Blow-up estimates for a reaction-diffusion equation with a special medium void and variable exponents

Tan Duc Do*, Nguyen Ngoc Trong, Bui Le Trong Thanh

Abstract

Let $d \in \{3, 4, 5, \ldots\}$ and $\Omega \subset \mathbb{R}^d$ be open bounded with Lipschitz boundary. Let $p, m \in C(\overline{\Omega})$ be such that

$$2 \le m^- \le m(\cdot) \le m^+ < p^- \le p(\cdot) \le p^+ < \infty,$$

where $p^- := \operatorname{ess\,inf}_{x\in\Omega} p(x)$ and $p^+ := \operatorname{ess\,sup}_{x\in\Omega} p(x)$. Suppose m satisfies a certain log-continuity condition. Consider the reaction-diffusion parabolic problem

$$(P) \quad \begin{cases} \dfrac{u_t}{|x|^2} + \Delta_{m(x)} u = k(t)\, |u|^{p(x)-2} u & (x,t) \in \Omega \times (0,T), \\ u(x,t) = 0, & (x,t) \in \partial\Omega \times (0,T), \\ u(x,0) = u_0(x), & x \in \Omega, \end{cases}$$

where $T > 0$, $0 \ne u_0 \in W_0^{1,m(\cdot)}(\Omega) \cap L^{p(\cdot)}(\Omega)$ and $\Delta_{m(x)} u = \operatorname{div}(|\nabla u|^{m(x)-2} \nabla u)$. We investigate the upper and lower bounds on the blow-up time of a weak solution to (P).

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Home institution:

1. Division of Applied Mathematics, Thu Dau Mot University, Binh Duong Province, Vietnam
   Email: doductan@tdmu.edu.vn

*. Corresponding author

2. Ho Chi Minh City University of Education, Ho Chi Minh City, Vietnam
   Email: trongnn@hcmue.edu.vn

3. Faculty of Mathematics and Computer Science, University of Science, Ho Chi Minh City, Vietnam
   Vietnam National University, Ho Chi Minh City, Vietnam
   Email: bltthanh@hcmus.edu.vn

1 Introduction

Differential equations with variable-exponent growth originates from the study of non-linear elasticity, rheological and electrorheological fluids [10, 1, 6, 17]. They also occur in image processing [15]. Therefore, a systematic study of such equations is of practical interest and potentializes their further applications in real-life situations. Recently one can observe an exciting movement in this direction, from which many results have been established. The development up to date has reached a state that core ideas are well-documented. We refer the readers to [1], [5] and the references therein for a general mathematical framework on spaces with variable exponents.

Meanwhile for various types of reaction-diffusion partial differential equations, blow-up behaviors of the solutions are common. This means that the solutions to these equations exist only in a finite time and their energy functionals blow up when the maximal existence time has been reached. Known methods for the investigation include the first eigenvalue method by Kaplan in 1963, the potential well method by Levine and Payne in 1970, the comparison method and other methods involving integration. A recent overview of the account can be found in the monograph [13]. Also confer the surveys [9] and [16] for the blow-up properties of more general evolution problems.

In this paper, we are motivated by the work [14] on a class of nonlinear heat equation with nonlinearities of variable-exponent type on the one hand and the works [19] and [11] on a reaction-diffusion equation with a special diffusion process on the other hand. It is our aim to extend the models in [19] and [11] to the setting of variable exponents.

Specifically, let $d \in \{3, 4, 5, \ldots\}$ and $\Omega \subset \mathbb{R}^d$ be open bounded with Lipschitz boundary. Let $p, m \in C(\overline{\Omega})$ be such that

$$2 \leq m^- \leq m(\cdot) \leq m^+ < p^- \leq p(\cdot) \leq p^+ < \infty, \tag{1}$$

where

$$p^- := \operatorname*{ess\,inf}_{x\in\Omega} p(x) \quad \text{and} \quad p^+ := \operatorname*{ess\,sup}_{x\in\Omega} p(x).$$

Assume further that m satisfies the so-called Zhikov-Fan condition that there exist constants $A > 0$ and $\delta \in (0, 1)$ such that

$$|m(x) - m(y)| \leq \frac{A}{\log\left(\frac{1}{|x-y|}\right)} \tag{2}$$

for all $x, y \in \Omega$ such that $|x - y| < \delta$.

Next let k be a function with the following properties:

$$k \in C^1[0, \infty), \quad k(0) > 0 \quad \text{and} \quad k'(t) \geq 0 \;\text{ for all } t \in [0, \infty). \tag{3}$$

In this paper, we consider the following reaction-diffusion parabolic problem:

$$(P) \quad \begin{cases} \dfrac{u_t}{|x|^2} + \Delta_{m(x)} u = k(t)\,|u|^{p(x)-2}u, & (x,t) \in \Omega \times (0,T), \\ u(x,t) = 0, & (x,t) \in \partial\Omega \times (0,T), \\ u(x,0) = u_0(x), & x \in \Omega, \end{cases}$$

where $T > 0$, $0 \neq u_0 \in W_0^{1,m(\cdot)}(\Omega) \cap L^{p(\cdot)}(\Omega)$ and

$$\Delta_{m(x)} u := \operatorname{div}(|\nabla u|^{m(x)-2} \nabla u).$$

When m and p are constants, our model reduces to that of [12] and [19]. Whereas, if $\frac{u_t}{|x|^2}$ is replaced by u_t and k is constantly 1, (P) restores the model investigated in [14]. For other models along our line, one may consult [3], [18], [7] and the references therein.

Our aim here is to provide upper and lower bounds on the blow-up time of a weak solution to (P) the precise definitions of which is given next.

Definition 1.1. Let $p, m \in C(\overline{\Omega})$ be given by (1) and $0 \neq u_0 \in W_0^{1,m(\cdot)}(\Omega) \cap L^{p(\cdot)}(\Omega)$. A function $u(x,t)$ is called a weak solution to (P) if $u \in L^2(0,T; W_0^{1,m(\cdot)}(\Omega) \cap L^{p(\cdot)}(\Omega))$ with $u(0) = u_0$,

$$\int_0^T \int_\Omega \frac{|u_t|^2}{|x|^2}\, dx\, dt < \infty$$

and $u(x,t)$ satisfies

$$\left(\frac{u_t}{|x|^2}, \varphi \right) + (|\nabla u|^{m(x)-2} \nabla u, \nabla \varphi) = k(t) \left(|u|^{p(x)-1} u, \varphi \right) \tag{4}$$

for all $\varphi \in W_0^{1,m(\cdot)}(\Omega) \cap L^{p(\cdot)}(\Omega)$ and $t \in [0,T)$.

