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UNIFORM CONVERGENCE ON SUBSPACES IN VON NEUMANN'S ERGODIC THEOREM WITH CONTINUOUS TIME

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Abstract. Power-law uniform (in the operator norm) convergence on vector subspaces with their own norms in von Neumann's ergodic theorem with continuous time is considered. All possible exponents of the considered power-law convergence are found; for each of these exponents, spectral criteria for such convergence are given and a complete description of all such subspaces is obtained. Uniform convergence over the entire space takes place only in trivial cases, which explains the interest in the uniform convergence just on subspaces.

In addition, along the way, the old convergence rate estimates in the von Neumann ergodic theorem for (semi)flows are generalized and refined.

Keywords: von Neumann's ergodic theorem; rates of convergence in ergodic theorems; power-law uniform convergence.

1. Introduction

1.1. Let $\mathcal{H}$ be a Hilbert space, and let $\{U^t\}_{t\in\mathbb{R}}$ be a group of unitary operators acting on $\mathcal{H}$. Suppose that the group $\{U^t\}_{t\in\mathbb{R}}$ is strongly continuous (see [1, Ch. VIII, Sec. 1]; such will be, for example, the considered in [2] Koopman's groups of unitary operators $\{U^t\}_{t\in\mathbb{R}}$ in $L_2(\Omega)$ generated by groups of invertible measure preserving transformations of a probability measure space Ω, i.e., by flows).

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For every vector $f \in \mathcal{H}$ and for all $t, s \in \mathbb{R}$ such that $t > s$, consider the ergodic averages

$$P_{t,s}f = \frac{1}{t-s}\int_s^t U^\tau f\,d\tau.$$

The classical von Neumann ergodic theorem [2] states for each vector $f \in \mathcal{H}$ the existence of a limit in $\mathcal{H}$

$$\lim_{t-s\to\infty} P_{t,s}f = f^* := Pf,$$

where P is the orthogonal projector onto the subspace of fixed vectors.

For the group $\{U^t\}_{t\in\mathbb{R}}$, by Stone's theorem [3, Ch. XII, Sec. 6], there exists an infinitesimal generator $\mathbf{B}$, i.e., $U^t = e^{it\mathbf{B}}$ for all $t \in \mathbb{R}$. Let $\{E(\lambda)\}_{\lambda\in\mathbb{R}}$ be the resolution of the identity corresponding to the group $\{U^t\}_{t\in\mathbb{R}}$ (its infinitesimal generator). Then (see, e.g., [2]), for any vectors $f, g \in \mathcal{H}$, we have a representation

$$(U^t f, g)_{\mathcal{H}} = \int_{-\infty}^{\infty} e^{it\lambda}\, d(E(\lambda)f, g)_{\mathcal{H}}$$

for all $t \in \mathbb{R}$; and one can introduce spectral measures $\sigma_{f,g}$, i.e., such finite Borel measures on the real line that for all its Borel subsets C

$$\sigma_{f,g}(C) = \int_C d(E(\lambda)f, g)_{\mathcal{H}};$$

denote $\sigma_f = \sigma_{f,f}$. As is well known (it has been proven and applied already in [2]), for all $t > s$, we have an integral representation

$$\|P_{t,s}f\|_{\mathcal{H}}^2 = \int_{-\infty}^{+\infty} \left(\frac{\sin\frac{(t-s)x}{2}}{\frac{(t-s)x}{2}}\right)^2 d\sigma_f(x) = \int_{-\infty}^{+\infty} F_{t-s}(x)\, d\sigma_f(x)$$

for the kernel

$$F_\tau(x) = \left(\frac{\sin\frac{\tau x}{2}}{\frac{\tau x}{2}}\right)^2,\ x \neq 0;\ F_\tau(0) = 1;$$

obviously, always $0 \le F_\tau(x) \le 1$, and $F_\tau(x) \le \frac{4}{(\tau x)^2}$ for all $x \neq 0$.

Considering that $P = E(0)$, for all $t > s$, we obtain the integral representation

$$\|P_{t,s}f - f^*\|_{\mathcal{H}}^2 = \|P_{t,s}(f - f^*)\|_{\mathcal{H}}^2 = \int_{\mathbb{R}} F_{t-s}(x)d\sigma_{f-f^*}(x) = \int_{\mathbb{R}\setminus\{0\}} F_{t-s}(x)d\sigma_f(x). \tag{1}$$

1.2. In a recent work [4] Ben-Artzi and Morisse discovered the existence of (power-law) uniform convergence on some special subspaces in the von Neumann ergodic theorem with continuous time. Namely, for some Banach subspace $\mathcal{X}$, densely and continuously embedded in $\mathcal{H}$, the estimate $\|P_{t,-t} - P\|_{\mathcal{X}\to\mathcal{H}} = \mathcal{O}(t^{-l})$ as $t \to \infty$ was obtained for some $l = l(\mathcal{X}) > 0$ (exact formulation we give in Theorem 4). Looking for its place in general theory of rates of convergence in ergodic theorems (see, for example, reviews [5, 6]) led to appearance of this our article.

In Theorem 1, we give the spectral criterion for uniform power-law convergence with exponent $\alpha \in [0, 2)$ on one-dimensional subspaces of $\mathcal{H}$. It turnes out (Remark 7) that power-law rate of convergence with the exponent $\alpha = 2$ is the maximum possible; Theorem 7 gives the spectral (in somewhat different terms) criterion for the presence of such the maximum rate on one-dimensional subspaces. Along the way, Theorems 1 and 7 generalize and refine the convergence rate estimates in the von Neumann ergodic theorem for (semi)flows, obtained earlier in [7]; while one of the new constants of these estimates turns out to be exact (Subsection 3.4).

Further, in Theorems 2 and 8 we transfer the criteria of Theorems 1 and 7 for the general case of multidimensional vector subspaces in $\mathcal{H}$ with their own norms. It is also possible to get (Theorem 3) full description of all such multidimensional (normed, with their own norms) subspaces with uniform power-law convergence for all possible exponents $\alpha \in [0, 2]$.

In Section 3, we show (Remark 3) that uniform convergence considered on the whole space $\mathcal{H}$ takes place only in the case of a spectral gap; in the von Neumann theorem for flows on a Lebesgue space, this is equivalent (Remark 4) to periodicity of a.e. trajectories with uniformly bounded periods, i.e., happens only in trivial cases. This fact already explains our interest to the uniform convergence precisely on subspaces in $\mathcal{H}$ with their own norms.

Section 4 also considers possible applications of the obtained results to the study of solutions of some equations of mathematical physics, carried out in [4].

2. One-dimensional subspaces with uniform power-law convergence

2.1. Complete description of all one-dimensional subspaces with uniform power-law (with exponents less than 2) convergence was actually given in [7, 8], where we have considered the ergodic averages for Koopman's semigroups of isometric operators $\{U^t\}_{t\geq 0}$ in $L_2(\Omega)$ generated by semigroups of measure-preserving transformations of a probability measure space Ω (i.e., usual von Neumann's ergodic theorem for semiflows). Since representation (1) is also valid for those ergodic averages (see, for example, [7] for details), and the constants of our estimates in both parts of Theorem 1 are more accurate than corresponding constants from [7], then the proof of Theorem 1 immediately gives a refinement of those old estimates.

The main clarification was obtained due to a different approach to proof of Lemma 1 below. In [7], to estimate the integral of the kernel F_τ with respect to the spectral measure σ_f, we used its expanding into a series (following the approach for the case of discrete time from [5, 9, 10]). Here, we employ to evaluate this integral more suitable method of integration by parts used in [8], which originally was proposed by V.F. Gaposhkin (for the case of discrete time) in [11]. Note also that, since representation (1) is also valid for (considered in [11]) averages for stationary in the wide sense processes with continuous time — see, for example, [12, Theorem 18.3.1], then our Theorem 1 has an obvious exact analog for those stochastic processes also.

In what follows, we need the following two technical lemmas refining the corresponding Lemmas 1 and 2 from [7].

Lemma 1. *For all $t > s$ the following inequality holds*

$$\|P_{t,s}f - f^*\|_{\mathcal{H}}^2 \leq \frac{8}{(t-s)^2} \int\limits_{\frac{2}{t-s}}^{\infty} x^{-3} \sigma_{f-f^*}(-x, x] dx.$$

Proof of Lemma 1. Without loss of generality, we assume that $f^* = 0$. Denoting $\tau = t - s$ and using representation (1), we get

$$\|P_{t,s}f - f^*\|^2_{\mathcal{H}} = \|P_{t,s}f\|^2_{\mathcal{H}} = \int\limits_{-\infty}^{+\infty} F_\tau(x)\, d(E(x)f, f)_{\mathcal{H}} =$$

$$= \int\limits_{(-\frac{2}{\tau},\frac{2}{\tau}]} F_\tau(x)\, d\sigma_f(x) + \int\limits_{(-\infty,-\frac{2}{\tau}]} F_\tau(x)\, d\sigma_f(x) + \int\limits_{(\frac{2}{\tau},\infty)} F_\tau(x)\, d\sigma_f(x) \le$$

$$\le \int\limits_{(-\frac{2}{\tau},\frac{2}{\tau}]} d\sigma_f(x) + \frac{4}{\tau^2} \int\limits_{(-\infty,-\frac{2}{\tau}]} x^{-2}\, d\sigma_f(x) + \frac{4}{\tau^2} \int\limits_{(\frac{2}{\tau},\infty)} x^{-2}\, d\sigma_f(x).$$

Let $G(x) = \sigma_f(-\infty,\ x]$; clearly that $G(x)$ is a monotonic function. Since σ_f is a finite Borel measure, then $G(x)$ is a function of bounded variation on any interval of real line. Thus, along with the Lebesgue–Stieltjes integral, we can consider the Riemann–Stieltjes integral over the function $G(x)$. If $h(x)$ is a continuous function on $[a, b]$, then it is integrable in both senses, and the value of its Riemann–Stieltjes integral is the same with the Lebesgue–Stieltjes integral:

$$\int\limits_{[a,b]} h(x) dG(x) = \int\limits_{[a,b]} h(x) d\sigma_f(x).$$

Therefore, if $h(x)$ is continuous and has bounded variation on $[a, b]$, then

$$\int\limits_{[a,b]} h(x) dG(x) = h(b)G(b) - h(a)G(a) - \int\limits_{[a,b]} G(x) dh(x).$$

This is an analogue of the formula for integration by parts for the Riemann–Stieltjes integral (see [13, Theorem 6.30]; and also [14, Ch. II, Sec. 6, Theorem 11]). Since the function G is right continuous, then the last formula is valid both for finite half-intervals $(a, b]$, and infinite $(-\infty, b]$ and $(a, +\infty)$. Using this approach, we obtain

$$\|P_{t,s}f\|^2_{\mathcal{H}} \le \int\limits_{(-\frac{2}{\tau},\frac{2}{\tau}]} dG(x) + \frac{4}{\tau^2} \int\limits_{(-\infty,-\frac{2}{\tau}]} x^{-2}\, dG(x) +$$

$$+ \frac{4}{\tau^2} \int\limits_{(\frac{2}{\tau},\infty)} x^{-2}\, dG(x) = G\left(\frac{2}{\tau}\right) - G\left(\frac{-2}{\tau}\right) +$$

$$+ \frac{4}{\tau^2} \left(x^{-2} G(x) \Big|_{-\infty}^{\frac{-2}{\tau}} + 2 \int\limits_{-\infty}^{\frac{-2}{\tau}} x^{-3} G(x)\, dx \right) +$$

$$+ \frac{4}{\tau^2} \left(x^{-2} G(x) \Big|_{\frac{2}{\tau}}^{\infty} + 2 \int\limits_{\frac{2}{\tau}}^{\infty} x^{-3} G(x)\, dx \right) =$$

$$= G\left(\frac{2}{\tau}\right) - G\left(\frac{-2}{\tau}\right) + G\left(\frac{-2}{\tau}\right) - G\left(\frac{2}{\tau}\right) +$$

$$+\frac{8}{\tau^2}\int_{\frac{2}{\tau}}^{\infty} x^{-3}(G(x)-G(-x))\,dx = \frac{8}{\tau^2}\int_{\frac{2}{\tau}}^{\infty} x^{-3}\sigma_f(-x,x]\,dx.$$

