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MSC 17C70DUAL COALGEBRA OF THE DIFFERENTIAL POLYNOMIAL
ALGEBRA IN ONE VARIABLE AND RELATED COALGEBRAS

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ABSTRACT. It is shown that the dual coalgebra of the polynomial algebra in one variable is a differential coalgebra which does not contain non-zero finite-dimensional differentially closed subcoalgebras if the characteristic of ground field is zero. Using this coalgebra, we construct a Novikov coalgebra which is the dual coalgebra of the left-symmetric Witt algebra of index one. Also, we construct a Jordan supercoalgebra which is dual to the Jordan superalgebra of vector type of the polynomial algebra in one variable. All these coalgebras do not contain non-zero finite-dimensional subcoalgebras if the characteristic of ground field is zero. It is shown that over a field of characteristic different from 2 the adjoint Lie coalgebra of the dual coalgebra of the left-symmetric Witt algebra of index one is isomorphic to the dual coalgebra of the Witt algebra of index one.

Key words: coalgebra, coderivation, associative commutative algebra, differential algebra, Novikov algebra, Lie algebra, Witt algebra, Jordan superalgebra, locally finite coalgebra

In general, an algebra is a vector space A over a ground field F equipped with a linear map $m : A \otimes A \rightarrow A$. The dual notion is known as a coalgebra. The theory of coalgebras has been initially developed within the framework of the theory of Hopf algebras [1]. The main result of the theory of associative coalgebras is the Fundamental Theorem on Coalgebras, which asserts that every associative coalgebra over a field is locally finite. The latter means that every finitely generated coalgebra is finite-dimensional.

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The interest to nonassociative coalgebras is related to the notion of a quantum group introduced by Drinfeld [2]. Initially, it was associated with the notion of a Lie bialgebra. The latter was introduced in [3] as one of the most important notions of quantum group theory. A Lie bialgebra is simultaneously a Lie algebra and a Lie coalgebra. In contrast to associative coalgebras there exist Lie coalgebras that are not locally finite [4].

It is known (see [1, 4]) that the dual algebra of an associative coalgebra or a Lie coalgebra is associative or Lie, respectively. In [5] (1994), the notion of a coalgebra related to some variety of algebras was introduced. In particular, alternative and Jordan coalgebras were defined, and their local finiteness was proved. An analogue of this result is true for structurable coalgebras [6], for Jordan copairs [8], for right alternative Malcev admissible coalgebras, and for binary $(-1, 1)$ -coalgebras [9]. In [7] (1995), some necessary and sufficient conditions for a Lie coalgebra to be locally finite were found.

In the papers [10, 11] we found the connection between Jordan and Lie (super)coalgebras, which is an analogue of the well-known Kantor–Koecher–Tits construction for usual (super)algebras. In [12, 13], it was shown that every Malcev coalgebra is embedded into a Lie coalgebra with triality.

As mentioned above, there exist non-locally finite Lie coalgebras. In [14], an example of a non-locally finite right-symmetric coalgebra was constructed. In contrast to Jordan coalgebras, non-locally finite Jordan supercoalgebras exist [11].

In [16], it was constructed an example of a non-locally finite differential coalgebra. On this differential coalgebra we can define a Lie comultiplication so that the Lie coalgebra obtained coincides with the example of Michaelis presented in [4]. The dual analogue of the Gelfand–Dorfman construction was proposed in [16], which implied the construction of Novikov coalgebras based on differential associative commutative coalgebras. Using this construction, an example of a non-locally finite Novikov coalgebra was built. A dual analogue of the Kantor construction for usual Jordan superalgebra was also presented in [16], as a corollary, a new example of a non-locally finite Jordan supercoalgebra was constructed.

In [15] and [16], examples non-locally finite right alternative coalgebras were constructed.

In [17], it was shown that the dual coalgebra W_1° of the Witt Lie algebra W_1 is a non-zero Lie coalgebra which does not contain non-zero finite-dimensional subcoalgebras. An analogue of this result for Jordan supercoalgebra was obtained in [19]. The structure of the dual Lie coalgebra of the Witt algebra was also described in [18].

It is known that both the Witt Lie algebra W_1 and the left-symmetric algebra \mathcal{L}_1 can be obtained from the differential polynomial algebra in one variable by means of appropriate constructions. The purpose of this paper is to obtain the dual analogues of these results for the dual coalgebras W_1° and \mathcal{L}_1° , accordingly.

In particular, we prove that the dual coalgebra P_1° of the differential polynomial algebra P_1 in one variable is a differential coalgebra. Moreover, if the characteristic of ground field is zero then the coalgebra P_1° does not contain finite-dimensional differentially closed subcoalgebras. The coderivation of the coalgebra P_1° induces a Lie comultiplication on the space P_1° , so that the Lie coalgebra obtained is a subcoalgebra of W_1° . Moreover, if the characteristic of ground field is not equal 2 then these coalgebras coincide.

We also show that over field of characteristic different from 2 the dual coalgebra of the left-symmetric Witt algebra of index one can be obtained from the coalgebra P_1^0 by means of the dual Gelfand–Dorfman construction.

Finally, we describe the dual supercoalgebra of a Jordan superalgebra obtained by the Kantor construction from the differential polynomial algebra in one variable.

1. COALGEBRAS AND CODERIVATIONS

Let F be an arbitrary field. Denote by $\underbrace{V \otimes \dots \otimes V}_n$ the n th tensor power of the vector space V over F . Denote by V^* the dual vector space of V , i. e., $V^* = \text{Hom}_F(V, F)$.