Definition 1.2. Let $u(t)$ be a weak solution to (P). Then $u(t)$ is said to blow up at a finite time T^* if $u(t)$ exists for all $t \in [0, T^*)$ and

$$\lim_{t \to T^*} \left\| \frac{u(t)}{|x|} \right\|^2_{L^2(\Omega)} = \infty. \tag{5}$$

Such a T^* is called the maximal existence time for $u(t)$. If (5) does not happen for any finite time T^*, then $u(t)$ is called a global solution and the maximal existence time of $u(t)$ is ∞.

Under the conditions (1) and (2) on the exponents p and m as well as (3) on the function k, the existence of a local weak solution follows from the standard ODE theory plus the technique of Faedo-Garlekin approximation, as is well-known in the literature. Also confer [14, Theorem 3.1] and [7, Theorem 3.1].

Our first result concerns an upper bound on the blow-up time for a weak solution to (P) when the initial energy functional is negative.

Theorem 1.3. *Let $d \in \{3, 4, 5, \ldots\}$ and $\Omega \subset \mathbb{R}^d$ be open bounded with Lipschitz boundary. Let $p, m \in C(\overline{\Omega})$ satisfy (1) and (2). Let k satisfy (3). Suppose further that*

$$J(u_0, 0) := \int_\Omega \frac{1}{m(x)} |\nabla u_0(x)|^{m(x)}\, dx - k(t) \int_\Omega \frac{1}{p(x)} |u_0(x)|^{p(x)}\, dx < 0.$$

Let u be a weak solution to (P) with $T > 0$. Then u blows up at a finite time T^* satisfying

$$T^* \leq \frac{\left\| \frac{u_0}{|x|} \right\|^2_{L^2(\Omega)}}{p^- \,(2 - p^-)\, J(u_0, 0)}.$$

The next result provides an upper bound on the blow-up time for a weak solution to (P) when the initial energy functional is positive.

Theorem 1.4. *Let $d \in \{3, 4, 5, \ldots\}$ and $\Omega \subset \mathbb{R}^d$ be open bounded with Lipschitz boundary. Let $p, m \in C(\overline{\Omega})$ satisfy* (1) *and* (2)*. Let k satisfy* (3)*. Suppose further that*

$$0 \leq C_1\, J(u_0, 0) + C_2 < \frac{1}{2} \left\| \frac{u_0}{|x|} \right\|^2_{L^2(\Omega)} =: L(0),$$

where

$$C_1 = \frac{m^+ p^+\, H_d}{(p^- - m^+)m^-}, \quad C_2 = H_d \left(1 - \frac{1}{m^-} - \frac{1}{m^+}\right) |\Omega| \quad \text{and} \quad H_d = \frac{4}{(d-2)^2}.$$

Let u be a weak solution to (P) with $T > 0$. Then u blows up at a finite time T^* satisfying

$$T^* \leq \frac{4p^+\, C_1\, L(0)}{(p^+ - 2)^2\, p^+\, (L(0) - C_1\, J(u_0, 0) - C_2)}.$$

Lastly we present a lower bound on the blow-up time.

Theorem 1.5. *Let $d \in \{3, 4, 5, \ldots\}$ and $\Omega \subset \mathbb{R}^d$ be open bounded with Lipschitz boundary. Let $p, m \in C(\overline{\Omega})$ satisfy* (1) *and* (2)*. Let k satisfy* (3)*. Suppose in addition that*

$$m^+ < d, \quad p^+ < \frac{2\, m^-}{d - m^-} \quad \text{and} \quad k_\infty := \lim_{t \to \infty} k(t) < \infty.$$

Let $u(t)$ be a weak solution to (P) with $T > 0$. Suppose u blows up at a time T^*. Then there exist $t_0 \in [0, T^*)$ and $C^* = C^*(\Omega, d, p^\pm, m^\pm, k_\infty)$ such that

$$T^* \geq t_0 + \frac{1}{C^*} \int_{L(t_0)}^\infty \frac{ds}{s^{\gamma^+} + s^{\gamma^-}}.$$

The paper is outlined as follows. In Section 2 we provide a brief summary of function spaces with variable exponents as well as collect fundamental estimates for later use. The upper and lower bounds on the blow-up time are investigated in Sections 3 and 4 respectively.

2 Preliminaries

In this section we discuss appropriate function spaces for our setting and some preliminary estimates to be used in the proof of the main results. We assume throughout that $\Omega \subset \mathbb{R}^d$ is open bounded with Lipschitz boundary.

2.1 Function spaces with variable exponents

For the sake of clarity, we provide the definitions for variable exponent Lebesgue and Sobolev spaces as well as the log-continuity condition in the sense of [5, Definition 4.1.1].

Definition 2.1. Let $s \in \mathcal{P}(\Omega)$ in the sense that $s : \Omega \longrightarrow [1, \infty]$ is measurable. The *variable exponent Lebesgue space* $L^{s(\cdot)}(\Omega)$ is defined to consist of all measurable functions $u : \Omega \longrightarrow \mathbb{R}$ such that

$$\varrho_{L^{s(\cdot)}(\Omega)}(u) := \int_\Omega |u(x)|^{s(x)}\, dx < \infty.$$

We endow $L^{s(\cdot)}(\Omega)$ with the Luxemburg norm

$$\|u\|_{L^{s(\cdot)}(\Omega)} := \inf\left\{\lambda > 0 : \varrho_{L^{s(\cdot)}(\Omega)}\left(\frac{u}{\lambda}\right) \leq 1\right\}.$$

Definition 2.2. Let $s \in \mathcal{P}(\Omega)$. The *variable exponent Sobolev space* $W^{1,s(\cdot)}(\Omega)$ is defined to consist of all $u \in L^{s(\cdot)}(\Omega)$ whose distributional derivative $\partial_j u \in L^{s(\cdot)}(\Omega)$ for all $j \in \{1, \ldots, d\}$.

