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ASYMPTOTICS OF SOLUTIONS FOR TWO ELASTIC PLATES WITH THIN JUNCTION

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Abstract. The paper concerns an equilibrium problem for two elastic plates connected by a thin junction (bridge) in a case of Neumann boundary conditions. An existence of solutions is proved. Passages to limits are justified with respect to the rigidity parameter of the junction. In particular, the rigidity parameter tends to infinity and to zero. Limit models are investigated.

Keywords: Thin junction, elastic plate, rigidity parameter, non-coercive boundary value problem, thin inclusion

1. Introduction

Boundary value problems describing elastic structures have a lot of applications. As a rule, a complex geometry of the structure implies additional difficulties from the standpoint of mathematical modeling. In this paper, we analyze an equilibrium problem of two Kirchhoff-Love elastic plates connected by a thin elastic junction assuming that boundary conditions for one plate have the Neumann type. The junction can be seen as an elastic inclusion for both plates with a suitable equilibrium equation. In cases with inclusions, equilibrium equations for the elastic body are fulfilled outside the inclusion, i.e. in a nonsmooth domain with cuts. In addition to this, Neumann boundary conditions provide a non-coercivity for the problem considered. To overcome difficulties related to the non-coercivity, we have to find solutions in suitable subspaces. The boundary value problems considered in this paper, along with non-coercivity, are formulated in nonsmooth domains.

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General approaches to non-coercive boundary value problems can be found in [1]. Last years, a lot of papers are published concerning equilibrium problems for elastic bodies with inclusions of different nature [2-13]. We can also mention different results related to boundary value problems for elastic bodies in nonsmooth domains [14-19], see also [20-22].

The structure of the paper is as follows. In Section 2, we provide variational and differential formulations of the problem. Passages to limits, as a rigidity parameter of the junction tends to infinity and to zero, are investigated in Sections 3,4. Limit models are analyzed.

2. Problem formulation

Let $\Omega_L, \Omega_R \subset \mathbb{R}^2$ be bounded domains with Lipschitz boundaries Γ^1, Γ^2, respectively, such that $\bar{\Omega}_L \cap \bar{\Omega}_R = \emptyset$. Assume that Γ^2 is divided into two smooth parts Γ^2_N and Γ^2_D, meas $\Gamma^2_D > 0$, and denote $\gamma = (-2,2) \times \{0\}$, $\gamma_1 = (-2,-1) \times \{0\}$, $\gamma_2 = (1,2) \times \{0\}$, $\gamma_0 = (-1,1) \times \{0\}$. Moreover, $\gamma_i \subset \Omega_i$, and γ crosses Γ^i, $i = 1,2$, see Fig. 1. Denote by $\nu = (0,1)$, $n = (n_1, n_2)$ unit normal vectors to γ, Γ^i, respectively, $\tau = (1,0)$, and set $\Omega = \Omega_L \cup \Omega_R$, $\Omega_\gamma = \Omega \setminus \bar{\gamma}$.

The set Ω_γ fits to two elastic Kirchhoff-Love plates, and γ corresponds a thin elastic junction (bridge) with suitable properties. We describe γ in the frame of Euler-Bernoulli beam model. Thus, the junction γ is partially incorporated into Ω_i, and it can be seen as a bridge between Ω_L and Ω_R. The junction γ would be characterized by a positive rigidity parameter δ. At the first step this parameter is fixed, and in the sequel we analyze passages to the limit as δ goes to infinity and to zero.

We denote $w_n = \frac{\partial w}{\partial n}$, $w_\nu = \frac{\partial w}{\partial \nu}$. If $m = \{m_{ij}\}, i,j = 1,2$, then $\nabla\nabla m = m_{ij,ij}$. For a scalar function w we put $\nabla\nabla w = \{w_{,ij}\}, i,j = 1,2$. Summation convention over repeated indices is used; all functions with two lower indices are assumed to be symmetric in those indices.

By $A = \{a_{ijkl}\}, i,j,k,l = 1,2$, we denote a given elasticity tensor with the usual properties of symmetry and positive definiteness,

$$a_{ijkl} \in L^\infty(\Omega),$$
$$A\xi \cdot \xi \geq c_0|\xi|^2 \quad \forall \xi = \{\xi_{ij}\},\ c_0 = const > 0\ .$$

Introduce notations for a bending moment m_n and a transverse force $t^n = t^n(m)$,

$$m_n = -m_{ij}n_j n_i;\ t^n = -m_{ij,j}n_i - m_{ij,k}\tau_k\tau_j n_i\ ,\ (\tau_1,\tau_2) = (-n_2, n_1). \tag{1}$$

Then for smooth functions $w, m = \{m_{ij}\}, i,j = 1,2$, the following Green's formula holds

$$-\int_{\Omega_i} m \cdot \nabla\nabla w = -\int_{\Omega_i} w\nabla\nabla m + \int_{\Gamma^i} m_n w_n - \int_{\Gamma^i} t^n w,\ i = 1,2.$$