The map

$$\rho : \underbrace{V^* \otimes \dots \otimes V^*}_n \rightarrow \underbrace{(V \otimes \dots \otimes V)^*}_n$$

defined by

$$\rho(f_1 \otimes \dots \otimes f_n) \left(\sum_{i_1, \dots, i_n} v_{i_1} \otimes \dots \otimes v_{i_n} \right) = \sum_{i_1, \dots, i_n} f_1(v_{i_1}) \dots f_n(v_{i_n})$$

is injective. Therefore, we can assume that

$$\underbrace{V^* \otimes \dots \otimes V^*}_n \subseteq \underbrace{(V \otimes \dots \otimes V)^*}_n.$$

If $\phi : V \rightarrow U$ is a linear map of vector spaces then the transpose $\phi^* : U^* \rightarrow V^*$ of ϕ is defined by the rule $\phi^*(u^*)(v) = u^*(\phi(v))$, where $v \in V$, $u^* \in U^*$.

Definition 1. A pair (C, Δ) , where C is a vector space over F and $\Delta : C \otimes C \rightarrow C$ is a linear map, is called a coalgebra. The map Δ is said to be a comultiplication of C . For an element $a \in C$, we will use the Sweedler's notation (see [1]), namely, $\Delta(a) = \sum_a a_{(1)} \otimes a_{(2)}$.

If (C, Δ) is a coalgebra then the rule $f \otimes g \mapsto fg$, where

$$(fg)(a) = \rho(f \otimes g)(\Delta(a)) = \sum_a f(a_{(1)})g(a_{(2)}), \quad f, g \in C^*, \quad a \in C,$$

defines a product on C^* , so that C^* is an algebra. The algebra C^* is called the *dual algebra* of (C, Δ) .

The dual algebra C^* has natural left and right actions on the initial coalgebra C . Namely,

$$\alpha \cdot a = \sum_{(a)} a_{(1)} \alpha(a_{(2)}), \quad a \cdot \alpha = \sum_{(a)} \alpha(a_{(1)}) a_{(2)},$$

for $\alpha \in C^*$, $a \in C$. Hence, C is a C^* -bimodule.

A linear map $d : C \rightarrow C$ is called a *coderivation* of a coalgebra (C, Δ) if

$$\Delta d = (d \otimes id + id \otimes d) \Delta,$$

i. e., for every $a \in C$

$$\Delta(d(a)) = \sum_a d(a_{(1)}) \otimes a_{(2)} + a_{(1)} \otimes d(a_{(2)}).$$

Lemma 1. *Let d be a coderivation of a coalgebra (C, Δ) . Then its transpose map d^* is a derivation of the dual algebra C^* , i. e.,*

$$d^*(fg) = d^*(f)g + fd^*(g)$$

holds in the algebra C^* for all $f, g \in C^*$.

Proof. Let $f, g \in C^*$, $a \in C$. Then

$$\begin{aligned} (d^*(fg))(a) &= (fg)(d(a)) = \rho(f \otimes g)\Delta(d(a)) = \\ &= \sum_a f(d(a_{(1)}))g(a_{(2)}) + f(a_{(1)})g(d(a_{(2)})) = \\ &= \sum_a (d^*(f))(a_{(1)})g(a_{(2)}) + f(a_{(1)})(d^*(g))(a_{(2)}) = (d^*(f)g + fd^*(g))(a). \end{aligned}$$

Thus we obtain $d^*(fg) = d^*(f)g + fd^*(g)$. \square

A coalgebra equipped with a coderivation is called a *differential coalgebra*.

A coalgebra (C, Δ) is said to be associative (coassociative) if

$$(\Delta \otimes id - id \otimes \Delta)\Delta = 0,$$

i. e., for every $a \in C$

$$\sum_a (\Delta(a_{(1)}) \otimes a_{(2)} - a_{(1)} \otimes \Delta(a_{(2)})) = 0.$$

It is known that a coalgebra (C, Δ) is associative if and only if its dual algebra C^* is associative. By this reason, the following definition of a coalgebra related to some variety of algebras was stated in [5].

Definition 2. *Let \mathcal{M} be a variety of algebras. Then a coalgebra (C, Δ) is called an \mathcal{M} -coalgebra if its dual algebra C^* is an algebra of \mathcal{M} .*

Let V be a vector space, and let the linear map $\tau : V \otimes V \rightarrow V \otimes V$ is defined by $\tau(x \otimes y) = y \otimes x$, $x, y \in V$.

A coalgebra (C, Δ) is commutative (cocommutative) if

$$\Delta = \tau\Delta,$$

i. e.,

$$\sum_a a_{(1)} \otimes a_{(2)} = \sum_a a_{(2)} \otimes a_{(1)}$$

for every $a \in C$.

Let (C, Δ) be an arbitrary coalgebra. A vector subspace B of C is a *subcoalgebra* of (C, Δ) if $\Delta(B) \subseteq B \otimes B$.

It is known (see [5]) that a vector space B of C is a subcoalgebra if and only if B is a submodule of the C^* -bimodule C . Therefore, the intersection of a family of subcoalgebras of (C, Δ) is again a subcoalgebra of C .