The space $W^{1,s(\cdot)}(\Omega)$ is a Banach space under the norm

$$\|u\|_{W^{1,s(\cdot)}(\Omega)} := \|u\|_{L^{s(\cdot)}(\Omega)} + \sum_{j=1}^{d} \|\partial_j u\|_{L^{s(\cdot)}(\Omega)}.$$

The following smoothness condition on the exponent $s(\cdot)$ is well-known in the literature (cf. [5, Definition 4.1.1]).

Definition 2.3. We say that $\alpha : \Omega \longrightarrow \mathbb{R}$ is *locally log-Holder continuous* if there exists a $c_1 > 0$ such that

$$|\alpha(x) - \alpha(y)| \leq \frac{c_1}{\log(e + 1/|x - y|)}$$

for all $x, y \in \Omega$.

We say that $\alpha : \Omega \longrightarrow \mathbb{R}$ satisfies the *log-Holder decay condition* if there exist constants $\alpha_\infty \in \mathbb{R}$ and $c_2 > 0$ such that

$$|\alpha(x) - \alpha_\infty| \leq \frac{c_2}{\log(e + |x|)}$$

for all $x \in \Omega$.

We say that $\alpha : \Omega \longrightarrow \mathbb{R}$ is *globally log-Holder continuous* if it is locally log-Holder continuous and satisfies the log-Holder decay condition.

The class $\mathcal{P}^{\log}(\Omega)$ is defined to consist of all $s \in \mathcal{P}(\Omega)$ such that $\frac{1}{s}$ is globally log-Holder continuous.

It is well-known that $L^{s(\cdot)}(\Omega)$ and $W^{1,s(\cdot)}(\Omega)$ are Banach spaces. In the sequel we will implicitly make use of the following convenient facts. A thorough account can be found in [5].

(i) If $s \in \mathcal{P}(\Omega)$ with $s^+ < \infty$, then $C_c^\infty(\Omega)$ is dense in $L^{s(\cdot)}(\Omega)$.

(ii) If $s \in \mathcal{P}^{\log}(\Omega)$, then $C_c^\infty(\Omega)$ is dense in $W^{1,s(\cdot)}(\Omega)$.

(iii) Let $r, s \in \mathcal{P}(\Omega)$ be such that $r \geq s$. Define $w \in \mathcal{P}(\Omega)$ by

$$\frac{1}{w(\cdot)} = \frac{1}{s(\cdot)} - \frac{1}{r(\cdot)}.$$

Then $L^{r(\cdot)}(\Omega) \hookrightarrow L^{s(\cdot)}(\Omega)$ provided that $1 \in L^{w(\cdot)}(\Omega)$. The condition $1 \in L^{w(\cdot)}(\Omega)$ is automatic when $|\Omega| < \infty$ due to [5, Lemma 3.2.12].

(iv) If $s \in C(\overline{\Omega})$ and $r \in \mathcal{P}(\Omega)$ are such that $r^+ < \infty$ and

$$\operatorname*{ess\,inf}_{x \in \Omega} \big(s^*(x) - r(x)\big) > 0,$$

then

$$W^{1,s(\cdot)}(\Omega) \hookrightarrow\hookrightarrow L^{r(\cdot)}(\Omega).$$

Here

$$s^*(x) := \begin{cases} \frac{d\, s(x)}{d - s(x)} & \text{if } s(x) < d, \\ \infty & \text{otherwise.} \end{cases}$$

(Confer [8] and [5, Theorem 8.4.6].)

We also need the following result on zero-trace functions.

Lemma 2.4. *Let $s \in \mathcal{P}^{\log}(\Omega)$ and $u \in W_0^{1,s(\cdot)}(\Omega)$. Then u, $u\,\mathbb{1}_{[u \geq 1]}$, $u\,\mathbb{1}_{[u<1]}$ all belong to $W_0^{1,s^-}(\Omega)$.*

Proof. It suffices to show that $u \in W_0^{1,s^-}(\Omega)$. Clearly $u \in W^{1,s^-}(\Omega)$. By identifying u with $u\,\mathbb{1}_\Omega$, we also have $u \in W_0^{1,s^-}(\Omega)$ by [4, Lemma 9.5]. The rest is similar. □

Note that the assumptions on the exponents p and m in Theorems 1.3, 1.4 and 1.5 are sufficient for us to apply the results of this subsection in what follows. In particular, the conditions $p, m \in C(\overline{\Omega})$, (1) and (2) together imply $p \in \mathcal{P}(\Omega)$ and $m \in \mathcal{P}^{\log}(\Omega)$.

2.2 Fundamental inequalities

Next we present two crucial inequalities for a later development. Let us begin with the following Hardy inequality.

Lemma 2.5. *Let $d \geq 3$ and $u \in H_0^1(\Omega)$. Then $\dfrac{u}{|x|} \in L^2(\Omega)$ and*

$$\int_\Omega \frac{|u|^2}{|x|^2}\, dx \leq \frac{4}{(d-2)^2} \int_\Omega |\nabla u|^2\, dx =: H_d \int_\Omega |\nabla u|^2\, dx.$$

Proof. This follows at once from [2, Theorem 4.2.2]. □

The next result is the well-known Gagliardo-Nirenberg inequality.