The last equality follows from the replacement $y=-x$ in the integral $\int_{-\infty}^{\frac{-2}{\tau}} x^{-3}G(x)\,dx$ and taking into account that $G(x)-G(-x)=\sigma_f(-x,x]$. □

Lemma 2. *Let $\varepsilon\in(0,\pi)$; then for all $t>s$ the following inequality holds*

$$\sigma_{f-f^*}\left(-\frac{2\varepsilon}{t-s},\frac{2\varepsilon}{t-s}\right] \le \frac{\varepsilon^2}{\sin^2\varepsilon}\|P_{t,s}f-f^*\|^2_{\mathcal{H}}.$$

Proof of Lemma 2. Again, without loss of generality, we assume that $f^*=0$. Denoting $\tau=t-s$ and using representation (1), we get

$$\|P_{t,s}f\|^2_{\mathcal{H}} = \int_{-\infty}^{+\infty} F_\tau(x)d\sigma_f(x) \ge \int_{\left(-\frac{2\varepsilon}{\tau},\frac{2\varepsilon}{\tau}\right]} F_\tau(x)d\sigma_f(x) \ge$$

$$\ge \min_{0<|x|\le\frac{2\varepsilon}{\tau}}\left(\frac{\sin\frac{\tau x}{2}}{\frac{\tau x}{2}}\right)^2\sigma_f\left(-\frac{2\varepsilon}{\tau},\frac{2\varepsilon}{\tau}\right] =$$

$$= \min_{0<|y|\le\varepsilon}\left(\frac{\sin y}{y}\right)^2\sigma_f\left(-\frac{2\varepsilon}{\tau},\frac{2\varepsilon}{\tau}\right] = \frac{\sin^2\varepsilon}{\varepsilon^2}\sigma_f\left(-\frac{2\varepsilon}{\tau},\frac{2\varepsilon}{\tau}\right].$$

□

2.2. Power-law convergence criterion with exponent $\alpha<2$. For $\alpha\in[0,2]$ we set

$$\rho(\alpha)=\inf_{x>0}\frac{x^{2-\alpha}}{\sin^2 x}.$$

It is easy to see that for $\alpha=0$ the infimum is arrived as $x\to 0+$, and therefore $\rho(0)=1$. For $\alpha=2$ the infimum is obtained at the points $x_n=\pi/2+\pi n, n\ge 0$ and is also equal to one: $\rho(2)=1$. For $\alpha\in(0,2)$ the infimum is reached at the first positive solution of the equation $\operatorname{tg} x=\frac{2x}{2-\alpha}$. This root not exceeds $\pi/2$, so $\rho(\alpha)\le\left(\frac{\pi}{2}\right)^{2-\alpha}$ for all $\alpha\in[0,2]$. This estimate is a good approximation of $\rho(\alpha)$ for α close to 2. One can also easily check that $1\le\rho(\alpha)\le\sin^{-2}(1)$; while the value $\sin^{-2}(1)$ is obtained at $\alpha=2(1-\operatorname{ctg}(1))$.

Theorem 1. *Let $\alpha\in[0,2)$; fix $f\in\mathcal{H}$. Then:*

1. If the spectral measure σ_{f-f^*} has a power-law singularity at zero, i.e., if for some positive constant A for all $\delta>0$ the following inequality holds

$$\sigma_{f-f^*}(-\delta,\delta]\le A\delta^\alpha,$$

then the rate of convergence of ergodic averages $P_{t,s}f$ is power-law with the same exponent, i.e., for all $t>s$

$$\|P_{t,s}f-f^*\|^2_{\mathcal{H}}\le B(t-s)^{-\alpha},$$

where we can put $B=\frac{2^{\alpha+1}}{2-\alpha}A$.

2. If the rate of convergence of ergodic averages $P_{t,s}f$ is a power-law, i.e., if for some positive constant B for all $t>s$ the following inequality holds

$$\|P_{t,s}f-f^*\|^2_{\mathcal{H}}\le B(t-s)^{-\alpha},$$

then the spectral measure σ_{f-f^*} has power-law singularity at zero (with the same exponent), i.e., for all $\delta > 0$

$$\sigma_{f-f^*}(-\delta,\delta] \leq A\delta^{\alpha}, \text{ where } A = \frac{\rho(\alpha)}{2^{\alpha}}B\left(\leq \frac{\pi^{2-\alpha}}{4}B\right).$$

Proof of Theorem 1. Without loss of generality, we assume that $f^* = 0$. Denoting $\tau = t - s$ and using Lemma 1, we obtain

$$\|P_{t,s}f\|^2_{\mathcal{H}} \leq \frac{8}{\tau^2}\int\limits_{\frac{2}{\tau}}^{\infty} x^{-3}\sigma_f(-x,x]dx =$$

$$= \frac{8A}{\tau^2}\int\limits_{\frac{2}{\tau}}^{\infty} x^{-3}x^{\alpha}dx = -\frac{8A}{(-2+\alpha)\tau^2}\left(\frac{2}{\tau}\right)^{-2+\alpha} = \frac{2^{\alpha+1}}{2-\alpha}A\tau^{-\alpha}.$$

We now prove the second part of the theorem. By Lemma 2, for each $\varepsilon \in (0,\pi)$, for all $\tau > 0$, we get

$$\sigma_f\left(-\frac{2\varepsilon}{\tau},\frac{2\varepsilon}{\tau}\right] \leq \frac{\varepsilon^2}{\sin^2\varepsilon}B\tau^{-\alpha}.$$

Representing an arbitrary $\delta > 0$ as $\delta = \frac{2\varepsilon}{\tau}$ with $\varepsilon \in (0,\pi)$ and $\tau > 0$, we obtain

$$\sigma_f(-\delta,\delta] \leq \frac{\varepsilon^{2-\alpha}}{\sin^2\varepsilon}\frac{B}{2^{\alpha}}\delta^{\alpha}.$$

Minimization of the constant in the right side along $\varepsilon \in (0,\pi)$ leads to the required inequality. This completes the proof of Theorem 1. $\square$

As it will be shown below (Proposition 4), the constant $\rho(\alpha)/2^{\alpha}$ in the second part of Theorem 1 is exact (in [7] the corresponding constant was slightly worse, and had the form $(\frac{\pi}{2})^{2-\alpha}/2^{\alpha}$).

Of course, analogs of spectral criterion of the power-law convergence rates of Theorem 1 can also be obtained for a wider rates range. The following range was proposed by V.F. Gaposhkin in [11].

Remark 1. *Let $\alpha \in [0,2)$, and a function $\varphi(u)$ be weakly oscillating on $[1,\infty)$, i.e., the function $\varphi(u)u^{\delta}$ is monotonically increasing and the function $\varphi(u)u^{-\delta}$ decreases monotonically for each $\delta > 0$. Then an analogue of the statement of Theorem 1 can also be obtained for all rates of convergence of the form $t^{-\alpha}\varphi(t)$ (the statement of Theorem 1 corresponds to the case $\varphi(t) \equiv 1$; for $\alpha = 0$ and $\varphi(t) = \ln^{\beta} t$, $\beta \geq 0$ we obtain an analog of this statement for logarithmic rates; the range of rates under consideration also includes all the rates of the form $t^{-\alpha}$ $ln^{\beta}t$ for all $\alpha \in (0,2)$ and for all β).*

Proofs of analogues of Theorem 1 for the above rates can be obtained in the same way as the proof of Theorem 1, by concretizing estimates of Lemmas 1 and 2 for each of these rates; and so, we formulated these estimates in the form of two separate lemmas (taking them out from the proof of Theorem 1). In this work, we are interested in the power-law rates only.

2.3. Power-law convergence with exponent $\alpha \geq 2$. For the exhaustive solution of the question about the presence of the power-law convergence rates considered in Theorem 1, it remains to analyze the case $\alpha \geq 2$. It turns out that the power-law rate of convergence with the exponent $\alpha > 2$ does not exist (except for the degenerate case $f = f^*$). The proof of this fact given for the von Neumannn ergodic theorem for semiflows in [7, Remark 3] (obtained there by transferring the corresponding result of V.F. Gaposhkin for a discrete time [15, Corollary 5] to the continuous time) — can be transferred to our case almost word-to-word. In Remark 7 of Section 4, we will give an independent simple proof that already the rate of convergence $o(t-s)^{-2}$ as $t-s \to \infty$ does not exist, i.e., that the rate of convergence $O(t-s)^{-2}$ is the maximum possible. The problem of finding a criterion for the presence of such a maximum possible (power-law with exponent $\alpha = 2$) convergence rate will be solved further in Section 4 by Theorem 7. This will require a correction of the approach of Theorem 1.

Let's see what we can get with $\alpha = 2$ using the old approach. An analog of the second part of Theorem 1 is also valid in this case: the proof goes through without change. But the analogue of the first part of this theorem does not hold, with any constants: the condition $\sigma_{f-f^*}(-\delta,\delta] = O(\delta^2)$ as $\delta \to 0$ is not enough for the relation $\|P_{t,s}f - f^*\|^2_{\mathcal{H}} = O(t-s)^{-2}$ as $t-s \to \infty$ (see [7, Remark 2]). However, the following weakened version of this relation holds.

Proposition 1. *Let $f \in \mathcal{H}$ and $\alpha = 2$. If for some positive constant A the inequality*

$$\sigma_{f-f^*}(-\delta,\delta] \le A_f \delta^2$$

holds for all $\delta > 0$, then

$$\|P_{t,s}f - f^*\|^2_{\mathcal{H}} \le B(t-s)^{-2}\ln(t-s)$$

for all $t \ge s+2$, where we can put $B = 8A + \frac{4}{\ln 2}\|f-f^*\|^2_{\mathcal{H}}$.