Note that the orthogonal complement

$$B^\perp = \{\alpha \in C^* \mid B \subseteq \ker \alpha\}$$

of a subcoalgebra B of (C, Δ) is an ideal of the algebra C^* . Conversely, the orthogonal complement I^\perp of an ideal I of C^* is a subcoalgebra of (C, Δ) .

Indeed, let B be a subcoalgebra of (C, Δ) and let $\beta \in B^\perp$, $\alpha \in C^*$. Then for every $b \in B$

$$(\beta\alpha)(b) = \sum_{(b)} \beta(b_{(1)})\alpha(b_{(2)}) = 0,$$

since $\sum_{(b)} b_{(1)} \otimes b_{(2)} \in B \otimes B$. Therefore, $\beta\alpha \in B^\perp$. Similarly, $\alpha\beta \in B^\perp$, so B^\perp is an ideal of C^* .

Conversely, let $B = I^\perp$, where I is an ideal of C^* . Then for $W = I \otimes C^* + C^* \otimes I \subseteq C^* \otimes C^* \subseteq (C \otimes C)^*$ we have $W^\perp = B \otimes B$. Obviously, $\rho(\alpha \otimes \beta)(\Delta(a)) = (\alpha\beta)(a) = 0$ if $\alpha \in I$, $\beta \in C^*$ and $a \in B$. Similarly, $\rho(\beta \otimes \alpha)(\Delta(a)) = 0$, so $\Delta(a) \in W^\perp = B \otimes B$.

Let S be a subset of C . The smallest subcoalgebra which contains S is called the *subcoalgebra generated by S* and denoted by $\text{Coalg}(S)$. In other words, $\text{Coalg}(S)$ is the submodule of the C^* -bimodule C , generated by S . If S is a finite set then $\text{Coalg}(S)$ is called finitely generated coalgebra.

Definition 3. A coalgebra (C, Δ) is called *locally finite* if every finitely generated subcoalgebra of C is finite-dimensional.

Let (C, Δ) be a coalgebra. Denote by $\text{Loc}(C)$ the sum of all locally finite subcoalgebras of C . It is clear that $\text{Loc}(C)$ is a locally finite coalgebra.

Let A be an algebra over a field F with a multiplication $m : A \otimes A \rightarrow A$, i. e., $m(a \otimes b) = ab$ for $a, b \in A$. Then $m^* : A^* \rightarrow (A \otimes A)^*$ is the transpose map of m . A vector subspace V of A^* is called *good* if $m^*(V) \subseteq \rho(V \otimes V)$. If V is a good subspace then we can define the comultiplication Δ_V on V by the rule $\Delta_V = \rho^{-1}m^*$.

Let A° be the sum of all good subspaces of A^* . Then A° is the largest good subspace of A^* , and hence the pair (A°, Δ°) is a coalgebra with the comultiplication $\Delta^\circ = \Delta_{A^\circ}$ [4, 5]. The coalgebra (A°, Δ°) is called the *finite dual coalgebra* (or simply *dual coalgebra*) of A . For every $a, b \in A$ and for every $f \in A^\circ$ we have $f(ab) = \sum_f f_{(1)}(a)f_{(2)}(b)$, where $\Delta^\circ(f) = \sum_f f_{(1)} \otimes f_{(2)}$.

If A is finite dimensional then $A^* \otimes A^* \cong \rho(A^* \otimes A^*) = (A \otimes A)^*$ and the dual space A^* is a coalgebra with the comultiplication $\Delta = \rho^{-1}m^*$. Therefore, $(A^\circ, \Delta^\circ) = (A^*, \Delta)$.

In [5], it was shown that the dual coalgebra A° is a \mathcal{M} -coalgebra if A is an algebra of variety \mathcal{M} .

Lemma 2. Let A be an algebra over a field F , and let (V, Δ) be a coalgebra, where $V \subseteq A^*$. If $f(ab) = \rho(\Delta(f))(a \otimes b)$ holds for every $a, b \in A$ and for every $f \in V$ then V is a good subspace of A^* .

Proof. By the definition of m^* , we have $m^*(f)(a \otimes b) = f(ab)$ for all $a, b \in A$, $f \in A^*$. If $f(ab) = \rho(\Delta(f))(a \otimes b)$ holds for $f \in V$ then

$$m^*(f)(a \otimes b) = \rho(\Delta(f))(a \otimes b).$$

Consequently, $m^*(f) = \rho(\Delta(f))$. Thus $m^*(V) \subseteq \rho(V \otimes V)$. \square

Let \mathcal{I} be the set of finite-codimensional ideals of an algebra A . Let us define

$$A_{\mathcal{I}}^* = \{\alpha \in A^* \mid \text{exists } I \in \mathcal{I} \text{ such that } I \subseteq \ker \alpha\}.$$

As it was shown in [5], $A_{\mathcal{I}}^* = \text{Loc}(A^\circ)$. It is known that if A is an associative algebra then $A_{\mathcal{I}}^* = \text{Loc}(A^\circ) = A^\circ$.

2. DUAL COALGEBRA OF THE DIFFERENTIAL POLYNOMIAL ALGEBRA $F[x]$

Let $P_1 = F[x]$ be the polynomial algebra in one variable equipped with the standard derivation $d = \frac{d}{dx}$. Then

$$P_1^\circ = \{\alpha \in P_1^* \mid \text{exists } f(x) \in F[x], f(x) \neq 0 \text{ such that } f(x)F[x] \subseteq \ker \alpha\}.$$

Let $\Delta^\circ = \Delta_{P_1^\circ}$. For $\alpha \in P_1^*$ and $a \in P_1$ put $\langle \alpha, a \rangle = \alpha(a)$. Also, let us denote $d^\circ = d^*|_{P_1^\circ}$, where d^* is the transpose map of the derivation d . Let $(P_1, d)^\circ$ be the sum of all good subspaces of P_1^* which are differential coalgebras with the coderivation d° .