Lemma 2.6. *Let $r \in [2, \infty)$, $d > r$ and $r < q < \left(\frac{1}{r} - \frac{1}{d}\right)^{-1}$. Then there exists a $N = N(\Omega, d, q, r) > 0$ such that*

$$\|u\|_{L^q(\Omega)}^q \leq N\, \|\nabla u\|_{L^r(\Omega)}^{\alpha q}\, \|u\|_{L^2(\Omega)}^{(1-\alpha)q}$$

for all $u \in W_0^{1,r}(\Omega)$, where

$$\alpha = \left(\frac{1}{2} - \frac{1}{q}\right)\left(\frac{1}{2} + \frac{1}{d} - \frac{1}{r}\right)^{-1} \in (0,1). \tag{6}$$

2.3 Energy estimates

Let $p, m \in C(\overline{\Omega})$ satisfy (1) and (2). Let k satisfy (3). For each $u \in W_0^{1,m(\cdot)}(\Omega) \cap L^{p(\cdot)}(\Omega)$ and $t \in [0,T)$ define the following:

- Energy functional:

$$J(u,t) = \int_\Omega \frac{1}{m(x)} |\nabla u(x)|^{m(x)}\, dx - k(t) \int_\Omega \frac{1}{p(x)} |u|^{p(x)}\, dx;$$

- Nehari functional:

$$I(u,t) = \int_\Omega |\nabla u(x)|^{m(x)}\, dx - k(t) \int_\Omega |u(x)|^{p(x)}\, dx.$$

The roles of the energy and Nehari functionals are fundamental to our analysis. The following identities hold for them.

Lemma 2.7. *Let u be a weak solution to (P). Then the following identities hold.*

(i) *For a.e. $t_0 \in [0,T)$ one has*

$$J(u(t_0),t_0) + \int_0^{t_0} \left(\left\| \frac{u_t(s)}{|x|} \right\|^2_{L^2(\Omega)} + k'(s) \int_\Omega \frac{1}{p(x)} |u(x,s)|^{p(x)}\, dx \right) ds = J(u_0, 0).$$

(ii) *For a.e. $t_0 \in [0,T)$ one has*

$$\frac{d}{dt}\left(\frac{1}{2} \left\| \frac{u(t_0)}{|x|} \right\|_2^2 \right) = \left(\frac{u(t_0)}{|x|^2}, u_t(t_0) \right) = -I(u(t_0),t_0).$$

Proof. Regarding (i), first suppose that $u_t \in L^2(0,T; W_0^{1,m(\cdot)}(\Omega))$. Then by using u_t as a test function in (4) we obtain

$$\left\| \frac{u_t}{|x|} \right\|^2_{L^2(\Omega)} + (|\nabla u|^{m(x)-2} \nabla u, \nabla u_t) = k(t)\, (|u|^{p(x)-2}\, u, u_t).$$

On the other hand, direct calculations give

$$\begin{aligned} \frac{d}{dt} J(u(\tau),\tau) &= \left(|\nabla u(\tau)|^{m(x)-2} \nabla u(\tau), \nabla u_t(\tau)\right) - k(\tau)\, (|u(\tau)|^{p(x)-2}\, u(\tau), u_t(\tau)) \\ &\quad - k'(t) \int_\Omega \frac{1}{p(x)} |u(x,t)|^{p(x)}\, dx \end{aligned}$$

for each $\tau \in (0,T)$. Combining these two identities together yields that

$$\frac{d}{dt} J(u(\tau),\tau) = - \left\| \frac{u_t(\tau)}{|x|} \right\|^2_{L^2(\Omega)} - k'(\tau) \int_\Omega \frac{1}{p(x)} |u(x,t)|^{p(x)}\, dx \tag{7}$$

for each $\tau \in (0,T)$. Now (i) follows by integrating both sides of (7) with respect to t over $(0,t_0)$, where $t_0 \in (0,T)$.

To finish, we observe that (7) holds without the assumption that $u_t \in L^2(0,T; W_0^{1,m(\cdot)}(\Omega))$ by an approximation argument.

The proof of (ii) follows the same line and hence is omitted. □

The next concavity argument is classic and is used extensively in the literature for a sufficient condition of blow-up time.

Lemma 2.8 ([16])**.** *Let $\theta > 0$. Let $\psi \geq 0$ be weakly twice-differentiable on $(0, \infty)$ such that $\psi(0) > 0$, $\psi'(0) > 0$ and*

$$\psi''(t)\,\psi(t) - (1+\theta)\,(\psi'(t))^2 \geq 0$$

for all $t \in (0, \infty)$. Then there exists a $T > 0$ such that

$$\lim_{t \to T^-} \psi(t) = \infty$$

and

$$T \leq \frac{\psi(0)}{\theta\,\psi'(0)}.$$

3 Upper bound for blow-up time

In this section we work with the upper bounds for the blow-up time. These are the contents of Theorems 1.3 and 1.4. To this end, it is convenient to denote

$$L(t) = \frac{1}{2} \left\| \frac{u(t)}{|x|} \right\|^2_{L^2(\Omega)}$$

for each $t \in [0, T)$.

We start with the proof of Theorem 1.3 which deals with the case of negative initial energy functional.

Proof of Theorem 1.3. Let $T^* \geq 0$ be the maximal existence time of u. We aim to show that $T^* < \infty$ and then to provide an upper bound for T^*.

For this purpose, set

$$K(t) = -J(u(t), t)$$

for each $t \in [0, T^*)$. By hypothesis $L(0) > 0$ and $K(0) > 0$.

Also Lemma 2.7 gives

$$K'(t) = -\frac{d}{dt} J(u(t), t) = \left\| \frac{u_t(t)}{|x|} \right\|^2_{L^2(\Omega)} + k'(t) \int_\Omega \frac{1}{p(x)} \, |u(x,t)|^{p(x)} \, dx \geq 0 \tag{8}$$

for each $t \in [0, T^*)$, whence K is increasing on $[0, T^*)$. Consequently, $K(t) \geq K(0) > 0$ for all $t \in [0, T^*)$.