Proof of Proposition 1. Assume that $f^* = 0$ and denote $\tau = t-s$. By Lemma 1 for $0 < \frac{2}{\tau} \le 1$, we get

$$\|P_{t,s}f\|^2_{\mathcal{H}} \le \frac{8}{\tau^2}\int_{\frac{2}{\tau}}^{\infty} x^{-3}\sigma_f(-x,x]dx \le$$

$$\le \frac{8}{\tau^2}\left(\int_{\frac{2}{\tau}}^{1} x^{-3}\sigma_f(-x,x]dx + \int_{1}^{\infty} x^{-3}\sigma_f(-x,x]dx\right) \le$$

$$\le \frac{8}{\tau^2}\left(A\int_{\frac{2}{\tau}}^{1} x^{-1}dx + \int_{1}^{\infty} x^{-3}\|f\|^2_{\mathcal{H}}dx\right) =$$

$$= \frac{8A\ln(\tau)}{\tau^2} - \frac{8A\ln 2}{\tau^2} + \frac{4\|f\|^2_{\mathcal{H}}}{\tau^2} \le \frac{8A\ln(\tau)}{\tau^2} + \frac{4\|f\|^2_{\mathcal{H}}}{\tau^2}.$$

□

Consider now the case $\alpha > 2$. Proposition 2 as an analogue of the first part of Theorem 1 for the case $\alpha > 2$ gives a sufficient condition of the power-law convergence rate with the maximum possible exponent equal to 2.

Proposition 2. *Let $f \in \mathcal{H}$ and $\alpha > 2$. If for some positive constant A the inequality*

$$\sigma_{f-f^*}(-\delta,\delta] \le A_f \delta^{\alpha}$$

holds for all $\delta > 0$, then

$$\|P_{t,s}f - f^*\|^2_{\mathcal{H}} \le B(t-s)^{-2}$$

for all $t \geq s+2$, *where we can put* $B = \frac{8}{\alpha-2}A + 4\|f-f^*\|^2_{\mathcal{H}}$.

Proof of Proposition 2. Again assume that $f^* = 0$ and set $\tau = t-s$. By Lemma 1 for $0 < \frac{2}{\tau} \leq 1$, we get

$$\|P_{t,s}f\|^2_{\mathcal{H}} \leq \frac{8}{\tau^2}\int_{\frac{2}{\tau}}^{\infty} x^{-3}\sigma_f(-x,x]dx \leq \frac{8}{\tau^2}\left(\int_{\frac{2}{\tau}}^{1} x^{-3}\sigma_f(-x,x]dx + \int_{1}^{\infty} x^{-3}\sigma_f(-x,x]dx\right) \leq$$

$$\leq \frac{8}{\tau^2}\left(A\int_{\frac{2}{\tau}}^{1} x^{-3+\alpha}dx + \int_{1}^{\infty} x^{-3}\|f\|^2_{\mathcal{H}}dx\right) = \frac{8A}{(\alpha-2)\tau^2}\left(1-\frac{2^{\alpha-2}}{\tau^{\alpha-2}}\right) + \frac{4\|f\|^2_{\mathcal{H}}}{\tau^2} =$$

$$= \frac{8A}{(\alpha-2)\tau^2} - \frac{2^{\alpha+1}A}{(\alpha-2)\tau^{\alpha}} + \frac{4\|f\|^2_{\mathcal{H}}}{\tau^2} \leq \left(\frac{8A}{\alpha-2} + 4\|f\|^2_{\mathcal{H}}\right)\frac{1}{\tau^2}.$$

□

3. General multidimensional case

3.1. Power-law uniform convergences on subspaces. The following theorem is a corollary (obvious reformulation for the general multidimensional case) of Theorem 1.

Theorem 2. *Let* $\alpha \in [0,2)$, $\mathcal{X} \subseteq \mathcal{H}$ *be a vector subspace of* $\mathcal{H}$ *with its own norm* $\|\cdot\|_{\mathcal{X}}$. *Then:*

1. If there exists a positive constant A *such that for all* $f \in \mathcal{X}$ *for all* $\delta > 0$ *the following inequality holds*

$$\sigma_{f-f^*}(-\delta,\delta] \leq A\|f\|^2_{\mathcal{X}}\delta^{\alpha},$$

then there is a power-law (with the same exponent) uniform convergence on the space $\mathcal{X}$ *in the von Neumann ergodic theorem: for all* $t > s$

$$\|P_{t,s} - P\|^2_{\mathcal{X}\to\mathcal{H}} \leq B(t-s)^{-\alpha},$$

where we can put $B = \frac{2^{\alpha+1}}{2-\alpha}A$.

2. If there is a power-law uniform convergence on the space $\mathcal{X}$ *in the von Neumann ergodic theorem, i.e., for some positive constant* B *for all* $t > s$ *the following inequality holds*

$$\|P_{t,s} - P\|^2_{\mathcal{X}\to\mathcal{H}} \leq B(t-s)^{-\alpha},$$

then for all $f \in \mathcal{X}$ *the spectral measure* σ_{f-f^*} *has a power-law (with the same exponent* α*) singularity at zero, i.e., for all* $\delta > 0$

$$\sigma_{f-f^*}(-\delta,\delta] \leq A\|f\|^2_{\mathcal{X}}\delta^{\alpha}, \text{ where } A = \frac{\rho(\alpha)}{2^{\alpha}}B\left(\leq \frac{\pi^{2-\alpha}}{4}B\right).$$

Proof of Theorem 2. It follows from the first part of Theorem 1 that

$$\|(P_{t,s} - P)f\|^2_{\mathcal{H}} \leq B(t-s)^{-\alpha}\|f\|^2_{\mathcal{X}},$$

where $B = \frac{2^{\alpha+1}}{2-\alpha}A$.

Therefore, for all $t > s$

$$\|P_{t,s} - P\|^2_{\mathcal{X}\to\mathcal{H}} = \sup_{f\in\mathcal{X}: f\neq 0}\frac{\|(P_{t,s} - P)f\|^2_{\mathcal{H}}}{\|f\|^2_{\mathcal{X}}} \leq B(t-s)^{-\alpha}.$$

Let us prove the second part of the theorem. From the uniform convergence of ergodic averages, we obtain that for all $f \in \mathcal{X}$ for all $t > s$

$$\frac{\|(P_{t,s} - P)f\|^2_{\mathcal{H}}}{\|f\|^2_{\mathcal{X}}} \leq B(t-s)^{-\alpha}.$$

Then it follows from the second part of Theorem 1 that for all $\delta > 0$

$$\sigma_{f-f^*}(-\delta, \delta] \leq A\delta^\alpha \|f\|^2_{\mathcal{X}}, \text{ where } A = \frac{\rho(\alpha)}{2^\alpha} B.$$

□

In Theorem 8 of Section 4 below, we will give a criterion for a presence of the power-law with exponent $\alpha = 2$ (maximum possible — according to Remark 7 of the same Section 4) uniform convergence on subspaces. In the meantime, let us reformulate for the multidimensional case statements of Propositions 1 and 2 of the previous section.

Proposition 3. *Let $\alpha \geq 2$, $\mathcal{X} \subseteq \mathcal{H}$ be a vector subspace of $\mathcal{H}$ with its own norm $\|\cdot\|_{\mathcal{X}}$, which is continuously embedded in $\mathcal{H}$, i.e., without loss of generality we assume $\|\cdot\|_{\mathcal{H}} \leq \|\cdot\|_{\mathcal{X}}$.*

If there exists a positive constant A such that for all $f \in \mathcal{X}$ for all $\delta > 0$ the following inequality holds

$$\sigma_{f-f^*}(-\delta, \delta] \leq A\|f\|^2_{\mathcal{X}}\delta^2,$$

then there is a uniform convergence on the space $\mathcal{X}$ in the von Neumann ergodic theorem:

1. In the case $\alpha = 2$ for all $t \geq s + 2$

$$\|P_{t,s} - P\|^2_{\mathcal{X}\to\mathcal{H}} \leq B\frac{\ln(t-s)}{(t-s)^2},$$

where we can put $B = 8A + \frac{4}{\ln 2}$.

2. In the case $\alpha > 2$ for all $t \geq s + 2$

$$\|P_{t,s} - P\|^2_{\mathcal{X}\to\mathcal{H}} \leq B(t-s)^{-2},$$

where we can put $B = \frac{8}{\alpha-2}A + 4$.

3.2. The spaces $\mathcal{X}_\alpha$ and $\mathcal{Y}$. For any $\alpha > 0$ we denote by $\mathcal{X}_\alpha$ the set of all $f \in \mathcal{H}$ for which the spectral measure σ_f has power-law with exponent α singularity at zero, i.e.,

$$\mathcal{X}_\alpha = \{f \in \mathcal{H} | \; \exists A > 0 \; \forall \delta > 0 \;\; \sigma_f(-\delta, \delta] \leq A\delta^\alpha\}.$$

Let us show that a local power-law estimate for the spectral measure is equivalent to the global power-law estimate.

Lemma 3. *Let $\alpha > 0$ and $f \in \mathcal{H}$. Then the following conditions are equivalent:*

(1)There exists a constant $A > 0$ such that $\sigma_f(-\delta, \delta] \leq A\delta^\alpha$ for all $\delta > 0$;

(2) There exist a number $r > 0$ and a constant $A_r > 0$ such that $\sigma_f(-\delta, \delta] \leq A_r\delta^\alpha$ for all $\delta \in (0, r)$.

Proof of Lemma 3. The implication (1) ⇒ (2) is obvious; show (2) ⇒ (1). Using equality $\sigma_f(\mathbb{R}) = \|f\|^2_{\mathcal{H}}$, for all $\delta \geq r$ we get

$$\sigma_f(-\delta, \delta] \leq \sigma_f(\mathbb{R}) = \|f\|^2_{\mathcal{H}} \leq \frac{\|f\|^2_{\mathcal{H}}}{r^\alpha}\delta^\alpha.$$

Therefore, for all $\delta > 0$

$$\sigma_f(-\delta, \delta] \le A\delta^\alpha,$$

where $A = \max\left\{A_r, \frac{\|f\|^2_{\mathcal{H}}}{r^\alpha}\right\}$. □

It is easy to verify that the sets $\mathcal{X}_\alpha$ form a decreasing chain (in the order by inclusion); namely, $\mathcal{X}_{\alpha_1} \subseteq \mathcal{X}_{\alpha_2}$ for $\alpha_1 > \alpha_2$. Indeed, let $f \in \mathcal{X}_{\alpha_1}$; then by Lemma 3 it suffices to obtain a local power-law estimate. Indeed, for all $\delta \in (0,1)$ we have $\sigma_f(-\delta, \delta] \le A\delta^{\alpha_1} \le A\delta^{\alpha_2}$; hence, $f \in \mathcal{X}_{\alpha_2}$.

Proposition 4. *For any $\alpha > 0$ the set $\mathcal{X}_\alpha$ is a vector subspace in $\mathcal{H}$, orthogonal to the subspace of fixed vectors of the group $\{U^t\}_{t\in\mathbb{R}}$; it can be endowed with the norm*

$$||f||^2_{\mathcal{X}_\alpha} = \inf\{A | \forall\delta > 0 \ \sigma_f(-\delta, \delta] \le A\delta^\alpha\}.$$

Proof of Proposition 4. Let us verify the validity of all three axioms of the norm.

1) Let $f \in \mathcal{X}_\alpha$ be such that $\|f\|^2_{\mathcal{X}_\alpha} = 0$, i.e., $\sigma_f(-\delta, \delta] = 0$ for all $\delta > 0$. Let us show that $f = 0$. To do this, we introduce the sequence $\sigma_n = \sigma_f(-n, n]$. On the one hand, $\lim\limits_{n\to\infty} \sigma_n = 0$. On the other hand, $\lim\limits_{n\to\infty} \sigma_f(-n, n] = \sigma_f(\mathbb{R}) = (f, f)_{\mathcal{H}}$. This implies that $f = 0$.