Lemma 3. *The inclusion $d^\circ(P_1^\circ) \subseteq P_1^\circ$ holds. The triple $(P_1^\circ, \Delta^\circ, d^\circ)$ is an associative and commutative differential coalgebra with the coderivation d° . Moreover, $(P_1^\circ, \Delta^\circ, d^\circ) = (P_1, d)^\circ$.*

Proof. Let us show that $d^\circ(P_1^\circ) \subseteq P_1^\circ$. Suppose $u \in P_1^\circ$, then there exists $I = f(x)P_1$, $f(x) \neq 0$, such that $\langle u, I \rangle = 0$. Consider $J = I^2$: this is an ideal of finite codimension in P_1 , and $\langle d^*(u), J \rangle = \langle u, d(J) \rangle \subseteq \langle u, I \rangle = 0$ since $d(I^2) \subseteq Id(I) \subseteq I$.

Obviously, $(P_1^\circ, \Delta^\circ, d^\circ)$ is a differential coalgebra with coderivation d° . Hence, $(P_1^\circ, \Delta^\circ, d^\circ) = (P_1, d)^\circ$. \square

Theorem 1. *Let F be a field of characteristic zero. Then the coalgebra $(P_1^\circ, \Delta^\circ, d^\circ)$ does not contain non-zero finite-dimensional differentially closed subcoalgebras.*

Proof. Let B be a finite-dimensional differentially closed subcoalgebra of the differential coalgebra $(P_1^\circ, \Delta^\circ, d^\circ)$. Then the orthogonal complement B^\perp is a differentially closed ideal of P_1 . Moreover, the ideal B^\perp has finite codimension. Since the characteristic of F is zero then P_1 is a differentially simple algebra. Therefore, either $B^\perp = 0$ or $B^\perp = P_1$. The first option is impossible since P_1 is an infinite-dimensional algebra. Consequently, $B^\perp = P_1$ and $B = 0$. \square

In the algebra P_1 we put $x_i = x^{i+1}$, $i = -1, 0, 1, \dots$. Then we have $x_i \cdot x_j = x_{i+j+1}$ in the algebra P_1 .

We identify elements of P_1^* with sequences of elements of F . Namely, every $\alpha \in P_1^*$ corresponds to $(\alpha(x_n))_{n \geq -1}$.

Following [18], we say that a sequence $(a_n)_{n \geq -1}$ of elements of F is (F) -linearly recursive if there exist $\beta_0, \beta_1, \dots, \beta_r \in F$, not all equal to zero, and a number k such that $\sum_{i=0}^r \beta_i a_{n+i} = 0$ for all $n \geq k$.

Let W_1 be the Witt algebra of index one, i. e., $W_1 = \text{Der}_F(P_1)$; P_1^* and W_1^* are isomorphic as vector spaces.

Lemma 4. *The space P_1° is the space of linearly recursive sequences. Therefore, $P_1^\circ \subseteq W_1^\circ$. Moreover, if F has characteristic 0 or characteristic $p \neq 2$, then $P_1^\circ = W_1^\circ$.*

Proof. Let $\alpha \in P_1^*$ and $a_n = \alpha(x_n)$, where $n \geq -1$.

Assume $\alpha \in P_1^\circ$. We show that the sequence $(a_n)_{n \geq -1}$ is linearly recursive. There exists $f(x) \in P_1$, $f(x) \neq 0$, such that $\alpha(f(x)g(x)) = 0$ for all $g(x) \in P_1$. Let $f(x) = \beta_r x_{r-1} + \dots + \beta_0 x_{-1}$, where $\beta_0, \beta_1, \dots, \beta_r \in F$. Since $f(x) \neq 0$, not all $\beta_0, \beta_1, \dots, \beta_r$ are zero. Then for all $n \geq 0$ we have

$$\sum_{i=0}^r \beta_i a_{n+i} = \sum_{i=0}^r \beta_i \alpha(x_{n+i}) = \sum_{i=0}^r \beta_i \alpha(x^{n+i+1}) = \alpha(f(x)x^{n+1}) = 0.$$

Hence, the sequence $(a_n)_{n \geq -1}$ is linearly recursive.

Let $(a_n)_{n \geq -1}$ be a linearly recursive sequence. Then there exist $\beta_0, \beta_1, \dots, \beta_r \in F$, not all zero, and a number k , such that $\sum_{i=0}^r \beta_i a_{n+i} = 0$ for all $n \geq k$. Put $f(x) = \beta_r x_{k+r} + \dots + \beta_0 x_k$. Then for all $j \geq -1$ we get

$$\alpha(f(x)x_j) = \sum_{i=0}^r \beta_i \alpha(x_{k+i+j+1}) = \sum_{i=0}^r \beta_i a_{k+i+j+1} = 0,$$

since $k+j+1 \geq k$. Consequently, $f(x)P_1 \subseteq \ker \alpha$.