Let $t \in [0, T^*)$. By the same token,

$$\begin{aligned}
L'(t) &= \left(\frac{u(t)}{|x|^2}, u_t(t) \right) = -I(u(t), t) \\
&= \int_\Omega \frac{p(x) - m(x)}{m(x)} \, |\nabla u(x,t)|^{m(x)} \, dx \\
&\quad - \int_\Omega p(x) \left(\frac{1}{m(x)} \, |\nabla u(x,t)|^{m(x)} - k(t) \int_\Omega \frac{1}{p(x)} \, |u(x,t)|^{p(x)} \right) dx \\
&\geq -p^- \, J(u(t), t) = p^- \, K(t).
\end{aligned} \tag{9}$$

Therefore,

$$L(t)\,K'(t) \geq \frac{1}{2}\left\|\frac{u(t)}{|x|}\right\|_2^2 \left\|\frac{u_t(t)}{|x|}\right\|_2^2 \geq \frac{1}{2}\left(\frac{u(t)}{|x|^2}, u_t(t)\right)^2 = \frac{1}{2}\,(L'(t))^2$$
$$\geq \frac{p^-}{2}\,L'(t)\,K(t).$$

With the above in mind, one has

$$\left(K(t)\,L^{-p^-/2}(t)\right)' = L^{-(p^-+2)/2}\left(K'(t)\,L(t) - \frac{p^-}{2}\,K(t)\,L'(t)\right) \geq 0.$$

This implies $K\,L^{-p^-/2}$ is strictly increasing on $[0, T^*)$, from which it follows that

$$0 < \xi_0 := K(0)\,L^{-p^-/2}(0) < K(t)\,L^{-p^-/2}(t)$$
$$\leq \frac{1}{p^-}\,L'(t)\,L^{-p^-/2}(t) = \frac{2}{p^-\,(2-p^-)}\,\left(L^{(2-p^-)/2}(t)\right)',$$

where we used (9) in the second-to-last step. By integrating this last display with respect to t over $(0, \tau)$, where $\tau \in [0, T^*)$, we arrive at

$$\xi_0\,\tau \leq \frac{2}{p^-\,(2-p^-)}\,\left[L^{(2-p^-)/2}(\tau) - L^{(2-p^-)/2}(0)\right].$$

From this we deduce that $T^* < \infty$ since this inequality holds for a finite time only. Moreover,

$$0 \leq L^{(2-p^-)/2}(\tau) \leq L^{(2-p^-)/2}(0) + \frac{p^-\,(2-p^-)}{2}\,\xi_0\,\tau$$

for all $\tau \in [0, T^*)$. This in turn yields that

$$T^* \leq -\frac{2}{p^-\,(2-p^-)\,\xi_0}\,L^{(2-p^-)/2}(0) = \frac{2\,L(0)}{p^-\,(2-p^-)\,J(u_0,0)}.$$

This completes the proof. □

Next we prove Theorem 1.4 which deals with the case of positive initial energy functional.

Proof of Theorem 1.4. Let $T^* \geq 0$ be the maximal existence time of u. We aim to show that $T^* < \infty$ and then to provide an upper bound for T^*.

To begin with, for each $t \in [0, T^*)$ observe that

$$\int_\Omega |\nabla u(x,t)|^{m(x)}\,dx = \int_{[\nabla u(t) \geq 1]} |\nabla u(x,t)|^{m(x)}\,dx + \int_{[\nabla u(t) < 1]} |\nabla u(x,t)|^{m(x)}\,dx$$
$$\geq \int_{[\nabla u(t) \geq 1]} |\nabla u(x,t)|^{m^-}\,dx + \int_{[\nabla u(t) < 1]} |\nabla u(x,t)|^{m^+}\,dx$$

and

$$\|\nabla u(t)\|^2_{L^2([\nabla u(t)\geq 1])} \leq \left(\int_{[\nabla u(t)\geq 1]} |\nabla u(x,t)|^{m^-}\, dx\right)^{\frac{2}{m^-}} \left|[\nabla u(t)\geq 1]\right|^{1-\frac{2}{m^-}}$$
$$\leq \frac{2}{m^-}\left(\int_{[\nabla u(t)\geq 1]} |\nabla u(x,t)|^{m^-}\, dx\right) + \left(1-\frac{2}{m^-}\right)|\Omega|$$

by Holder and Young inequalities, where

$$[\nabla u(t)\geq 1] := \{x\in\Omega : \nabla u(x,t)\geq 1\}\,.$$

Similarly,

$$\|\nabla u(t)\|^2_{L^2([\nabla u(t)< 1])} \leq \frac{2}{m^+}\left(\int_{[\nabla u(t)< 1]} |\nabla u(x,t)|^{m^+}\, dx\right) + \left(1-\frac{2}{m^+}\right)|\Omega|$$

for all $t\in[0,T^*)$.

As a consequence, we obtain

$$\begin{aligned}\int_\Omega |\nabla u(x,t)|^{m(x)}\, dx &\geq \frac{m^-}{2}\left[\|\nabla u(t)\|^2_{L^2([\nabla u(t)\geq 1])} - \left(1-\frac{2}{m^-}\right)|\Omega|\right] \\ &\quad + \frac{m^+}{2}\left[\|\nabla u(t)\|^2_{L^2([\nabla u(t)< 1])} - \left(1-\frac{2}{m^+}\right)|\Omega|\right] \\ &\geq \frac{m^-}{2}\left[\|\nabla u(t)\|^2_{L^2(\Omega)} - 2\left(1-\frac{1}{m^-}-\frac{1}{m^+}\right)|\Omega|\right] \\ &\geq \frac{m^-}{2}\left[\frac{1}{H_d}\left\|\frac{u(t)}{|x|}\right\|^2_{L^2(\Omega)} - 2\left(1-\frac{1}{m^-}-\frac{1}{m^+}\right)|\Omega|\right]\end{aligned}$$

for all $t\in[0,T^*)$, where we used Lemma 2.5 in the last step.

Combining this with (9) we derive

$$\begin{aligned}L'(t) &= \int_\Omega \left(\frac{p(x)}{m(x)}-1\right)|\nabla u(x,t)|^{m(x)}\, dx \\ &\quad - \int_\Omega p(x)\left(\frac{1}{m(x)}|\nabla u(x,t)|^{m(x)} - k(t)\int_\Omega \frac{1}{p(x)}|u(x,t)|^{p(x)}\right) dx \\ &\geq \left(\frac{p^-}{m^+}-1\right)\int_\Omega |\nabla u(x,t)|^{m(x)}\, dx - p^+\, J(u(t),t) \\ &\geq \left(\frac{p^-}{m^+}-1\right)\frac{m^-}{2}\left[\frac{1}{H_d}\left\|\frac{u(t)}{|x|}\right\|^2_{L^2(\Omega)} - 2\left(1-\frac{1}{m^-}-\frac{1}{m^+}\right)|\Omega|\right] - p^+\, J(u(t),t) \\ &= \left(\frac{p^-}{m^+}-1\right)\frac{m^-}{H_d}\left[L(t) - H_d\left(1-\frac{1}{m^-}-\frac{1}{m^+}\right)|\Omega| - \frac{m^+p^+\, H_d}{(p^--m^+)m^-}\, J(u(t),t)\right] \\ &= \left(\frac{p^-}{m^+}-1\right)\frac{m^-}{H_d}\left[L(t) - C_2 - C_1\, J(u(t),t)\right] =: \left(\frac{p^-}{m^+}-1\right)\frac{m^-}{H_d}\, M(t)\end{aligned}$$

for each $t \in (0, T^*)$.