2) Let us show that $\|\lambda f\|^2_{\mathcal{X}_\alpha} = |\lambda|^2 \|f\|^2_{\mathcal{X}_\alpha}$ for all $f \in \mathcal{X}_\alpha, \lambda \in \mathbb{C}$. For all $\delta > 0$ we have

$$\sigma_{\lambda f}(-\delta, \delta] = \int\limits_{(-\delta,\delta]} d(E(x)\lambda f, \lambda f)_{\mathcal{H}} = |\lambda|^2 \int\limits_{(-\delta,\delta]} d(E(x)f, f)_{\mathcal{H}} = |\lambda|^2 \sigma_f(-\delta, \delta].$$

Thus, we get that if $\sigma_f(-\delta, \delta] \le A_f\delta^\alpha$, then $\sigma_{\lambda f}(-\delta, \delta] \le |\lambda|^2 A_f \delta^\alpha$. Taking the infimum over such A_f, we obtain $\|\lambda f\|^2_{\mathcal{X}_\alpha} \le |\lambda|^2 \|f\|^2_{\mathcal{X}_\alpha}$.

It is easy to verify that this inequality automatically implies equality, namely:

$$\|f\|_{\mathcal{X}_\alpha} = \|\lambda \frac{1}{\lambda} f\|_{\mathcal{X}_\alpha} \le |\lambda| \|\frac{1}{\lambda} f\|_{\mathcal{X}_\alpha} \le |\lambda| |\frac{1}{\lambda}| \|f\|_{\mathcal{X}_\alpha}.$$

3) Let $f, g \in \mathcal{X}_\alpha$; then $\sigma_f(-\delta, \delta] \le \|f\|^2_{\mathcal{X}_\alpha}\delta^\alpha$ and $\sigma_g(-\delta, \delta] \le \|g\|^2_{\mathcal{X}_\alpha}\delta^\alpha$ for all $\delta > 0$. Using the inequality $|\sigma_{f,g}(-\delta, \delta]|^2 \le \sigma_f(-\delta, \delta]\sigma_g(-\delta, \delta]$, valid for all $f, g \in \mathcal{H}$ (see, e.g., [16, 5.5]), we get

$$\sigma_{f+g}(-\delta, \delta] = \sigma_f(-\delta, \delta] + \sigma_{f,g}(-\delta, \delta] + \sigma_{g,f}(-\delta, \delta] + \sigma_g(-\delta, \delta] \le$$
$$\le \sigma_f(-\delta, \delta] + \sqrt{\sigma_f(-\delta, \delta]\sigma_g(-\delta, \delta]} + \sqrt{\sigma_g(-\delta, \delta]\sigma_f(-\delta, \delta]} + \sigma_g(-\delta, \delta] \le$$
$$\le \|f\|^2_{\mathcal{X}_\alpha}\delta^\alpha + 2\|f\|_{\mathcal{X}_\alpha}\|g\|_{\mathcal{X}_\alpha}\delta^\alpha + \|g\|^2_{\mathcal{X}_\alpha}\delta^\alpha = (\|f\|_{\mathcal{X}_\alpha} + \|g\|_{\mathcal{X}_\alpha})^2\delta^\alpha.$$

Hence, $f + g \in \mathcal{X}_\alpha$, and

$$||f + g||^2_{\mathcal{X}_\alpha} \le (||f||_{\mathcal{X}_\alpha} + \|g\|_{\mathcal{X}_\alpha})^2,$$

which is what was required.

It is easy to see that for any vector $f \in \mathcal{X}_\alpha$ the value of its spectral measure at zero point $\sigma_f(\{0\}) = \|f^*\|^2_{\mathcal{H}} = 0$. Therefore, $Pf = f^* = 0$. □

Defining $\mathcal{X}_0$ in the same way as $\mathcal{X}_\alpha$, i.e., for $\alpha = 0$, it is easy to see that then $\mathcal{X}_0 = \mathcal{H}$ and $\|\cdot\|_{\mathcal{X}_0} = \|\cdot\|_{\mathcal{H}}$. Consider one more subspace $\mathcal{Y}$, consisting of image vectors $\mathcal{R}(\mathbf{B})$ of generator $\mathbf{B}$, with norm defined by equality

$$\|f\|_{\mathcal{Y}} = \sup_{t>0} \left\| \int_0^t U^\tau f \, d\tau \right\|_{\mathcal{H}}, \quad f \in \mathcal{R}(\mathbf{B}).$$

It is well known [1, Ch. VIII, Sec. 1, Lemma 7] that $\|f\|_{\mathcal{Y}} < \infty$ for any vector $f \in \mathcal{R}(\mathbf{B})$; namely, for any $g \in \mathrm{Dom}(\mathbf{B})$ and all $t > 0$

$$\left\| \int_0^t U^\tau \mathbf{B} g \, d\tau \right\|_{\mathcal{H}} = \|(U^t - I)g\|_{\mathcal{H}} \le 2\|g\|_{\mathcal{H}}.$$

For some semigroups, the converse is also true: if $\|f\|_{\mathcal{Y}} < \infty$, then $f \in \mathcal{R}(\mathbf{B})$. In particular, for a unitary group, this will be true: see [17, Theorem 2.6], where this property is proved for dual (consisting of Banach conjugate operators) strongly continuous semigroups. It is also easy to check the validity of all properties of the norm for $\|\cdot\|_{\mathcal{Y}}$.

3.3. Characterization of subspaces with power-law uniform convergence. Using subspaces $\mathcal{X}_\alpha, \alpha \in [0, 2)$ and $\mathcal{Y}$, we can describe all embedded in $\mathcal{H}$ normed spaces on which there is a uniform power-law convergence in the von Neumann ergodic theorem. Recall that (Remark 7 of Section 4) such a rate of convergence with an exponent greater than 2 does not exist.

Theorem 3. *Let $\alpha \in [0, 2]$, $\mathcal{X} \subseteq \mathcal{H}$ be a vector subspace in $\mathcal{H}$ with its own norm $\|\cdot\|_{\mathcal{X}}$. On the space $\mathcal{X}$, there is the uniform power-law convergence with exponent α in the von Neumann ergodic theorem if and only if $I - P$ is a continuous embedding of $\mathcal{X}$ in $\mathcal{X}_\alpha$ for $\alpha \in [0, 2)$; and, respectively, it is a continuous embedding of $\mathcal{X}$ in $\mathcal{Y}$ for $\alpha = 2$.*

Proof of Theorem 3. Consider first the case $\alpha \in [0, 2)$. Suppose there exist a constant $B > 0$ such that $\|P_{t,s} - P\|^2_{\mathcal{X}\to\mathcal{H}} \le B(t-s)^{-\alpha}$ for all $t > s$. From the proof of the second part of Theorem 2 it follows that for $\alpha \in [0, 2)$ (and for $\alpha = 2$) for all $f \in \mathcal{X}$ the spectral measure of the vector $f - f^* = (I - P)f$ has a power-law singularity at zero with exponent α; namely, for all $\delta > 0$

$$\sigma_{(I-P)f}(-\delta, \delta] \le \frac{\rho(\alpha)}{2^\alpha} B \|f\|^2_{\mathcal{X}} \delta^\alpha.$$

Hence, $(I - P)f \in \mathcal{X}_\alpha$ and $\|(I-P)f\|^2_{\mathcal{X}_\alpha} \le \frac{\rho(\alpha)}{2^\alpha} B\|f\|^2_{\mathcal{X}}$ for all $f \in \mathcal{X}$, what was required.

Let us prove the statement in the opposite direction. Suppose $(I - P)\mathcal{X} \subseteq \mathcal{X}_\alpha$ and there exists a positive constant A such that $\|I - P\|^2_{\mathcal{X}\to\mathcal{X}_\alpha} \le A$. Then for any $\delta > 0$

$$\sigma_{(I-P)f}(-\delta, \delta] \le \|(I-P)f\|^2_{\mathcal{X}_\alpha} \delta^\alpha \le A\|f\|^2_{\mathcal{X}} \delta^\alpha,$$

and the first part of Theorem 2 gives for all $t > s$ the required estimate

$$\|P_{t,s} - P\|^2_{\mathcal{X}\to\mathcal{H}} \le \frac{2^{\alpha+1}}{2-\alpha} A(t-s)^{-\alpha}.$$

Let now $\alpha = 2$ and there exists a constant $B > 0$ such that for all $t > s$ the inequality $\|P_{t,s} - P\|^2_{\mathcal{X}\to\mathcal{H}} \le B(t-s)^{-2}$ holds. When $s = 0$, we obtain for all $t > 0$ the estimate

$$\left\| \int_0^t U^\tau (I-P) f \, d\tau \right\|^2_{\mathcal{H}} \le B\|f\|^2_{\mathcal{X}}.$$

Hence, $(I - P)f \in \mathcal{Y}$ and $\|(I-P)f\|^2_{\mathcal{Y}} \le B\|f\|^2_{\mathcal{X}}$, which is needed to be proven. On the other hand: suppose $(I - P)\mathcal{X} \subseteq \mathcal{Y}$ and there is a positive constant A such that $\|I - P\|^2_{\mathcal{X}\to\mathcal{Y}} \le A$. Then for all $t > s$

$$\|P_{t,s}f - Pf\|^2_{\mathcal{H}} = \frac{1}{(t-s)^2} \left\| \int_s^t U^\tau (I-P) f d\tau \right\|^2_{\mathcal{H}} =$$

$$= \frac{1}{(t-s)^2} \left\| U^s \int_0^{t-s} U^\tau (I-P) f d\tau \right\|_{\mathcal{H}}^2 = \frac{1}{(t-s)^2} \left\| \int_0^{t-s} U^\tau (I-P) f d\tau \right\|_{\mathcal{H}}^2 \leq$$
$$\leq \frac{1}{(t-s)^2} \|(I-P)f\|_{\mathcal{Y}}^2 \leq A \|f\|_{\mathcal{X}}^2 (t-s)^{-2}.$$

□

Remark 2 (M. Lin [18])**.** *Uniform convergence in the von Neumann ergodic theorem on the entire space $\mathcal{H}$ holds if and only if the image $\mathcal{R}(\mathbf{B})$ of the group generator $\mathbf{B}$ is closed.*

Moreover: $\mathcal{H} = \mathcal{R}(\mathbf{B}) \oplus \{x : U^t x = x \ \forall t > 0\}$, restriction $\mathbf{B}_1$ of the generator $\mathbf{B}$ to $\mathrm{Dom}\,(\mathbf{B}) \cap \mathcal{R}(\mathbf{B})$ will be invertible, and $\|\mathbf{B}_1^{-1}\| < \infty$. Therefore (see proof in [18]) in this case, the convergence rate will be maximum possible. Namely, for any vector $f \in \mathcal{H}$, since $(I-P)f \in \mathcal{R}(\mathbf{B}) = \mathcal{R}(\mathbf{B}_1)$, for of all $t > s$

$$\|P_{t,s} f - Pf\|_{\mathcal{H}} = \frac{1}{t-s} \left\| U^s \int_0^{t-s} U^\tau (I-P) f d\tau \right\|_{\mathcal{H}} =$$
$$= \frac{1}{t-s} \left\| \int_0^{t-s} U^\tau (I-P) f d\tau \right\|_{\mathcal{H}} = \frac{1}{t-s} \|(U^{t-s} - I)\mathbf{B}^{-1}(I-P)f\|_{\mathcal{H}} \leq$$
$$\leq \frac{2}{t-s} \|\mathbf{B}_1^{-1}\| \|(I-P)f\|_{\mathcal{H}} \leq \frac{2\|\mathbf{B}_1^{-1}\| \|f\|_{\mathcal{H}}}{t-s}.$$

This criterion can be reformulated in terms of the existence of a spectral gap; see also discussion for general semigroups in [19, 1.2.15].

