinks otherwise he has to convince the reader in the truth of his statement.

The second confused statement is contained in the main theorem cited above. One can conclude that the number n is the same for all semigroups of variety $\mathbf{V}$ ("there exists a number n such that every $S \in \mathbf{V}$ satisfies the following conditions") but I haven't found any reason for such a conclusion.

I'll point out a few more places I have objections to.

1. The author cites Theorem 2.1. with a comment that it is the classic theorem which describes completely simple semigroups. Such a recommendation is not enough. In such cases, they usually point to a well-known book that contains this theorem.

2. The expression "(i.e. $V_S(S') = V_S(S)$)" must be in the next paragraph.

3. The last expression in the proof of Lemma must be "Thus, G^∞ is not G^∞-equationally Noetherian", that is, G^∞ instead of G.

4. We read "Proposition 3.7. ([1], Theorem 2.54)...". The reference [1] is "O. Andersen, Ein Bericht uber die Struktur abstrakter Halbgruppen, Thesis Staatsexamensarbeit, Hamburg (1952) ". But this work is not published. As for "Theorem 2.54", it is formulated and proved in the well-known book of Clifford and Preston "The algebraic theory of semigroups" with reference to the Andersen's article. The reference must be given to this book.

5. We read in the proof of Lemma 3.10 :"According to the theory of abelian groups, if var(G) does not satisfy any identity $x^n = 1$, it contains free abelian groups." What is the var(G) in the sentence quoted above and what is the var(G) in the sentence following it?

6.The author uses the same letter I for a set of indexes and for an ideal, and the same letter P for a sandwich matrix and for an element of S^n. It is not convenient for a reader.

7. Finally, I wondered whether there are varieties (other than abelian groups of course) described by the main theorem.

The problem raised by the author is of interest, the results are new and deserve publication, however, only after the author dispels the misunderstandings noted above.

Grigori Zhitomirski