With the above inequality in mind, observe that

$$M'(t) = L'(t) - C_1 \frac{d}{dt} J(u(t), t) \geq L'(t) \geq \left(\frac{p^-}{m^+} - 1\right) \frac{m^-}{H_d} M(t)$$

for each $t \in (0, T^*)$, where we used (8) in the second step. Furthermore,

$$M(0) = L(0) - C_1 \, J(u_0, 0) - C_2 > 0$$

by assumption. As a consequence, an application of Gronwall's inequality yields

$$M(t) \geq M(0) \, \exp\left(\left(\frac{p^-}{m^+} - 1\right) \frac{m^-}{H_d} \, t\right) > 0.$$

This in turn implies $L'(t) > 0$ for each $t \in (0, T^*)$. That is, L is strictly increasing on $[0, T^*)$ and hence

$$L(t) > L(0) \tag{10}$$

for each $t \in (0, T^*)$.

Next fix $\tau \in [0, T^*)$ as well as

$$\beta \in \left(0, \frac{p^+}{(p^+ - 1) \, C_1} \, M(0)\right) \quad \text{and} \quad \sigma \in \left(\frac{L(0)}{(p^+ - 2) \, \beta}, \infty\right). \tag{11}$$

The choices of β and σ are justified below by (13) and (14) respectively. Define the nonnegative functional

$$G(h) = \int_0^h L(s) \, ds + (\tau - h) \, L(0) + \beta \, (h + \sigma)^2,$$

where $h \in [0, \tau]$. Then

$$G'(h) = L(h) - L(0) + 2\beta \, (h + \sigma) = 2 \int_0^h \left(\frac{u(s)}{|x|^2}, u_t(s)\right) \, ds + 2\beta \, (h + \sigma)$$

and

$$\begin{aligned}
G''(h) &= 2 \left(\frac{u(h)}{|x|^2}, u_t(h)\right) + 2\beta = -2 \, I(u(h), h) + 2\beta \\
&\geq -2p^+ \, J(u(h), h) + 2 \left(\frac{p^-}{m^+} - 1\right) \left(\int_\Omega |\nabla u(x, h)|^{m(x)}\right) + 2\beta \\
&\geq -2p^+ \left[J(u_0, 0) - \int_0^h \left(\left\|\frac{u_t(s)}{|x|}\right\|_{L^2(\Omega)}^2 + k'(s) \int_\Omega \frac{1}{p(x)} \, |u(x, s)|^{p(x)} \, dx\right) \, ds\right] \\
&\quad + 2 \left(\frac{p^-}{m^+} - 1\right) \left(\int_\Omega |\nabla u(x, h)|^{m(x)} \, dx\right) + 2\beta
\end{aligned}$$

$$\geq -2p^+ \left[J(u_0,0) - \int_0^h \left(\left\| \frac{u_t(s)}{|x|} \right\|^2_{L^2(\Omega)} + k'(s) \int_\Omega \frac{1}{p(x)} |u(x,s)|^{p(x)} \, dx \right) ds \right]$$
$$+ 2 \left(\frac{p^-}{m^+} - 1 \right) \frac{m^-}{2} \left[\frac{1}{H_d} \left\| \frac{u(h)}{|x|} \right\|^2_{L^2(\Omega)} - 2 \left(1 - \frac{1}{m^-} - \frac{1}{m^+} \right) |\Omega| \right] + 2\beta$$
$$= -2p^+ \left[J(u_0,0) - \int_0^h \left(\left\| \frac{u_t(s)}{|x|} \right\|^2_{L^2(\Omega)} + k'(s) \int_\Omega \frac{1}{p(x)} |u(x,s)|^{p(x)} \, dx \right) ds \right]$$
$$+ 2 \left(\frac{p^-}{m^+} - 1 \right) \frac{m^-}{2} \left[\frac{2}{H_d} L(h) - 2 \left(1 - \frac{1}{m^-} - \frac{1}{m^+} \right) |\Omega| \right] + 2\beta \qquad (12)$$

for each $h \in [0, \tau]$, where we used Lemma 2.7 in the fourth step.

In what follows it is convenient to denote

$$\theta(h) = \left(2 \int_0^h L(s) \, ds + \beta \, (h + \sigma)^2 \right) \left(\int_0^h \left\| \frac{u_t(s)}{|x|} \right\|^2_{L^2(\Omega)} ds + \beta \right)$$
$$- \left(\int_0^h \left(\frac{u(s)}{|x|^2}, u_t(s) \right) ds + \beta \, (h + \sigma) \right)^2 \geq 0$$

for each $h \in [0, \tau]$, where we used Cauchy-Schwartz inequality to verify the last step.