Remark 3. *Uniform convergence in the von Neumann ergodic theorem on the entire space $\mathcal{H}$ holds if and only if for some $\gamma > 0$ there are no spectrum points of operator $\mathbf{B}$ in the set $(-\gamma, \gamma) \setminus \{0\}$.*

Indeed, if the convergence is uniform over the entire space, then by the previous remark, the spectrum of the generator $\mathbf{B}$ coincides with spectrum of the operator $\mathbf{B}_1$ in union with the point 0 (if $\{x : U^t x = x \ \forall t > 0\} = \ker \mathbf{B} \neq \{0\}$). Since for the operator $\mathbf{B}_1$ the point 0 is regular value (because $\mathcal{R}(\mathbf{B}_1) = \mathcal{R}(\mathbf{B})$ and $\|\mathbf{B}_1^{-1}\| < \infty$), while the set of regular values for closed operator is open, then there is a punctured neighborhood of zero point containing only regular generator values.

Conversely, if there exists such $\gamma > 0$, then, using representation (1), we obtain for each vector $f \in \mathcal{H}$ and for all $t > s$ the estimate

$$\|P_{t,s} f - Pf\|_{\mathcal{H}}^2 = \frac{4}{(t-s)^2} \int_{\mathbb{R} \setminus \{0\}} \frac{\sin^2(\frac{(t-s)x}{2})}{x^2} \, d\sigma_f(x) =$$
$$= \frac{4}{(t-s)^2} \int_{\mathbb{R} \setminus (-\gamma, \gamma)} \frac{\sin^2(\frac{(t-s)x}{2})}{x^2} \, d\sigma_f(x) \leq \frac{4\|f\|_{\mathcal{H}}^2}{(t-s)^2 \gamma^2}.$$

This completes the proof of Remark 2.

Since the generator of the unitary group generated by the flow (i.e., by the group of invertible measure-preserving transformations) has its own specifics (see, e.g., [20]), here we are interested in the description of flows, for which the uniform convergence on the whole space in the von Neumann ergodic theorem exists. In the discrete case, it is known that such a situation has place only for a periodic automorphism (see the discussion in [21]). The following Remark 4 contains an analogue of this statements for continuous time.

Let $\mathbf{T} = \{T^t\}_{t\in\mathbb{R}}$ be a flow on a Lebesgue space (Ω,μ) with non-atomic measure μ, and $U^t_{\mathbf{T}} f = f \circ T^t$ be its group of unitary Koopman operators in $L_2(\Omega,\mu)$. Let $\mathcal{P}(\omega)$ be a period of a point $\omega \in \Omega$ with respect to the flow $\mathbf{T}$, i.e., such a minimum number $t > 0$ with $T^t\omega = \omega$; we set for non-periodic points $\mathcal{P}(\omega) = \infty$.

Remark 4. *Uniform convergence on the entire space in the von Neumann ergodic theorem for unitary group $\{U^t_{\mathbf{T}}\}_{t\in\mathbb{R}}$ on $L_2(\Omega,\mu)$ holds if and only if $\mathcal{P} \in L_\infty(\Omega,\mu)$.*

Indeed, if there is uniform convergence on the entire space, then by Remark 3 the spectrum of the group generator has a gap. Then $\mathcal{P} \in L_\infty(\Omega,\mu)$, since otherwise, as shown in [22] (for the case of an aperiodic flow, see also [23]) the spectrum of the generator is whole $\mathbb{R}$.

Conversely, let $\|\mathcal{P}\|_\infty < \infty$. Then $f^*(\omega) = \frac{1}{\mathcal{P}(\omega)} \int_0^{\mathcal{P}(\omega)} f(T^\tau\omega)\, d\tau$ for a.e. $\omega \in \Omega$; setting $0 < t-s = N\mathcal{P}(\omega) + r$, $0 \le r < \mathcal{P}(\omega)$, $N \in \mathbb{N} \cup \{0\}$, for all $t > s$ we get

$$\|P_{t,s}f - f^*\|_2 = \left\| \frac{1}{t-s} \int_0^{t-s} f(T^{\tau+s}\omega)\, d\tau - f^*(\omega) \right\|_2 =$$

$$= \left\| \frac{1}{t-s} \int_0^{t-s} f(T^{\tau}\omega)\, d\tau - f^*(\omega) \right\|_2 =$$

$$= \left\| \frac{1}{t-s} \left(\sum_{k=0}^{N-1} \int\limits_{k\mathcal{P}(\omega)}^{(k+1)\mathcal{P}(\omega)} f(T^\tau\omega)\, d\tau + \int\limits_{N\mathcal{P}(\omega)}^{t-s} f(T^\tau\omega)\, d\tau \right) - \frac{1}{\mathcal{P}(\omega)} \int\limits_0^{\mathcal{P}(\omega)} f(T^\tau\omega)\, d\tau \right\|_2 =$$

$$= \left\| \left(\frac{N}{t-s} - \frac{1}{\mathcal{P}(\omega)} \right) \int_0^{\mathcal{P}(\omega)} f(T^\tau\omega)\, d\tau + \frac{1}{t-s} \int_0^r f(T^\tau\omega)\, d\tau \right\|_2 \le$$

$$\le \frac{2}{t-s} \left\| \int_0^{\mathcal{P}(\omega)} |f(T^\tau\omega)|\, d\tau \right\|_2 \le \frac{2}{t-s} \left\| \int_0^{\|\mathcal{P}\|_\infty} |f(T^\tau\omega)|\, d\tau \right\|_2 \le \frac{2\|\mathcal{P}\|_\infty \|f\|_2}{t-s}.$$

If the function $\mathcal{P}$ is not essentially bounded, but only integrable with some degree, then we can get uniform convergence with the maximum rate on the corresponding classical subspaces of integrable functions. Namely, the following statement is true.

Remark 5. *Let $\mathcal{P} \in L_{2q}(\Omega,\mu)$ for $1 \le q < \infty$. Then there is power-law convergence with the exponent $\alpha = 2$ in the von Neumann ergodic theorem for the group $\{U^t_{\mathbf{T}}\}_{t\in\mathbb{R}}$ on the subspace $L_{2p}(\Omega,\mu)$, where $\frac{1}{q} + \frac{1}{p} = 1$.*

Indeed, setting $f^{**}(\omega) = \sup\limits_{t>0} \frac{1}{t} \int_0^t |f(T^\tau\omega)|\, d\tau$ and applying Hölder's inequality, from the calculations of the previous remark for all $t - s > 0$ we obtain

$$\|P_{t,s}f - f^*\|_2 \le \frac{2}{t-s} \left\| \int\limits_0^{\mathcal{P}(\omega)} |f(T^\tau\omega)|\, d\tau \right\|_2 \le \frac{2}{t-s} \|\mathcal{P}(\omega) f^{**}(\omega)\|_2 \le \frac{2\|\mathcal{P}\|_{2q} \|f^{**}\|_{2p}}{t-s}.$$

For $q > 1, p < \infty$ the assertion to be proved follows from this by dominant inequality $\|f^{**}\|_{2p} \le \frac{2p}{2p-1} \|f\|_{2p}$; for $q = 1, p = \infty$ the obvious inequality $\|f^{**}\|_\infty \le \|f\|_\infty$ works.

3.4. On the unimprovability of constant $\frac{\rho(\alpha)}{2^\alpha}$ in estimates of Theorems 1 and 2. Let's again use the subspaces $\mathcal{X}_\alpha$ from Subsection 3.2 above. On the Hilbert space $\mathcal{H} = L_2(\mathbb{R})$, consider the group of unitary multiplication operators

$$U^t f(x) = e^{ixt} \cdot f(x), \ \ f \in L_2(\mathbb{R}).$$

The infinitesimal generator $\mathbf{B}$ of this group is a self-adjoint operator of multiplication by x with natural domain of definition, i.e.,

$$\mathbf{B}f(x) = x \cdot f(x),\ f \in \operatorname{Dom} \mathbf{B} = \{g \in L_2(\mathbb{R}) :\ xg(x) \in L_2(\mathbb{R})\}.$$

Obviously, for any Borel subset $C \subseteq \mathbb{R}$ spectral projectors are defined by the equality $E(C)f(x) = \chi_C(\mathbf{B})f(x) = \chi_C(x) \cdot f(x)$. Then

$$\sigma_f(C) = (E(C)f, f) = \int_{\mathbb{R}} \chi_C(x)|f(x)|^2\, dx = \int_C |f(x)|^2\, dx,$$

i.e., the spectral measure σ_f is absolutely continuous, and has density $|f|^2$. The spaces $\mathcal{X}_\alpha$ in this case are described as follows:

$$\mathcal{X}_\alpha = \left\{ f \in L_2(\mathbb{R}) :\ \sup_{\delta>0} \frac{1}{\delta^\alpha} \int_{-\delta}^{\delta} |f(x)|^2\, dx < \infty \right\}.$$

For example, functions of the form $f(x) = \chi_{(a,b)}(x)|x|^{\alpha/2}, b > a > 0$ are representatives of this space. Indeed, for any $\delta > 0$ we get

$$\sigma_f(-\delta, \delta] = \begin{cases} 0 & 0 < \delta \le a \\ \frac{\delta^{\alpha+1} - a^{\alpha+1}}{\alpha+1} & a < \delta \le b \\ \frac{b^{\alpha+1} - a^{\alpha+1}}{\alpha+1} & b < \delta. \end{cases}$$

Hence, we conclude that $\sigma_f(-\delta, \delta] \le \frac{b^{\alpha+1} - a^{\alpha+1}}{b^\alpha} \frac{\delta^\alpha}{\alpha+1}$ for all $\delta > 0$. Denote by $\tilde{\mathcal{X}}_\alpha$ vector subspace of linear combinations of such functions, i.e.,

$$\tilde{\mathcal{X}}_\alpha = \{ f \in L_2(\mathbb{R}) :\ f(x) = |x|^{\alpha/2} \sum_{i=1}^{n} c_i \chi_{(a_i,b_i)}, n \in \mathbb{N}, a_1 < b_1 \le \ldots \le a_n < b_n \}.$$

Using this subspace, we will show that the constant $\frac{\rho(\alpha)}{2^\alpha}$ in the second parts of Theorems 1 and 2 cannot be improved. Namely, the following assertion is true.