It was shown in [18] that the space of linearly recursive sequences is a subspace of W_1° . Therefore, $P_1^\circ \subseteq W_1^\circ$. Moreover, if F has characteristic 0 or characteristic $p \neq 2$, then W_1° is the space of linearly recursive sequences. Hence, $P_1^\circ = W_1^\circ$. \square

Recall that the vector space P_1 is a Lie algebra relative to the operation

$$[f, g]_d = fd(g) - d(f)g,$$

this Lie algebra is isomorphic to the Witt algebra W_1 .

Define the new comultiplication $\Delta_{d^\circ}^{(-)}$ on the vector space P_1° :

$$\Delta_{d^\circ}^{(-)}(\alpha) = (id \otimes d^\circ - d^\circ \otimes id)\Delta^\circ(\alpha).$$

Then

$$\langle \alpha, [f, g]_d \rangle = \langle \alpha, fd(g) - d(f)g \rangle = \rho(\Delta_{d^\circ}^{(-)}(\alpha))(f \otimes g)$$

for $\alpha \in P_1^\circ$ and $f, g \in P_1$. Therefore, P_1° is a good subspace of $(P_1, [,])^*$. Consequently, we can assume that $(P_1^\circ, \Delta_{d^\circ}^{(-)})$ is a subcoalgebra of $(W_1^\circ, \Delta_{W_1^\circ})$.

Therefore, the following statement holds.

Theorem 2. *Define the comultiplication*

$$\Delta_{d^\circ}^{(-)}(\alpha) = (id \otimes d^\circ - d^\circ \otimes id)\Delta^\circ(\alpha).$$

on the vector space P_1° . Then $(P_1^\circ, \Delta_{d^\circ}^{(-)})$ is a subcoalgebra of the dual coalgebra of the Witt algebra $(W_1^\circ, \Delta_{W_1^\circ})$. Moreover, if F is a field of characteristic $\neq 2$ then $(P_1^\circ, \Delta_{d^\circ}^{(-)}) = (W_1^\circ, \Delta_{W_1^\circ})$.

Following [17], we define elements y_n , of W_1^* , $n \geq -1$, by $y_n(x_i) = \delta_{n,i}$. Define $Y = \text{span}(y_n \mid n \geq -1)$, and let Δ_Y be the restriction of m^* in P_1^* on the space Y . Then

$$\Delta_Y(y_n) = \sum_{i+j=n-1} y_i \otimes y_j.$$

By Lemma 2, Y is a good subspace of P_1^* . Consequently, (Y, Δ_Y) is a subcoalgebra of (P_1^*, Δ°) .

Now following [18], define $y_{a,j} \in P_1^*$, for $0 \neq a \in F$, $j \geq -1$, by putting $y_{a,j}(x_i) = a^i \binom{i+1}{j+1}$. Define $Y_a = \text{span}(y_{a,j} \mid j \geq -1)$, and let Δ_{Y_a} be the restriction of m^* in P_1^* on the space Y_a . For the sake of uniformity, we set $Y_0 = Y$.

Lemma 5. *The space Y_a is a good subspace of P_1^* , and $(Y_a, \Delta_{Y_a}, d^\circ)$ is a differential coalgebra. Consequently, $(Y_a, \Delta_{Y_a}, d^\circ) \subseteq (P_1, d)^\circ$. If F is an algebraically closed field of characteristic zero then $P_1^\circ = \bigoplus_{a \in F} Y_a$*

Proof. First let $a = 0$. We prove that $d^\circ(y_k) = (k+2)y_{k+1}$ for $k \geq -1$. Indeed, if $i > -1$ then

$$d^\circ(y_k)(x_i) = y_k(d((x_i))) = (i+1)y_k(x_{i-1}) = (i+1)y_{k+1}(x_i) = (k+2)y_{k+1}(x_i).$$

If $i = -1$ then $d^\circ(y_k)(x_{-1}) = y_k(d(1)) = 0$. On the other hand $y_{k+1}(x_{-1}) = \delta_{k+1,-1} = 0$. Hence, $d^\circ(y_k)(x_{-1}) = (k+2)y_{k+1}(x_{-1})$.

It is clearly that d° is a coderivation of (Y, Δ_Y) . Therefore, (Y, Δ_Y, d°) is a good subspace of $(P_1, d)^*$. Consequently, $(Y, \Delta_Y, d^\circ) \subseteq (P_1, d)^\circ$.

Let $0 \neq a \in F$. Recall that for binomial coefficients we have

$$\binom{r+s+2}{k+1} = \sum_{i+j=k-1, i, j \geq -1} \binom{r+1}{i+1} \binom{s+1}{j+1},$$

where $r, s, k \geq -1$.

The comultiplication Δ_{Y_a} on the space Y_a is given by

$$\Delta_{Y_a}(y_{a,n}) = \sum_{i+j=n-1, i, j \geq -1} a y_{a,i} \otimes y_{a,j}.$$

Indeed,

$$\begin{aligned} \rho(\Delta_{Y_a}(y_{a,k}))(x_r \otimes x_s) &= \sum_{i+j=k-1, i, j \geq -1} a y_{a,i}(x_r) y_{a,j}(x_s) = \\ &= \sum_{i+j=k-1, i, j \geq -1} a^{r+s+1} \binom{r+1}{i+1} \binom{s+1}{j+1} = a^{r+s+1} \binom{r+s+2}{k+1} = \\ &= y_{a,k}(x_{r+s+1}) = y_{a,k}(x_r \cdot x_s). \end{aligned}$$

Therefore, Y_a is a good subspace of P_1^* . Consequently, (Y_a, Δ_{Y_a}) is a subcoalgebra of $(P_1^\circ, \Delta^\circ)$.