In view of Lemma 2.8, consider

$$G(h) \, G''(h) - \frac{p+1}{2} \, (G'(h))^2$$
$$= G(h) \, G''(h) - 2p^+ \left[\int_0^h \left(\frac{u(s)}{|x|^2}, u_t(s) \right) ds + \beta \, (h + \sigma) \right]^2$$
$$= G(h) \, G''(h) + 2p^+ \left[\theta(h) - \big(G(h) - (\tau - h) \, L(0) \big) \left(\int_0^h \left\| \frac{u_t(s)}{|x|} \right\|^2_{L^2(\Omega)} ds + \beta \right) \right]$$
$$\geq G(h) \, G''(h) - 2p^+ \, G(h) \left(\int_0^h \left\| \frac{u_t(s)}{|x|} \right\|^2_{L^2(\Omega)} ds + \beta \right)$$
$$\geq G(h) \left[G''(h) - 2p^+ \left(\int_0^h \left\| \frac{u_t(s)}{|x|} \right\|^2_{L^2(\Omega)} ds + \beta \right) \right]$$
$$\geq G(h) \left[-2p^+ \, J(u_0,0) + m^- \left(\frac{p^-}{m^+} - 1 \right) \left(\frac{2}{H_d} \, L(h) - 2 \left(1 - \frac{1}{m^-} - \frac{1}{m^+} \right) |\Omega| \right) - 2(p^+ - 1)\beta \right]$$
$$\geq G(h) \left[-2p^+ \, J(u_0,0) + m^- \left(\frac{p^-}{m^+} - 1 \right) \left(\frac{2}{H_d} \, L(0) - 2 \left(1 - \frac{1}{m^-} - \frac{1}{m^+} \right) |\Omega| \right) - 2(p^+ - 1)\beta \right]$$
$$= 2p^+ \, G(h) \left[-J(u_0,0) + \frac{m^-}{p^+ \, H_d} \left(\frac{p^-}{m^+} - 1 \right) L(0) - \frac{m^-}{p^+} \left(\frac{p^-}{m^+} - 1 \right) \left(1 - \frac{1}{m^-} - \frac{1}{m^+} \right) |\Omega| \right.$$

$$
\left. - \frac{(p^+ - 1)\beta}{p^+} \right]
$$

$$
= 2p^+ \, G(h) \, \left[-J(u_0, 0) + \frac{1}{C_1} \, L(0) - \frac{C_2}{C_1} - \frac{(p^+ - 1)\beta}{p^+} \right] \geq 0 \tag{13}
$$

for all $h \in [0, \tau]$, where we used (12) and (10) in the fifth and sixth steps respectively.

Next observe that

$$
G(0) = \tau \, L(0) + \beta \, \sigma^2 > 0
$$

and

$$
G'(0) = 2\beta\sigma > 0.
$$

Consequently, Lemma 2.8 implies

$$
\tau \leq \frac{2G(0)}{(p^+ - 2) \, G'(0)} = \frac{2 \, (\tau \, L(0) + \beta \, \sigma^2)}{2(p^+ - 2) \, \beta\sigma} = \frac{L(0)}{(p^+ - 2) \, \beta\sigma} \, \tau + \frac{\sigma}{p^+ - 2}.
$$

This in turn yields

$$
\tau \, \left(1 - \frac{L(0)}{(p^+ - 2) \, \beta\sigma} \right) \leq \frac{\sigma}{p^+ - 2}
$$

or equivalently

$$
\tau \leq \frac{\sigma}{p^+ - 2} \, \left(1 - \frac{L(0)}{(p^+ - 2) \, \beta \, \sigma} \right)^{-1} = \frac{\beta \, \sigma^2}{(p^+ - 2)\beta\sigma - L(0)}. \tag{14}
$$

Minimizing this last display over the range of σ in (11) leads to

$$
\tau \leq \frac{4 \, L(0)}{(p^+ - 2)^2 \, \beta}. \tag{15}
$$

Then we minimize (15) over the the range of β in (11) to see that

$$
\tau \leq \frac{4p^+ \, C_1 \, L(0)}{(p^+ - 2)^2 \, p^+ \, M(0)}. \tag{16}
$$

Lastly, (16) holds for all $\tau \in (0, T^*)$, from we deduce that

$$
T^* \leq \frac{4p^+ \, C_1 \, L(0)}{(p^+ - 2)^2 \, p^+ \, M(0)}
$$

as required. □

4 Lower bound for blow-up time

In this section we provide a lower bound for the blow-up time, which is Theorem 1.5. Recall from Section 3 that we define

$$
L(t) = \frac{1}{2} \, \left\| \frac{u(t)}{|x|} \right\|_{L^2(\Omega)}^2
$$

for each $t \in [0, T)$.

Proof of Theorem 1.5. Recall from the hypothesis that T^* is the blow-up time of the solution u.

Observe that

$$\int_\Omega |u(x,h)|^{p(x)}\,dx = \int_\Omega |u(x,h)\,\mathbb{1}_{[u(h)\geq 1]}|^{p(x)}\,dx + \int_\Omega |\nabla u(x,h)\,\mathbb{1}_{[u(h)<1]}|^{p(x)}\,dx$$
$$\leq \int_\Omega |u(x,h)\,\mathbb{1}_{[u(h)\geq 1]}|^{p^+}\,dx + \int_\Omega |u(x,h)\mathbb{1}_{[u(h)<1]}|^{p^-}\,dx$$

for all $h \in [0,T^*)$, where

$$[u(h)\geq 1] := \{x\in\Omega : u(x,h)\geq 1\}.$$

Next

$$\int_{[\nabla u(h)<1]} |\nabla u(x,h)|^{m^-}\,dx \leq \big|[\nabla u(h)<1]\big|^{1-\frac{m^-}{m^+}} \left(\int_{[\nabla u(h)<1]} |\nabla u(x,h)|^{m^+}\,dx\right)^{\frac{m^-}{m^+}}$$
$$\leq S_0 + \int_{[\nabla u(h)<1]} |\nabla u(x,h)|^{m^+}\,dx$$

for all $h\in[0,T^*)$, where $S_0 = S_0(\Omega, m^\pm) > 0$ and we used Young"s inequality in the second step. This in turn implies

$$\int_\Omega |\nabla u(x,h)|^{m(x)}\,dx = \int_{[\nabla u(h)\geq 1]} |\nabla u(x,h)|^{m(x)}\,dx + \int_{[\nabla u(h)<1]} |\nabla u(x,h)|^{m(x)}\,dx$$
$$\geq \int_{[\nabla u(h)\geq 1]} |\nabla u(x,h)|^{m-}\,dx + \int_{[\nabla u(h)<1]} |\nabla u(x,h)|^{m^+}\,dx$$
$$\geq \int_\Omega |\nabla u(x,h)|^{m^-}\,dx - S_0$$

for all $h\in[0,T^*)$.