Proposition 5. *There exist a Hilbert space $\mathcal{H}$, a unitary group $\{U^t\}_{t\in\mathbb{R}}$, acting on $\mathcal{H}$, and a vector subspace $\mathcal{X} \subseteq \mathcal{H}$ with its own norm $\|\cdot\|_{\mathcal{X}}$ such that:*
1) $\|P_{t,s} - P\|^2_{\mathcal{X}\to\mathcal{H}} = B(t-s)^{-\alpha}$ for all $t > s$;
2) for any $\varepsilon \in (0,1)$ there exist a function $f \in \mathcal{X}$ and $\delta > 0$, for which

$$\sigma_{f-f^*}(-\delta, \delta] > \varepsilon \frac{\rho(\alpha)}{2^\alpha} B \|f\|^2_{\mathcal{X}} \delta^\alpha.$$

Proof of Proposition 5. Let's take $\mathcal{H} = L_2(\mathbb{R})$, group $U^t f(x) = e^{itx} \cdot f(x)$, and the subspace $\mathcal{X} = \tilde{\mathcal{X}}_\alpha, \alpha \in [0,2]$. Define the norm in $\mathcal{X}$ by the equality

$$\|f\|_{\mathcal{X}} := \|f(x)|x|^{-\alpha/2}\|_2.$$

Since the only fixed vector of the group under consideration is the zero function, then $P = 0$. Let us find corresponding norm of the operator $P_{t,s}$, where

$$P_{t,s}f = \frac{1}{t-s} \int_s^t U^\tau f(x)\, d\tau = \frac{e^{itx} - e^{isx}}{ix(t-s)} f(x).$$

Setting $2\tau = t - s$, we get

$$\begin{aligned}\|P_{t,s}f\|^2_{\mathcal{H}} &= \int_{\mathbb{R}} \frac{4\sin^2(x(t-s)/2)}{x^2(t-s)^2}|f(x)|^2\,dx = \sum_{i=1}^n |c_i|^2 \int_{a_i}^{b_i} F_{t-s}(x)|x|^\alpha\,dx = \\ &= \frac{1}{\tau^{\alpha+1}}\sum_{i=1}^n |c_i|^2 \int_{\tau a_i}^{\tau b_i} \frac{\sin^2 y}{y^{2-\alpha}}\,dy \le \frac{1}{\tau^\alpha}\sum_{i=1}^n |c_i|^2(b_i - a_i) \sup_{x\in(\tau a_i,\tau b_i)} \frac{\sin^2 y}{y^{2-\alpha}} \le \\ &\le \frac{1}{\tau^\alpha}\sup_{x>0}\frac{\sin^2 y}{y^{2-\alpha}}\sum_{i=1}^n |c_i|^2(b_i-a_i) = \frac{2^\alpha}{\rho(\alpha)}\|f\|^2_{\mathcal{X}}(t-s)^{-\alpha}.\end{aligned}$$

In this way, $\|P_{t,s}\|^2_{\mathcal{X}\to\mathcal{H}} \le \frac{2^\alpha}{\rho(\alpha)}(t-s)^{-\alpha}$. In fact, we have equality here, since the supremum (in the definition of operator norm) will be arrived as $\nu \to 0+$ on the family of functions $f_\nu(x) = |x|^{\alpha/2}\chi_{(\frac{r-\nu}{\tau},\frac{r+\nu}{\tau})}(x)$, where $\rho(\alpha) = \inf\limits_{x>0}\frac{x^{2-\alpha}}{\sin^2 x} = \frac{r^{2-\alpha}}{\sin^2 r}$.

Let us now fix $\varepsilon \in (0,1)$. We need to present the function $f \in \mathcal{X}$ and $\delta > 0$ such that $\sigma_f(-\delta,\delta] > \varepsilon\|f\|^2_{\mathcal{X}}\delta^\alpha$.

Take $f(x) = |x|^{\alpha/2}\chi_{(1,1+\nu)}$ and $\delta = 1+\nu$; then (as we already calculated above) $\sigma_f(-\delta,\delta] = \frac{(1+\nu)^{\alpha+1}-1}{\alpha+1}$ and $\|f\|^2_{\mathcal{X}} = \nu$. Therefore, it remains to prove that there exists a sufficiently small number $\nu = \nu(\varepsilon) > 0$ such that

$$\frac{(1+\nu)^{\alpha+1}-1}{\alpha+1} > \varepsilon\nu(1+\nu)^\alpha.$$

Applying Lagrange's mean value theorem to the function $x^{1+\alpha}$ on $[1, 1+\nu]$, for some $c \in (1, 1+\nu)$ we rewrite the last inequality in the form $c^\alpha > \varepsilon(1+\nu)^\alpha$. For $\alpha = 0$ the inequality holds for any $\nu > 0$. For $\alpha \in (0,2]$ we can take any $\nu > 0$ satisfying the inequality $1 \ge \varepsilon(1+\nu)^\alpha$, i.e., $\nu \le \varepsilon^{-1/\alpha} - 1$. □

4. Applications

4.1. As already mentioned in the introduction, in [4] the following theorem was proved (with applications to obtaining decay estimates for time averages of solutions of the linear Schrödinger and wave equations).

Theorem 4. *Let $\mathcal{X} \subset \mathcal{H}$ be a Banach space that is dense in $\mathcal{H}$ and is continuously embedded in it, i.e., $\|\cdot\|_{\mathcal{H}} \le \|\cdot\|_{\mathcal{X}}$, and has the following properties:*

1) there exist a number $r \in (0,1)$ and a function $\psi : [-r,r] \to \mathbb{R}$, which is strictly positive a.e. on the interval $I_r = [-r,r]$ and bounds from above the density of the spectral measure on the set $I_r \setminus \{0\}$:

$|\frac{d}{d\lambda}(E(\lambda)f,g)| \le \psi(\lambda)\|f\|_{\mathcal{X}}\|g\|_{\mathcal{X}}$ for all $f,g \in \mathcal{X}$ and all $\lambda \in I_r \setminus \{0\}$;

2) there exists a number $q > 0$ such that $|\lambda|^{-q}\psi(\lambda) \in L_1(I_r)$.

Then, for $l = \min\{q,2\}$, a power-law uniform convergence on the space $\mathcal{X}$ in the von Neumann ergodic theorem holds:

$$\|P_{t,-t} - P\|_{\mathcal{X}\to\mathcal{H}} \le Ct^{-l/2} \textit{ for all } t > 1,$$

where we can put $C = \sqrt{\Psi_q(r) + \frac{1}{r^2}}$, $\Psi_q(r) = \int_{I_r}|\lambda|^{-q}\psi(\lambda)d\lambda$.

Remark 6. *From the conditions of Theorem 4, it follows a local power with exponent q estimate of the spectral measure, namely: for all $\delta \in (0, r)$*

$$\sigma_{f-f^*}(-\delta, \delta] = \sigma_f(-\delta, \delta] - \|f^*\|_{\mathcal{H}}^2 =$$

$$= \int\limits_{(-\delta,\delta]\setminus\{0\}} d(E(\lambda)f, f) = \int\limits_{(-\delta,\delta]\setminus\{0\}} \frac{d}{d\lambda}(E(\lambda)f, f)d\lambda \leq$$

$$\leq \int\limits_{(-\delta,\delta]\setminus\{0\}} \psi(\lambda)\|f\|_{\mathcal{X}}^2 d\lambda = ||f||_{\mathcal{X}}^2 \int\limits_{(-\delta,\delta]\setminus\{0\}} |\lambda|^q|\lambda|^{-q}\psi(\lambda)d\lambda \leq$$

$$\leq ||f||_{\mathcal{X}}^2\delta^q \int_{I_r} |\lambda|^{-q}\psi(\lambda)d\lambda = \|f\|_{\mathcal{X}}^2\Psi_q(r)\delta^q.$$

4.2. Local power singularity of the spectral measures. Theorem 4 and Remark 6 lead to the natural problem of constructing of analogues of Theorem 2 and Proposition 3 for the case when there is only a local power estimate of the spectral measure. Lemma 3 allows to transfer our results to this case of local estimates.

Theorem 5. *Let $\alpha \geq 0$, $\mathcal{X} \subseteq \mathcal{H}$ be a vector subspace of $\mathcal{H}$ with its own norm $\|\cdot\|_{\mathcal{X}}$, which is continuously embedded in $\mathcal{H}$, i.e., without loss of generality, we assume $\|\cdot\|_{\mathcal{H}} \leq \|\cdot\|_{\mathcal{X}}$.*

If for some $r > 0$ there exists a positive constant A such that for all $f \in \mathcal{X}$ and all $\delta \in (0, r)$ the following inequality holds:

$$\sigma_{f-f^*}(-\delta, \delta] \leq A\|f\|_{\mathcal{X}}^2\delta^\alpha,$$

then there is a uniform convergence on the space $\mathcal{X}$ in the von Neumann ergodic theorem: for $D = \max\{A, \frac{1}{r^\alpha}\}$

1) in the case $\alpha \in [0, 2)$ for all $t > s$

$$\|P_{t,s} - P\|_{\mathcal{X}\to\mathcal{H}}^2 \leq B(t-s)^{-\alpha}, \text{ where we can put } B = \frac{2^{\alpha+1}}{2-\alpha}D;$$

2) in the case $\alpha = 2$ for all $t \geq s + 2$

$$\|P_{t,s} - P\|_{\mathcal{X}\to\mathcal{H}}^2 \leq B\frac{\ln(t-s)}{(t-s)^2}, \text{ where we can put } B = 8D + \frac{4}{\ln 2};$$

3) in the case $\alpha > 2$ for all $t \geq s + 2$

$$\|P_{t,s} - P\|_{\mathcal{X}\to\mathcal{H}}^2 \leq B(t-s)^{-2}, \text{ where we can put } B = \frac{8}{\alpha-2}D + 4.$$

4.3. Borel measures on the real line. As Remark 6 shows, Theorem 4 for $q \neq 2$ is, up to the values of the constants, a very special case of Theorem 5. For example, Theorem 5 does not require the Banach property of the embedded normed space $\mathcal{X}$, nor density of the embedding, nor (local) absolute continuity of spectral measures with respect to the Lebesgue measure. In addition, following von Neumann's original formulations [2], our theorems work for general averages $P_{t,s}$ (for $P_{t,0}$, for example — not just for $P_{t,-t}$, as in [4]), and contain statements about the unimprovability of the obtained asymptotics of the convergence rates.

While the estimate in the case $q = 2$ under the conditions of Theorem 4 turns out to be more accurate than in part 2 of Theorem 5. It turns out that the point here is a specific statement of the problem: in [4] the singularity of the spectral measure at zero was estimated in terms of the finiteness of the integral of its density

$\phi(x)$ multiplied by the power function $|x|^{-q}$. In the final part of the article, we will analyze the relationship between these two and other possible approaches to problem setting.

We will carry out the reasoning in the general case, introducing for all $q \geq 0$ the following classes $\mathcal{K}_j^q$, $j = 1, 2, 3, 4$, of finite Borel measures μ on the real line.

The class $\mathcal{K}_1^q$ contains all finite Borel measures μ on $\mathbb{R}$ absolutely continuous with respect to the Lebesgue measure on the set $[-r, r] \setminus \{0\}$ for some $r \in (0, 1)$ whose density $\phi(x)$ satisfies the condition $\int_{-r}^{r} |x|^{-q} \phi(x)\, dx < \infty$.

The class $\mathcal{K}_2^q$ contains all finite Borel measures μ on $\mathbb{R}$ for which

$$\int_{\mathbb{R}\setminus\{0\}} |x|^{-q}\, d\mu(x) < \infty.$$

The class $\mathcal{K}_3^q$ contains all finite Borel measures μ on $\mathbb{R}$ for which

$$\int_{\mathbb{R}\setminus\{0\}} \frac{\sin^2(\tau x)}{(\tau x)^2}\, d\mu(x) \leq B\tau^{-q}$$

for all $\tau > 0$ and some constant $B > 0$.