Let us show that for the coderivation d° we have $d^\circ(y_{a,k}) = a^{-1}(k+2)y_{a,k+1}$. Indeed, if $i > -1$ then

$$\begin{aligned} d^\circ(y_{a,k})(x_i) &= y_{a,k}(d(x_i)) = (i+1)y_{a,k}(x_{i-1}) = a^{i-1}(i+1) \binom{i}{k+1} = \\ &= a^{i-1}(i+1) \frac{i!}{(k+1)!(i-k-1)!} = a^{-1}(k+2)a^i \frac{(i+1)!}{(k+2)!(i+1-k-2)!} = \\ &= a^{-1}(k+2)y_{a,k+1}(x_i). \end{aligned}$$

If $i = -1$ then $d^\circ(y_{a,k})(x_{-1}) = y_{a,k}(d(1)) = 0$. On the other hand, $y_{a,k+1}(x_{-1}) = a^{-1} \binom{0}{k+2} = 0$. Therefore, $d^\circ(y_{a,k})(x_{-1}) = a^{-1}(k+2)y_{a,k+1}(x_{-1})$.

Consequently, $(Y_a, \Delta_{Y_a}, d^\circ)$ is a good subspace of $(P_1, d)^*$, so $(Y_a, \Delta_{Y_a}, d^\circ) \subseteq (P_1, d)^\circ$.

If F is an algebraically closed field of characteristic zero then Theorems 4 and 5 from [18] imply $W_1^\circ = \bigoplus_{a \in F} Y_a$. Hence, $P_1^\circ = \bigoplus_{a \in F} Y_a$. \square

An algebra (A, \circ) with a multiplication operation \circ is called a *Novikov algebra* (see [20], [21]) if A satisfies the following identities:

$$\begin{aligned} x \circ (y \circ z) - (x \circ y) \circ z &= y \circ (x \circ z) - (y \circ x) \circ z \quad (\text{left symmetry}), \\ (x \circ y) \circ z &= (x \circ z) \circ y \quad (\text{right commutativity}). \end{aligned}$$

Define the operation of multiplication \circ on P_1 by the rule $f \circ g = fd(g)$, $f, g \in P_1$. Then (P_1, \circ) is a Novikov algebra (see [20]). Following [22], we will call it left-symmetric Witt algebra of index 1 and denote by \mathcal{L}_1 . Let $x_i = x^{i+1}$, $i = -1, 0, 1, \dots$. Then the equality $x_i \circ x_j = (j+1)x_{i+j}$ holds in the algebra \mathcal{L}_1 . It is known that the algebra \mathcal{L}_1 is simple, if F is field of characteristic zero.

Define the comultiplication Δ_N on P_1° by the rule

$$\Delta_N(\alpha) = (id \otimes d^\circ)\Delta^\circ(\alpha).$$

As it was shown in [16], the coalgebra (P_1°, Δ_N) is a Novikov coalgebra. Obviously, P_1° is a good subspace of \mathcal{L}_1^* . Therefore, $(P_1^\circ, \Delta_N) \subseteq (\mathcal{L}_1^\circ, \Delta_{\mathcal{L}_1^\circ})$.

The following Theorem holds.

Theorem 3. *Let $\Delta_N = (id \otimes d^\circ)\Delta^\circ$. Then $(P_1^\circ, \Delta_N) \subseteq (\mathcal{L}_1^\circ, \Delta_{\mathcal{L}_1^\circ})$. Therefore, the dual coalgebra $(\mathcal{L}_1^\circ, \Delta_{\mathcal{L}_1^\circ})$ of \mathcal{L}_1 is a non-zero Novikov coalgebra. If F is a field of characteristic zero then the coalgebra $(\mathcal{L}_1^\circ, \Delta_{\mathcal{L}_1^\circ})$ does not contain non-zero proper finite-dimensional subcoalgebras.*

Let $f, g \in \mathcal{L}_1$. It is easy to see that in the algebra \mathcal{L}_1 we have

$$[f, g] = f \circ g - g \circ f = fd(g) - d(f)g = [f, g]_d.$$

Consequently, the algebra $(\mathcal{L}_1, [,])$ is isomorphic to the Witt algebra W_1 .

Define the new comultiplication $\Delta_{\mathcal{L}_1^\circ}^{(-)} = (1 - \tau)\Delta_{\mathcal{L}_1^\circ}$ on the vector space \mathcal{L}_1° . Then for every $f, g \in P_1$ and $\alpha \in \mathcal{L}_1^\circ$ we have

$$\langle \alpha, [f, g] \rangle = \langle \alpha, f \circ g - g \circ f \rangle = \rho(\Delta_{\mathcal{L}_1^\circ}^{(-)}(\alpha))(f \otimes g).$$

Therefore, $(\mathcal{L}_1^\circ, \Delta_{\mathcal{L}_1^\circ}^{(-)})$ is a good subspace of $(\mathcal{L}_1, [,])^*$, and we can assume that $(\mathcal{L}_1^\circ, \Delta_{\mathcal{L}_1^\circ}^{(-)})$ is a good subspace of W_1^* . Consequently, $(\mathcal{L}_1^\circ, \Delta_{\mathcal{L}_1^\circ}^{(-)}) \subseteq (W_1^\circ, \Delta_{W_1^\circ})$.