By assumption $2 < p^- \leq p^+ < \dfrac{2\,m^-}{d-m^-}$, which leads to

$$0 < \alpha^+\,p^+ < m^- \quad \text{and} \quad 0 < \alpha^-\,p^- < m^-$$

where $\alpha^\pm$ are given in Lemma 2.6. Now in view of Lemma 2.4 we have

$$L'(h)$$
$$= \left(\frac{u(h)}{|x|^2}, u_t(h)\right) = -I(u(h),h) = k(h)\int_\Omega |u(x,h)|^{p(x)}\,dx - \int_\Omega |\nabla u(x,h)|^{m(x)}\,dx$$
$$\leq k_\infty \left(\int_\Omega |u(x,h)\,\mathbb{1}_{[u(h)\geq 1]}|^{p^+}\,dx + \int_\Omega |u(x,h)\mathbb{1}_{[u(h)<1]}|^{p^-}\,dx\right) - \int_\Omega |\nabla u(x,h)|^{m(x)}\,dx$$
$$\leq k_\infty \left(N_{p^+}\,\|\nabla u(h)\|^{\alpha^+ p^+}_{L^{m^-}([u(h)\geq 1])}\,\|u(h)\|^{(1-\alpha^+)\,p^+}_{L^2([u(h)\geq 1])} + N_{p^-}\,\|\nabla u(h)\|^{\alpha^- p^-}_{L^{m^-}([u(h)<1])}\,\|u(h)\|^{(1-\alpha^-)\,p^-}_{L^2([u(h)<1])}\right)$$

$$
\begin{aligned}
&- \int_\Omega |\nabla u(x,h)|^{m(x)}\,dx \\
&\le \int_{[\nabla u(h)\ge 1]} |\nabla u(x,h)|^{m^-}\,dx + \int_{[\nabla u(h)<1]} |\nabla u(x,h)|^{m^+}\,dx - \int_\Omega |\nabla u(x,h)|^{m(x)}\,dx \\
&\quad + \frac{2-\alpha^+ p^+}{2} \left(\frac{2}{k_\infty N_{p^+}\,\alpha^+ p^+}\right)^{-\alpha^+ p^+/(2-\alpha p^+)} \|u(h)\|_{L^2(\Omega)}^{2\gamma^+} \\
&\quad + \frac{2-\alpha^- p^-}{2} \left(\frac{2}{k_\infty N_{p^-}\,\alpha^- p^-}\right)^{-\alpha^- p^-/(2-\alpha p^-)} \|u(h)\|_{L^2(\Omega)}^{2\gamma^-} \\
&\le S_0 + \frac{2-\alpha^+ p^+}{2} \left(\frac{2}{k_\infty N_{p^+}\,\alpha^+ p^+}\right)^{-\alpha p^+/(2-\alpha^+ p^+)} \|u(h)\|_{L^2(\Omega)}^{2\gamma^+} \\
&\quad + \frac{2-\alpha^- p^-}{2} \left(\frac{2}{k_\infty N_{p^-}\,\alpha^- p^-}\right)^{-\alpha^- p^-/(2-\alpha^- p^-)} \|u(h)\|_{L^2(\Omega)}^{2\gamma^-} \\
&\le S_0 + \frac{2-\alpha p^+}{2} \left(\frac{2}{k_\infty N_{p^+}\,\alpha p^+}\right)^{-\alpha p^+/(2-\alpha p^+)} (\mathrm{diam}(\Omega))^{4\gamma^+} L(h)^{\gamma^+} \\
&\quad + \frac{2-\alpha p^-}{2} \left(\frac{2}{k_\infty N_{p^-}\,\alpha p^-}\right)^{-\alpha p^-/(2-\alpha p^-)} (\mathrm{diam}(\Omega))^{4\gamma^-} L(h)^{\gamma^-} \\
&\le S_0 + C_0 \left(L(h)^{\gamma^+} + L(h)^{\gamma^-}\right) \qquad (17)
\end{aligned}
$$

for all $h \in (0, T^*)$, where

$$
C_0 := \max\left\{ \frac{2-\alpha p^+}{2} \left(\frac{2}{k_\infty N_{p^+}\,\alpha p^+}\right)^{-\alpha p^+/(2-\alpha p^+)} (\mathrm{diam}(\Omega))^{4\gamma^+}, \right.
$$
$$
\left. \frac{2-\alpha p^-}{2} \left(\frac{2}{k_\infty N_{p^-}\,\alpha p^-}\right)^{-\alpha p^-/(2-\alpha p^-)} (\mathrm{diam}(\Omega))^{4\gamma^-} \right\},
$$

$$
\gamma^+ := \frac{(1-\alpha^+)\,p^+}{2} \left(1 - \frac{\alpha^+ p^+}{m^-}\right)^{-1} \quad \text{and} \quad \gamma^- := \frac{(1-\alpha^-)\,p^-}{2} \left(1 - \frac{\alpha^- p^+}{m^-}\right)^{-1}
$$

and we applied Lemma 2.6 in the fourth step and Young's inequality in the fifth step.

Since $\gamma^\pm > 1$ and $\lim_{t\to T^*} L(t) = \infty$, we may choose an $t_0 \in [0, T^*)$ such that

$$
S_0 \le L(h)^{\gamma^+} + L(h)^{\gamma^-}
$$

for all $h \in (t_0, T^*)$. Then we infer from (17) that

$$
L'(h) \le C^* \left(L(h)^{\gamma^+} + L(h)^{\gamma^-}\right)
$$

for all $h \in (t_0, T^*)$, where $C^* = C_0 + 1$. Equivalently one has

$$
\frac{L'(h)}{L(h)^{\gamma^+} + L(h)^{\gamma^-}} \le C^*,
$$

from which we obtain

$$\int_{L(t_0)}^{L(t)} \frac{ds}{s^{\gamma^+} + s^{\gamma^-}} \leq C^* (t - t_0)$$

Lastly, using $\gamma^\pm > 1$ and $\lim_{t \to T^*} L(t) = \infty$, we send $t \longrightarrow T^*$ in the above inequality to obtain

$$T^* \geq t_0 + \frac{1}{C^*} \int_{L(t_0)}^{\infty} \frac{ds}{s^{\gamma^+} + s^{\gamma^-}}$$

as required. □

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