RECOGNITION OF THE GROUP $E_6(5)$ BY PRIME GRAPH

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ABSTRACT. If n is an integer, then we denote by $\pi(n)$ the set of all prime divisors of n. Let G be a finite group. Then $\pi(|G|)$ is denoted by $\pi(G)$. We construct the prime graph of G, which is denoted by $\Gamma(G)$, as follows: the vertex set is $\pi(G)$ and two distinct primes p and q are joined by an edge if and only if G has an element of order pq. In this paper we prove the main result that if G is a finite group such that $\Gamma(G) = \Gamma(E_6(5))$, then $G \cong E_6(5)$.

keywords and phrases: finite groups, finite simple groups, element orders, prime graph of finite group.

1. Introduction

If n is an integer, then we denote by $\pi(n)$ the set of all prime divisors of n. Let G be a finite group. Then $\pi(|G|)$ is denoted by $\pi(G)$. We construct the prime graph of G, which is denoted by $\Gamma(G)$, as follows: the vertex set is $\pi(G)$ and two distinct primes p and q are joined by an edge if and only if G has an element of order pq. Let m and n be natural numbers. We write $m \sim n$ if and only if for all prime divisors $r \in \pi(m)$ and $s \in \pi(n)$, r is adjacent to s in $\Gamma(G)$. The spectrum of a finite group G, which is denoted by $\pi_e(G)$, is the set of its element orders. Let G be a finite group and $r \in \pi(G)$. We denote by $\rho(G)$ some independent set of vertices in $\Gamma(G)$ with the maximal number of elements. Also some independent set of vertices in $\Gamma(G)$ containing r with the maximal number of elements is denoted by $\rho(r, G)$. Put $t(G) = |\rho(G)|$ and $t(r, G) = |\rho(r, G)|$. We denote by $M(G)$, the set of orders of maximal abelian subgroups of G.

A finite nonabelian simple group P is called quasirecognizable by prime graph, if each finite group G with $\Gamma(G) = \Gamma(P)$ has a unique composition factor isomorphic to P. We denote by $k(\Gamma(G))$ the number of isomorphism classes of finite groups H satisfying $\Gamma(G) = \Gamma(H)$. Given a natural number r, a finite group G is called r-recognizable by prime graph if $k(\Gamma(G)) = r$ and unrecognizable if $k(\Gamma(G))$ is infinite. Usually a 1-recognizable group by prime graph is called a recognizable group by prime graph.

It is proved that if $q = 3^{2n+1}$ $(n > 0)$, then the simple group ${}^2G_2(q)$ is recognizable by prime graph [7, 26]. Also in [11] it is proved that $\mathrm{PSL}(2, p)$, where $p > 11$ is a prime number and $p \not\equiv 1 \pmod{12}$, is recognizable by prime graph. In [8, 12], finite groups with the same prime graph as $\mathrm{PSL}(2, q)$, where q is not prime, are determined. In [9, 10, 13, 14] finite groups with the same prime graph as $L_n(2)$, $U_n(2)$, $D_n(2)$ and ${}^2D_n(2)$ are obtained.

The authors in [18], proved that if G is a finite group such that $\Gamma(G) = \Gamma(B_p(3))$, where $p > 3$ is an odd prime, then $G \cong B_p(3)$ or $C_p(3)$. Also if $\Gamma(G) = \Gamma(B_3(3))$, then $G \cong B_3(3)$, $C_3(3)$, $D_4(3)$, or $G/O_2(G) \cong \mathrm{Aut}({}^2B_2(8))$. Also in [19] we prove that if G is a finite group such that $\Gamma(G) = \Gamma(B_n(3))$, where $n \geq 6$, then G has a unique nonabelian

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composition factor isomorphic to $B_n(3)$ or $C_n(3)$. Also if $\Gamma(G) = \Gamma(B_4(3))$, then G has a unique nonabelian composition factor isomorphic to $B_4(3)$, $C_4(3)$ or ${}^2D_4(3)$.

Note that if G is recognizable by $\Gamma(G)$, then it is also recognizable by $\pi_e(G)$ and $M(G)$, but the converse is not true in general. In [20], we proved that if q is not a Mersenne prime, then every finite group with the same orders of maximal abelian subgroups as $A_2(q)$, is isomorphic to $A_2(q)$ or an extension of $A_2(q)$ by a subgroup of the outer automorphism group of $A_2(q)$. In [22], we proved that if $L = \mathrm{PSU}_3(q)$, where q is not a Fermat prime, then every finite group with the same set of orders of maximal abelian subgroups as L is an almost simple group with socle $\mathrm{PSU}_3(q)$. In [21], we proved that if G is a finite group such that $M(G) = M(E_6(q))$, then G has a unique nonabelian composition factor which is isomorphic to $E_6(q)$. In [4, 16], it is proved that $E_6(2)$ and ${}^2E_6(2)$, are recoginable by their prime graphs. Also in [15], it is proved that $E_6(3)$ and ${}^2E_6(3)$ are recognizable by their prime graphs. In this paper we prove that $E_6(5)$ is recognizable by its prime graph. Throughout this paper, all groups are finite and by simple groups we mean nonabelian simple groups. All further unexplained notations are standard and refer to [2].

2. Preliminary Results

Lemma 2.1. ([24, Theorem 1]) *Let G be a finite group with $t(G) \geq 3$ and $t(2, G) \geq 2$. Then the following hold:*

(1) There exists a finite nonabelian simple group S such that $S \leq \overline{G} = G/K \leq \mathrm{Aut}(S)$ for a maximal normal solvable subgroup K of G.