Theorem 3 implies $(P_1^\circ, \Delta_{d^\circ}^{(-)}) \subseteq (\mathcal{L}_1^\circ, \Delta_{\mathcal{L}_1^\circ}^{(-)})$.

The following statement holds.

Theorem 4. *Let F be a field of characteristic $\neq 2$. Then $(P_1^\circ, \Delta_N) = (\mathcal{L}_1^\circ, \Delta_{\mathcal{L}_1^\circ})$ and $(W_1^\circ, \Delta_{W_1^\circ}) = (\mathcal{L}_1^\circ, \Delta_{\mathcal{L}_1^\circ}^{(-)})$, where $\Delta_{\mathcal{L}_1^\circ}^{(-)} = (1 - \tau)\Delta_{\mathcal{L}_1^\circ}$.*

Proof. By Theorem 3, we have $P_1^\circ \subseteq \mathcal{L}_1^\circ$. It was shown that $\mathcal{L}_1^\circ \subseteq W_1^\circ$. Let F be a field of characteristic $\neq 2$. Then, by Lemma 4, $P_1^\circ = W_1^\circ$. Consequently, $P_1^\circ = \mathcal{L}_1^\circ = W_1^\circ$. Therefore, $(W_1^\circ, \Delta_{W_1^\circ}) = (\mathcal{L}_1^\circ, \Delta_{\mathcal{L}_1^\circ}^{(-)})$.

Let us prove that $\Delta_{\mathcal{L}_1^\circ} = \Delta_N$. Suppose f, g are arbitrary polynomials in P_1 , and $\alpha \in \mathcal{L}_1^\circ$. Then

$$\langle \alpha, fd(g) \rangle = \rho(\Delta^\circ(\alpha))(f \otimes d(g)) = \rho((id \otimes d^\circ)\Delta^\circ(\alpha))(f \otimes g) = \rho(\Delta_N(\alpha))(f \otimes g).$$

On the other hand,

$$\langle \alpha, fd(g) \rangle = \langle \alpha, f \circ g \rangle = \rho(\Delta_{\mathcal{L}_1^\circ}(\alpha))(f \otimes g).$$

Hence, $\Delta_N = \Delta_{\mathcal{L}_1^\circ}$, so $(\mathcal{L}_1^\circ, \Delta_{\mathcal{L}_1^\circ}) = (P_1^\circ, \Delta_N)$. □

3. DUAL COALGEBRA OF THE KANTOR CONSTRUCTION OF POLYNOMIAL ALGEBRA IN ONE VARIABLE

Let G be the Grassmann algebra with identity. The algebra $G = G_0 + G_1$ is a \mathbb{Z}_2 -graded algebra. Let $J = J_0 + J_1$ be a \mathbb{Z}_2 -graded algebra. Then $G(J) = J_0 \otimes G_0 + J_1 \otimes G_1$ is a subalgebra of $J \otimes G$ and it is called the *Grassmann envelope* of J .

An algebra J is called a *Jordan superalgebra* if its Grassmann envelope is a Jordan algebra, i. e., $G(J)$ satisfies the identities

$$xy = yx, \quad (x^2y)x = x^2(xy).$$

Recall the Kantor constuction [23]. Let A be an associative commutative algebra over a field F with a derivation D . Denote by \bar{A} an isomorphic copy of the vector space A with an isomorphism $a \mapsto \bar{a}$. On the direct sum of the vector spaces $J(A, D) = A + \bar{A}$ define a product (\cdot) by

$$a \cdot b = ab, \quad a \cdot \bar{b} = \bar{ab}, \quad \bar{a} \cdot b = \bar{ab}, \quad \bar{a} \cdot \bar{b} = aD(b) - D(a)b,$$

where $a, b \in A$ and ab is the product of elemets in A . Then $J(A, D)$ is an Jordan superalgebra. The superalgebra $J(A, D)$ is said to be a superalgebra of *vector type*.

Let us give the dual analogue of the Kantor costruction for coalgebras. Suppose (C, Δ, d) is an associative commutative differential coalgebra with a coderivation d . Let \bar{C} be an isomorphic copy of the vector space C with an isomorphism $c \mapsto \bar{c}$. Define the comultiplication Δ_J on the direct sum of vector spaces $J^c(C, \Delta_J, d) = C + \bar{C}$ by

$$\begin{aligned} \Delta_J(c) &= \sum_{(c)} c_{(1)} \otimes c_{(2)} + \bar{c}_{(1)} \otimes \overline{d(c_{(2)})} - \overline{d(c_{(1)})} \otimes \bar{c}_{(2)}, \\ \Delta_J(\bar{c}) &= \sum_{(c)} \bar{c}_{(1)} \otimes c_{(2)} + c_{(1)} \otimes \bar{c}_{(2)}, \end{aligned}$$

where $c \in C$ and $\Delta(c) = \sum_{(c)} c_{(1)} \otimes c_{(2)}$. It is shown in [16] that the dual algebra of the coalgebra $J^c(C, \Delta_J, d)$ is a Jordan superalgebra of vector type $J(C^*, d^*)$, where d^* is the transpose map to d . Therefore, $J^c(C, \Delta_J, d)$ is a Jordan supercoalgebra.