Finally, the class $\mathcal{K}_4^q$ contains all finite Borel measures μ on $\mathbb{R}$ for which

$$\mu\{(-\delta, \delta] \setminus \{0\}\} \leq A\delta^{\alpha}$$

for all $\delta > 0$ and some constant $A > 0$.

It is clear that $\mathcal{K}_2^0 = \mathcal{K}_3^0 = \mathcal{K}_4^0$ and they coincide with the set of all finite Borel measures on $\mathbb{R}$. The class $\mathcal{K}_1^0$ consists of all locally (in a punctured neighborhood of zero) absolutely continuous measures. It is easy to see that Theorem 4 considers spectral measures from the class $\mathcal{K}_1^q$ and, in fact, it proved that these measures lie in the class $\mathcal{K}_3^{\min\{q,2\}}$. In our theorems, we consider spectral measures from the class $\mathcal{K}_4^q$ and also prove that they lie in the class $\mathcal{K}_3^q$, but only for $q \neq 2$. For $q = 2$ the inclusion holds in the reverse direction, as it is shown in the following theorem.

Theorem 6. *The following inclusions hold:*
(1) $\mathcal{K}_1^q \subsetneq \mathcal{K}_2^q$ *for any* $q > 0$;
(2) $\mathcal{K}_2^q \subsetneq \mathcal{K}_3^{\min\{q,2\}}$ *for any* $q \neq 2$, *and* $\mathcal{K}_2^2 = \mathcal{K}_3^2$;
(3) $\mathcal{K}_2^q \subsetneq \mathcal{K}_4^q$ *and* $\mathcal{K}_4^{q+p} \subsetneq \mathcal{K}_2^q$ *for all* $q, p > 0$;
(4) $\mathcal{K}_3^q = \mathcal{K}_4^q$ *for* $q \in [0, 2)$, *and* $\mathcal{K}_4^q \subsetneq \mathcal{K}_3^2 \subsetneq \mathcal{K}_4^2$ *for* $q > 2$;
(5) $\mathcal{K}_3^q$ *for any* $q > 2$ *consists of atomic measures supported at the point* 0.

Proof of Theorem 6. (1) Let $\mu \in \mathcal{K}_1^q$; then

$$\begin{aligned}\int_{\mathbb{R}\setminus\{0\}} |x|^{-q}\, d\mu(x) &= \int_{[-r,r]\setminus\{0\}} |x|^{-q}\, d\mu(x) + \int_{|x|>r} |x|^{-q}\, d\mu(x) \leq \\ &\leq \int_{[-r,r]\setminus\{0\}} |x|^{-q}\phi(x)\, dx + \int_{|x|>r} r^{-q}\, d\mu(x) \leq \\ &= \int_{[-r,r]\setminus\{0\}} |x|^{-q}\phi(x)\, dx + r^{-q}\mu(\mathbb{R}) < \infty;\end{aligned}$$

thus, $\mu \in \mathcal{K}_2^q$. As a measure $\mu \in \mathcal{K}_2^q \setminus \mathcal{K}_1^q$ we can take measure $d\mu(x) = x^q \chi_{[0,1]}(x) dk(x)$, where $dk(x)$ is a measure on interval $[0, 1]$, generated by the Cantor function $k(x)$.

(2) Let $\mu \in \mathcal{K}_2^q$ for $q \in [0,2]$; then for any $\tau > 0$

$$\int_{\mathbb{R}\setminus\{0\}} \frac{\sin^2(\tau x)}{(\tau x)^2}\, d\mu(x) = \int_{\mathbb{R}\setminus\{0\}} \frac{\sin^2(\tau x)}{(\tau|x|)^{2-q}} |\tau x|^{-q}\, d\mu(x) \le$$
$$\le \sup_{y>0} \frac{\sin^2 y}{y^{2-q}} \int_{\mathbb{R}\setminus\{0\}} |\tau x|^{-q}\, d\mu(x) = \frac{\tau^{-q}}{\rho(q)} \int_{\mathbb{R}\setminus\{0\}} |x|^{-q}\, d\mu(x).$$

Let now $q > 2$; then for all $\tau > 0$

$$\int_{\mathbb{R}\setminus\{0\}} \frac{\sin^2(\tau x)}{(\tau x)^2}\, d\mu(x) \le \int_{[-1,1]\setminus\{0\}} \frac{1}{(\tau x)^2}\, d\mu(x) + \int_{|x|>1} \frac{1}{(\tau x)^2}\, d\mu(x) \le$$
$$\le \int_{[-1,1]\setminus\{0\}} \frac{1}{\tau^2 |x|^q}\, d\mu(x) + \int_{|x|>1} \frac{1}{\tau^2}\, d\mu(x) \le$$
$$\le \tau^{-2} \left(\int_{\mathbb{R}\setminus\{0\}} |x|^{-q}\, d\mu(x) + \mu(\mathbb{R}) \right).$$

Thus, for any $q \ge 0$ the inclusion $\mathcal{K}_2^q \subset \mathcal{K}_3^{\min\{q,2\}}$ holds. And for $q \ne 2$ the measure $d\mu(x) = x^{q-1}\chi_{[0,1]} dx$ belongs to $\mathcal{K}_3^{\min\{q,2\}} \setminus \mathcal{K}_2^q$.

Let us now prove the equality $\mathcal{K}_2^2 = \mathcal{K}_3^2$. To do this, it suffices to show the reverse inclusion. Suppose $\mu \in \mathcal{K}_3^2$, i.e.,

$$\int_{\mathbb{R}\setminus\{0\}} \frac{\sin^2(\tau x)}{x^2}\, d\mu(x) \le B$$

for all $\tau > 0$ and some $B > 0$. Integrating this inequality over τ on the interval $(0,T)$, $T > 0$ and passing to the second repeated integral (by Tonelli's theorem we can do this), we get

$$2B \ge \frac{2}{T} \int_0^T \left(\int_{\mathbb{R}\setminus\{0\}} \frac{\sin^2(\tau x)}{x^2}\, d\mu(x) \right) d\tau = \frac{2}{T} \int_{\mathbb{R}\setminus\{0\}} x^{-2} \left(\int_0^T \sin^2(\tau x)\, d\tau \right) d\mu(x) =$$

$$= \frac{1}{T} \int_{\mathbb{R}\setminus\{0\}} x^{-2} \left(\int_0^T 1 - \cos(2\tau x)\, d\tau \right) d\mu(x) = \int_{\mathbb{R}\setminus\{0\}} x^{-2} \left(1 - \frac{\sin(2Tx)}{2Tx} \right) d\mu(x).$$

Passing to the limit inferior as $T \to +\infty$, by Fatou's lemma, we obtain the required inequality $2B \ge \int_{\mathbb{R}\setminus\{0\}} x^{-2}\, d\mu(x)$, i.e., $\mu \in \mathcal{K}_2^2$. Note that the idea of this reasoning we took from [24] (see also [25]).

(3) Let $\mu \in \mathcal{K}_2^q$ for $q \ge 0$; then for any $\delta > 0$

$$\mu\{(-\delta,\delta] \setminus \{0\}\} = \int_{(-\delta,\delta]\setminus\{0\}} d\mu(x) = \int_{(-\delta,\delta]\setminus\{0\}} |x|^q |x|^{-q} d\mu(x) \le$$
$$\le \delta^q \int_{(-\delta,\delta]\setminus\{0\}} |x|^{-q} d\mu(x) \le \delta^q \int_{\mathbb{R}\setminus\{0\}} |x|^{-q} d\mu(x);$$

thus, $\mu \in \mathcal{K}_4^q$.

Let now $\mu \in \mathcal{K}_4^{q+p}$ for $q, p > 0$; then

$$\int_{\mathbb{R}\setminus\{0\}} |x|^{-q} d\mu(x) = \int_{\mathbb{R}\setminus\{0\}} q \int_0^{|x|^{-1}} t^{q-1} dt d\mu(x) = q \int_0^{+\infty} t^{q-1} \int_{0<|x|<1/t} d\mu(x) dt =$$

$$= q \int_0^{+\infty} t^{q-1} \mu\{(-1/t, 1/t) \setminus \{0\}\} dt \leq$$

$$\leq q\mu(\mathbb{R}) \int_0^1 t^{q-1} dt + qA \int_1^{+\infty} t^{q-1} t^{-p-q} dt = \mu(\mathbb{R}) + A\frac{q}{p} < \infty;$$

thus $\mu \in \mathcal{K}_2^q$. As a measure $\mu \in \mathcal{K}_2^q \setminus \bigcup_{p>0} \mathcal{K}_4^{q+p}$ we can take measure

$$d\mu(x) = x^{q-1} |\ln x|^{-2} \chi_{[0,1/2]}(x) dx.$$

(4) Proof of class equality $\mathcal{K}_3^q = \mathcal{K}_4^q$ for $q \in [0, 2)$ word-to-word corresponds to the proof of Theorem 1, where spectral measures $\mu = \sigma_{f-f^*}$ were considered. Next, for $q = 2 + p > 2$, using inclusions of items (1) and (2), we obtain

$$\mathcal{K}_4^q \subsetneq \mathcal{K}_2^{2+p/2} \subseteq \mathcal{K}_3^2.$$

The inclusion $\mathcal{K}_3^2 \subset \mathcal{K}_4^2$ already was proved in Theorem 1 using Lemma 2.

(5) The assertion immediately follows from the following Lemma 4. □

Lemma 4. *Let* $\int_{\mathbb{R}\setminus\{0\}} \frac{\sin^2(\tau x)}{x^2} d\mu(x) = o(1)$ *as* $\tau \to +\infty$. *Then* $\mu(\mathbb{R} \setminus \{0\}) = 0$.

Proof of Lemma 4. In terms of the norm of the space $L_2(\mathbb{R} \setminus \{0\}, \mu)$, the condition of the lemma can be rewritten as $\left\| \frac{\sin(\tau x)}{x} \right\|_2 = o(1)$ as $\tau \to +\infty$. Then for any $\tau_0 > 0$ the following three asymptotic relations also hold as $\tau \to +\infty$:

$$\left\| \frac{\sin(\tau x)\sin(\tau_0 x)}{x} \right\|_2 = o(1), \quad \left\| \frac{\sin(\tau x)\cos(\tau_0 x)}{x} \right\|_2 = o(1), \quad \left\| \frac{\sin((\tau + \tau_0)x)}{x} \right\|_2 = o(1).$$

From the second and third relations, expanding the sine of the sum and using the triangle inequality for the norm, we get

$$\left\| \frac{\cos(\tau x)\sin(\tau_0 x)}{x} \right\|_2 = o(1).$$

Squaring this and adding it to the first of the three relations, we get the equality

$$\int_{\mathbb{R}\setminus\{0\}} \frac{\sin^2(\tau_0 x)}{x^2} d\mu(x) = 0,$$

which is valid for any $\tau_0 > 0$. Therefore,

$$\int_{\mathbb{R}\setminus\{0\}} \frac{\sin^2(x) + \sin^2(\pi x)}{x^2} d\mu(x) = 0.$$

Since the integrand is positive, then $\mu(\mathbb{R} \setminus \{0\}) = 0$. □

4.4. Maximum possible convergence rate: power-law with $\alpha = 2$. The following remark is well known (and can be proved for actions of groups of linear isometries on non-Hilbert spaces — see, for example, [26]).