(2) For every independent subset ρ of $\pi(G)$ with $|\rho| \geq 3$ at most one prime in ρ divides the product $|K| \cdot |\overline{G}/S|$. In particular, $t(S) \geq t(G) - 1$.

(3) One of the following holds:

(a) Every prime $r \in \pi(G)$ nonadjacent to 2 in $\Gamma(G)$ does not divide the product $|K| \cdot |\overline{G}/S|$; in particular, $t(2, S) \geq t(2, G)$.

(b) There exists a prime $r \in \pi(G)$ nonadjacent to 2 in $\Gamma(G)$; in which case $t(G) = 3$, $t(2, G) = 2$, and $S \cong Alt_7$ or $A_1(q)$ for some odd q.

Lemma 2.2. ([23, Lemma 1]) *Suppose that N is a normal elementary abelian subgroup of a finite group G and $H = G/N$. Define an authomorphism $\phi : H \to \mathrm{Aut}(N)$ as follows: $n^{\phi(gN)} = n^g$. Then $\Gamma(G) = \Gamma(N \rtimes_\phi H)$*

3. Main Results

Remark 3.1. By [2], we know that $\pi(E_6(5)) = \{2, 3, 5, 7, 11, 13, 19, 31, 71, 313, 601, 829\}$. Using [25, Prop 2.5, Prop 3.2, and Prop 4.5], we get that $t(E_6(5)) = 5$ and $t(2, E_6(5)) = 3$. Also $\rho(2, E_6(5)) = \{2, 19, 601, 829\}$.

Main Theorem 3.2. *If G is a finite group such that $\Gamma(G) = \Gamma(E_6(5))$, then $G \cong E_6(5)$.*

Proof. By Remark 3.1 and Lemma 2.1, there exists a nonabelian simple group S such that $S \leq \overline{G} = G/K \leq \mathrm{Aut}(S)$ for a maximal normal solvable subgroup K of G and $t(S) \geq 4$.
Step 1. We prove that K is nilpotent. We know that G acts on K by conjugation. Since by Remark 3.1, $829 \in \rho(2, G)$, so $829 \notin \pi(K)$ by Lemma 2.1. Suppose that $a \in G$ and $|a| = 829$. By [25, Prop 2.5, Prop 3.2, and Prop 4.5], 829 is nonadjacent to all vertices of $\pi(K)$. Therefore the action of a on K is fixed-point free. Hence by Thompson's theorem, K is nilpotent.

Step 2. We prove that $S \cong E_6(5)$. By Remark 3.1, 829 is the largest prime in $\pi(S)$. Using [27], S can be a group from the following groups:

$$L_3(5^3), L_4(5^3), L_2(829^2), S_4(829), U_3(829), E_6(5), U_4(829)$$

If S is $L_3(5^3)$, $L_2(829^2)$, $S_4(829)$, $U_3(829)$ or $U_4(829)$, then $601 \notin \pi(S)$. So we get a contradiction, since $601 \in \rho(2, G)$. If S is isomorphic to $L_4(5^3)$, then $t(L_4(5^3)) = 3$. Hence it is a contradiction, since $t(S) \geq 4$. Therefore $S \cong E_6(5)$.

Step 3. We prove that $G/K \cong E_6(5)$. By [2], $|\mathrm{Out}(E_6(5))| = 2$, so either $G/K \cong E_6(5)$ or $G/K \cong \mathrm{Aut}(E_6(5))$. Suppose that $G/K \cong \mathrm{Aut}(E_6(5))$. Let γ be a graph automorphism of order 2 of $E_6(5)$. By [1, Theorem 19.9], $C_L(\gamma) = F_4(5)$. Since $601 \in \pi(F_4(3))$ and $601 \in \rho(2, G)$, we get a contradiction. So $G/K \cong E_6(5)$.

Step 4. We show that $\pi(K) \subseteq \{5, 31\}$. Assume that $p \in \pi(K)$. Since K is nilpotent, by [13] K is an elementary abelian p-group. According to [17], ${}^3D_4(5) \leq G/K$. Now consider the action of ${}^3D_4(5)$ on K which is defined by ϕ in Lemma 2.2. Take an element $b \in {}^3D_4(5)$ of order 601. If $p \neq 5$, then b fixes an element in K by [28]. Therefore $601 \sim p$ in $\Gamma(G)$. So by [25], $p \in \{5, 31\}$.

Step 5. We prove that $\pi(K) \subseteq \{5\}$. Suppose that $31 \in \pi(K)$. Using [17], we see that $O_8(5) < E_6(5)$. According to the orders of the groups $O_8(5)$ and $E_6(5)$, we get that their Sylow 13-subgroups are isomorphic. By [6], we see that $E_6(5)$ has a torus that is a direct product of two cyclic groups of order $(5-1)(5^2+1)$. Since 13 divides into 5^2+1, hence we conclude that the Sylow 13-subgroup is non-cyclic. Now consider a Sylow 13-subgroup P of $E_6(5)$. Denote by $\widetilde{P}$ the full preimage of P in G. The conjugation action of 13-elements of $\widetilde{P}$ on K is fixed-point free, so $\widetilde{P}$ is a Frobenius group. By [3], P is cyclic, which is a contradiction.

Step 6. We prove that $K = 1$. Suppose that $K \neq 1$. Similarly to the above, we can assume that K is elementary abelian. By [17], $F_4(5) \leq G/K$. Consider the action of $F_4(5)$ on K by Lemma 2.2. According to [5], any element of order 601 in $F_4(5)$ fixes some non-identity element in K. Thus $601 \sim 5$ in $\Gamma(G)$, which is a contradiction. So $K = 1$. Therefore $G \cong E_6(5)$ and the proof of the Main Theorem is completed. □

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