Lemma 6. *Let $(P_1^\circ, \Delta^\circ, d^\circ)$ be the dual coalgebra of (P_1, d) . Then $J^c(P_1^\circ, \Delta^\circ, d^\circ) = J(P_1, d)^\circ$.*

Proof. Set $J = J(P_1, d)$, and let $(J^\circ, \Delta_{J^\circ})$ be the dual supercoalgebra of J . It was noted above that $J^c = J^c(P_1^\circ, \Delta_J^\circ, d^\circ)$ is a Jordan supercoalgebra. It is clear that J^c is a good subspace of J^* . Therefore, $J^c \subseteq J^\circ$.

Let us put $\langle \alpha, a \rangle = \alpha(a)$ for $\alpha \in J^*$, $a \in J$, as above. Since $J^c = P_1^\circ + \overline{P_1^\circ}$, where $\overline{P_1^\circ}$ is an isomorphic copy of the vector space P_1° with the isomorphism $a \mapsto \bar{a}$, we can assume $\langle \alpha, a \rangle = \langle \bar{\alpha}, \bar{a} \rangle$ for $\alpha \in P_1^\circ$ and $a \in P_1$.

The algebra J acts on J^* as on a bimodule by the rule

$$\langle \alpha \cdot a, b \rangle = \langle \alpha, ab \rangle, \quad \langle a \cdot \alpha, b \rangle = \langle \alpha, ba \rangle,$$

where $\alpha \in J^*$, $a, b \in J$.

Since $J = P_1 \oplus \overline{P_1}$, then we can assume that $J^* = P_1^* \oplus \overline{P_1^*}$ (e.g., $P_1^* \cong \overline{P_1}^\perp \subset J^*$). Moreover, P_1^* is a P_1 -subbimodule of J^* . Indeed, let $\alpha \in P_1^*$ and $f \in P_1$. Then

$$\langle \alpha \cdot f, \overline{P_1} \rangle = \langle \alpha, f\overline{P_1} \rangle \subseteq \langle \alpha, \overline{P_1} \rangle = 0.$$

Consequently, $\alpha \cdot f \in P_1^*$. Similarly, $f \cdot \alpha \in P_1^*$. Hence, P_1^* is a P_1 -subbimodule of J^* . In the same way, one may show that $\overline{P_1^*} \cong P_1^\perp \subset J^*$ is a P_1 -subbimodule of J^* .

Denote by V and W the projections of J° on the spaces P_1^* and $\overline{P_1^*}$, respectively. Since J° is a P_1 -subbimodule of J^* then V and W are also P_1 -subbimodules of J^* .

Let us show that V is a good subspace of P_1^* . For $\alpha \in V$ there exist $\gamma \in J^\circ$ and $\beta \in W$ such that $\gamma = \alpha + \beta$. Let $\Delta_{J^\circ}(\gamma) = \sum_\gamma \gamma_{(1)} \otimes \gamma_{(2)}$. Then $\gamma \cdot P_1 \subseteq \sum_\gamma \langle \gamma_{(1)}, P_1 \rangle \gamma_{(2)}$. Consequently, the space $\gamma \cdot P_1$ is finite-dimensional. The space $\alpha \cdot P_1$ is the projection of $\gamma \cdot P_1$ to V , therefore, $\alpha \cdot P_1$ is finite-dimensional. Similarly the space $P_1 \cdot \alpha$ is finite-dimensional. By Corollary 2.5 from [5] we get that V a good subspace of P_1^* .

Consequently, $V \subseteq P_1^\circ$. Since $P_1^\circ \subseteq J^c \subseteq J^\circ$ then $P_1^\circ \subseteq V$. Therefore, $P_1^\circ = V$. Then $V \subseteq J^\circ$ and $W \subseteq J^\circ$. Hence, the coalgebra J° is a \mathbb{Z}_2 -graded space, where $V = (J^\circ)_0$, $W = (J^\circ)_1$. Moreover, $\overline{V} = \overline{P_1^\circ} \subseteq W$.

Assume $w \in W$. Then $\Delta_{J^\circ}(w) = \sum_i v_i \otimes w_i + w_i \otimes v_i$, where $v_i \in V$, $w_i \in W$, and for all $a \in P_1$ we have

$$\langle w, \bar{a} \rangle = \langle w, \bar{1} \cdot a \rangle = \left\langle \sum_i \langle w_i, \bar{1} \rangle v_i, a \right\rangle = \left\langle \sum_i \langle w_i, \bar{1} \rangle \bar{v}_i, \bar{a} \right\rangle.$$

Since $W \subseteq \overline{P_1^*}$, we have $w = \sum_i \langle w_i, \bar{1} \rangle \bar{v}_i$. Consequently, $W = \overline{V}$, and thus $J^c(P_1^\circ, \Delta_{J^\circ}^\circ, d^\circ) = J^\circ$. \square

It is known (see [19]) that a Jordan superalgebra is simple if and only if it is simple as an algebra. Also, recall that the superalgebra $J(A, D)$ is simple if and only if the algebra A is a differentially simple algebra [24].

Therefore, the following Theorem is true.

Theorem 5. *Let $P_1 = F[x]$ be the polynomial algebra in the variable x , and $d = \frac{d}{dx}$ is the derivation with respect to the variable x . Consider the Jordan superalgebra $J(F[x], d)$ of vector type. Then $J(F[x], d)^\circ = J^c(P_1^\circ, \Delta_J^\circ, d^\circ)$. Moreover, if F is a field of characteristic 0 then $J(F[x], d)^\circ$ does not contain non-zero finite-dimensional subcoalgebras.*

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