Remark 7. *Using Lemma 4 and integral representation (1), it follows immediately that the rate of convergence $O((t-s)^{-2})$ in the von Neumann ergodic theorem is the maximum possible, i.e., that asymptotic relation $\|P_{t,s}f - f^*\|_{\mathcal{H}}^2 = o((t-s)^{-2})$ as $t - s \to \infty$ holds in the degenerate case $f = f^*$ only.*

Taking into account Remark 7, from proved in item 2 of Theorem 6 equality $\mathcal{K}_2^2 = \mathcal{K}_3^2$, we immediately obtain the next spectral criterion of the maximum rate of convergence in von Neumann's ergodic theorem: $\|P_{t,s}f - f^*\|_{\mathcal{H}}^2 = O(t-s)^{-2}$ as $t-s \to \infty$ if and only if the integral $\int_{\mathbb{R}} x^{-2}\, d\sigma_{f-f^*}(x)$ is finite. We will formulate this result in three variants: for one-dimensional subspaces (Theorem 7 — analogue of Theorem 1), for multidimensional subspaces (Theorem 8 — analogue of Theorem 2), and Theorem 9, which allows us to give a local version of Theorem 8.

Theorem 7. *Let's fix $f \in \mathcal{H}$. Then:*

1. If

$$\int_{\mathbb{R}} x^{-2}\, d\sigma_{f-f^*}(x) = A < \infty,$$

then the rate of convergence of ergodic averages $P_{t,s}f$ is power-law with exponent 2, i.e., for all $t > s$

$$\|P_{t,s}f - f^*\|_{\mathcal{H}}^2 \le B(t-s)^{-2},$$

where we can put $B = 4A$.

2. If the rate of convergence of ergodic averages $P_{t,s}f$ is power-law with exponent 2, i.e., if for some positive constant B for all $t > s$ the following inequality holds:

$$\|P_{t,s}f - f^*\|_{\mathcal{H}}^2 \le B(t-s)^{-2},$$

then

$$\int_{\mathbb{R}} x^{-2}\, d\sigma_{f-f^*}(x) \le A,$$

where we can put $A = 8B$.

Proof of Theorem 7. If $\int_{\mathbb{R}} x^{-2}\, d\sigma_{f-f^*}(x) = A < \infty$, then (see the proof of item 2 of Theorem 6) by representation (1) for all $t > s$

$$\|P_{t,s}f - f^*\|_{\mathcal{H}}^2 = \int_{\mathbb{R}} \left(\frac{\sin \frac{(t-s)x}{2}}{\frac{(t-s)x}{2}} \right)^2 d\sigma_{f-f^*}(x) \le$$

$$\le \frac{\left(\frac{t-s}{2}\right)^{-2}}{\rho(2)} \int_{\mathbb{R}} x^{-2}\, d\sigma_{f-f^*}(x) = 4A(t-s)^{-2}.$$

On the other hand, if for some constant $B > 0$ for all $t > s$

$$\|P_{t,s}f - f^*\|_{\mathcal{H}}^2 \le B(t-s)^{-2},$$

then this inequality, taking into account representation (1), can be rewritten as

$$\int_{\mathbb{R}} \left(\frac{\sin \frac{(t-s)x}{2}}{\frac{(t-s)x}{2}} \right)^2 d\sigma_{f-f^*}(x) \le 4B \left(\frac{t-s}{2} \right)^2,$$

and according to the estimates of the proof of item 2 of Theorem 6, we obtain $\int_{\mathbb{R}} x^{-2}\, d\sigma_{f-f^*}(x) \le 8B$. □

Theorem 8. *Let $\mathcal{X} \subseteq \mathcal{H}$ be a vector subspace of $\mathcal{H}$ with its own norm $\|\cdot\|_{\mathcal{X}}$. Then:*

1. If for some positive constant A for all $f \in \mathcal{X}$ the following inequality holds

$$\int_{\mathbb{R}} x^{-2}\, d\sigma_{f-f^*}(x) \le A\|f\|_{\mathcal{X}}^2,$$

then there is power-law with exponent 2 uniform convergence on the space $\mathcal{X}$ in the von Neumann ergodic theorem: for all $t > s$

$$\|P_{t,s} - P\|_{\mathcal{X}\to\mathcal{H}}^2 \le B(t-s)^{-2},$$

where we can put $B = 4A$.

2. If there is power-law with exponent 2 uniform convergence on the space $\mathcal{X}$ *in the von Neumann ergodic theorem, i.e., for some positive constant A for all* $f \in \mathcal{X}$ *the following inequality holds:*

$$\|P_{t,s} - P\|^2_{\mathcal{X}\to\mathcal{H}} \le B(t-s)^{-2},$$

then for all $f \in \mathcal{X}$

$$\int_{\mathbb{R}} x^{-2}\, d\sigma_{f-f^*}(x) \le A\|f\|^2_{\mathcal{X}},$$

where we can put $A = 8B$.

Proof of Theorem 8. Under the conditions of the first part of the theorem, from the statement of the first part of Theorem 7 we immediately obtain that for all $f \in \mathcal{X}$ and all $t > s$

$$\|(P_{t,s} - P)f\|^2_{\mathcal{H}} \le 4A\|f\|^2_{\mathcal{X}}(t-s)^{-2};$$

therefore,

$$\|P_{t,s} - P\|^2_{\mathcal{X}\to\mathcal{H}} = \sup_{f\in\mathcal{X}: f\neq 0} \frac{\|(P_{t,s} - P)f\|^2_{\mathcal{H}}}{\|f\|^2_{\mathcal{X}}} \le 4A(t-s)^{-2}.$$

Under the conditions of the second part of the theorem, for all $f \in \mathcal{X}$ and all $t > s$

$$\frac{\|(P_{t,s} - P)f\|^2_{\mathcal{H}}}{\|f\|^2_{\mathcal{X}}} \le B(t-s)^{-2},$$

and from the statement of second part of Theorem 7 it follows that

$$\int_{\mathbb{R}} x^{-2}\, d\sigma_{f-f^*}(x) \le 8B\|f\|^2_{\mathcal{X}}.$$

□

Theorem 9. *Let* $\mathcal{X} \subseteq \mathcal{H}$ *be a vector subspace of* $\mathcal{H}$ *with its own norm* $\|\cdot\|_{\mathcal{X}}$, *which is continuously embedded in* $\mathcal{H}$, *i.e., without loss of generality, we assume* $\|\cdot\|_{\mathcal{H}} \le \|\cdot\|_{\mathcal{X}}$.

If for some $r > 0$ *there exists a positive constant A such that for all* $f \in \mathcal{X}$

$$\int_{(-r,r]} x^{-2}\, d\sigma_{f-f^*}(x) \le A\|f\|^2_{\mathcal{X}},$$

then there is power-law with exponent 2 uniform convergence on the space $\mathcal{X}$ *in the von Neumann ergodic theorem: for all* $t > s$

$$\|P_{t,s} - P\|^2_{\mathcal{X}\to\mathcal{H}} \le B(t-s)^{-2},$$

where we can put $B = 4\left(A + \frac{1}{r^2}\right)$.

Proof of Theorem 9. By representation (1), for all $t > s$ we get

$$\|P_{t,s}f - f^*\|^2_{\mathcal{H}} = \int_{\mathbb{R}} \left(\frac{\sin\frac{(t-s)x}{2}}{\frac{(t-s)x}{2}}\right)^2 d\sigma_{f-f^*}(x) \le \frac{4}{(t-s)^2}\int_{\mathbb{R}} x^{-2}\, d\sigma_{f-f^*}(x) =$$

$$= \frac{4}{(t-s)^2}\int_{(-r,r]} x^{-2}\, d\sigma_{f-f^*}(x) + \frac{4}{(t-s)^2}\int_{(-\infty,-r]\cup(r,\infty)} x^{-2}\, d\sigma_{f-f^*}(x) \le$$

$$\le \frac{4}{(t-s)^2}\left(A\|f\|^2_{\mathcal{X}} + \frac{\|f\|^2_{\mathcal{H}}}{r^2}\right) \le \frac{4\|f\|^2_{\mathcal{X}}}{(t-s)^2}\left(A + \frac{1}{r^2}\right).$$

□

4.5. Let's make a few final remarks.

Remark 8. *Following the proofs of items (1) and (2) of Theorem 6, we can slightly improve the constant arising in Theorem 4 for $q \in [0, 2]$.*

Namely, under the conditions of Theorem 4 for all $q \in [0, 2]$ for any $f \in \mathcal{X}$ we have

$$\|P_{t,-t}f - Pf\|_{\mathcal{H}}^2 = \int_{\mathbb{R}\setminus\{0\}} \frac{\sin^2(tx)}{(tx)^2} d\sigma_f(x) \leq \frac{t^{-q}}{\rho(q)} \int_{\mathbb{R}\setminus\{0\}} |x|^{-q}\, d\sigma_f(x) \leq$$

$$\leq \frac{t^{-q}}{\rho(q)} \left(\|f\|_{\mathcal{X}}^2 \int_{[-r,r]\setminus\{0\}} |x|^{-q}\psi(x)\, dx + r^{-q}\sigma_f(\mathbb{R}) \right) =$$

$$= \frac{t^{-q}}{\rho(q)} \left(\|f\|_{\mathcal{X}}^2 \Psi_q(r) + r^{-q}\|f\|_{\mathcal{H}}^2 \right).$$

Therefore, for all $t > 0$

$$\|P_{t,-t} - P\|_{\mathcal{X}\to\mathcal{H}} \leq \sqrt{\frac{\Psi_q(r) + r^{-q}}{\rho(q)}} t^{-q/2}.$$

Theorem 9 gives exactly the same estimate for $q = 2$.

Remark 9. *The reasoning carried out in the proof of Theorem 7 above, allow us to introduce on the space $\mathcal{Y}$ (from the formulation of Theorem 3) a new norm $|||\cdot|||_{\mathcal{Y}}$, equivalent to the norm $\|\cdot\|_{\mathcal{Y}}$, based on the singularity of the spectral measures at zero; namely,*

$$|||f|||_{\mathcal{Y}}^2 = \int_{\mathbb{R}} x^{-2}\, d\sigma_f(x).$$

One of the important tools for studying of finite Borel measures μ on the real line is the Fourier transformation $\hat{\mu}(t) = \int_{\mathbb{R}} e^{itx}\, d\mu(x), t \in \mathbb{R}$. From its asymptotic behavior at infinity, one can conclude on whether the measure belongs to the classes introduced above. We give examples of such statements.

Remark 10. *Every finite Borel measure μ, for which the Fourier transform as $t \to +\infty$ has the power-law rate of convergence of order $O(t^{-q})$ for $q \in [0, 1)$, will belong to the class $\mathcal{K}_3^q$. For spectral measures of semiflows, this result was obtained in* [7, Theorem 2]*; the proof for arbitrary measures is carried out in a similar way.*

And if $\frac{1}{T}\int_0^T |\hat{\mu}(t)|^2\, dt$ has power-law asymptotics $O(T^{-q})$ as $T \to \infty$ for $q \in [0, 1]$, then measure μ is uniformly $q/2$-Hölder (i.e., there exists a constant $C > 0$ such that for any interval I with the Lebesgue measure $|I| < 1$ the inequality $\mu(I) \leq C|I|^{q/2}$ holds) [27, Theorem 3.1]. *In particular, such a measure belongs to the class $\mathcal{K}_4^{q/2}$.*

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