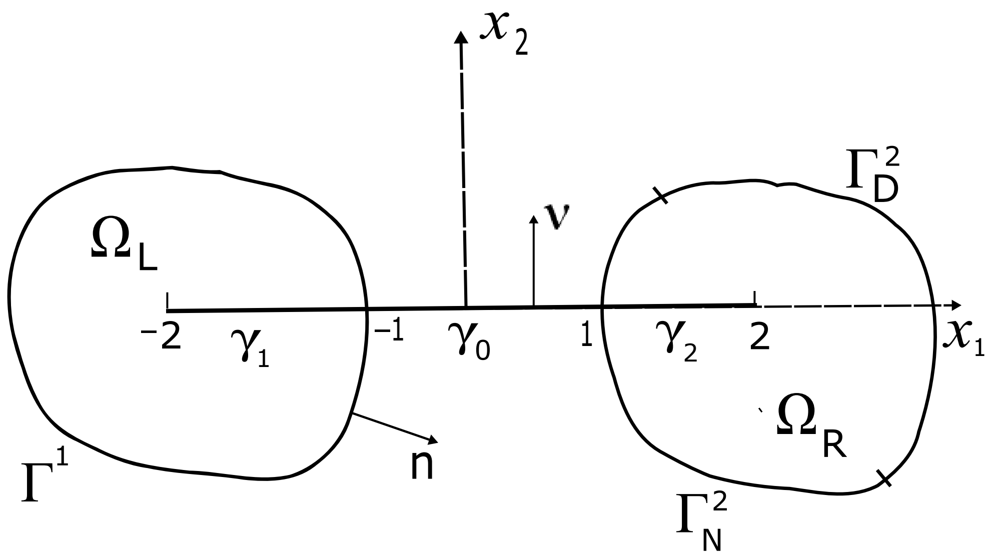

Рис. 1. Elastic plates Ω_L, Ω_R with thin junction γ

Hence, we can write Green's formula for Ω_γ with the cut $\gamma_1 \cup \gamma_2$,

$$-\int_{\Omega_\gamma} m \cdot \nabla\nabla w = -\int_{\Omega_\gamma} w \nabla\nabla m - \int_{\gamma_1 \cup \gamma_2} [m_\nu w_\nu] + \int_{\gamma_1 \cup \gamma_2} [t^\nu w] + \int_{\Gamma^1 \cup \Gamma^2} m_n w_n - \int_{\Gamma^1 \cup \Gamma^2} t^n w, \tag{2}$$

where $[\phi] = \phi^+ - \phi^-$ is a jump of a function ϕ on γ_i; $\phi^\pm$ are the traces of ϕ on the crack faces $\gamma_i^\pm$, $i = 1, 2$. The signs $\pm$ fit to positive and negative directions of ν; the values m_ν, t^ν with the normal vector ν are defined on γ similar to (1).

An equilibrium problem for the plates Ω_L, Ω_R and the junction γ is formulated as follows. For given external forces $f \in L^2(\Omega)$ acting on the plates, we have to find plates displacement w, a moment tensor $m = \{m_{ij}\}, i, j = 1, 2$, defined in Ω, Ω_γ, respectively, and a thin junction displacement v defined on γ such that

$$-\nabla\nabla m = f \text{ in } \Omega_\gamma, \tag{3}$$

$$m + A\nabla\nabla w = 0 \text{ in } \Omega, \tag{4}$$

$$v_{,1111} = 0 \text{ on } \gamma_0;\ v_{,1111} = -[t^\nu] \text{ on } \gamma_1 \cup \gamma_2\ , \tag{5}$$

$$w = w_n = 0 \text{ on } \Gamma_D^2;\ m_n = t^n = 0 \text{ on } \Gamma^1 \cup \Gamma_N^2, \tag{6}$$

$$v_{,11} = v_{,111} = 0 \text{ as } x_1 = -2, 2, \tag{7}$$

$$w = v,\ [w_\nu] = 0,\ [m_\nu] = 0 \text{ on } \gamma_1 \cup \gamma_2, \tag{8}$$

$$[v(\pm 1)] = [v_{,1}(\pm 1)] = 0,\ [v_{,11}(\pm 1)] = [v_{,111}(\pm 1)] = 0, \tag{9}$$

$$\int\limits_{\Omega_L} w \int\limits_{\Omega_L} x_2 + \int\limits_{\Omega_L} w_{,2} = 0. \tag{10}$$

Here $[h(a)] = h(a+0)) - h(a-0);\ w_{,1} = \frac{\partial w}{\partial x_1}$, $(x_1, x_2) \in \Omega$. Functions defined on γ we identify with functions of the variable x_1.

Relations (3)-(4) are the equilibrium equations for the elastic plates Ω_L, Ω_R and the constitutive law; (5) are the Euler-Bernoulli equilibrium equations for the junction parts γ_i. The right-hand sides $-[t^\nu]$ describe forces acting on $\gamma_1 \cup \gamma_2$ from the elastic plates. Relations (9) provide glue conditions at the points where the junction γ crosses the external boundaries of the elastic plates. Neumann conditions on Γ^1 correspond to a non-coercive boundary value problem. As for the condition (10) it provides a uniqueness of the solution.

We can provide a variational formulation of the problem (3)-(10). Introduce a space

$$V = \{(w, v) \in H^2(\Omega) \times H^2(\gamma)\ |$$
$$w = w_n = 0 \text{ on } \Gamma_D^2;\ w = v \text{ on } \gamma_1 \cup \gamma_2\}$$

and the energy functional $\Pi :\ V \to \mathbb{R}$,

$$\Pi(w, v) = \frac{1}{2}B(w, w) - \int\limits_\Omega f w + \frac{1}{2}\int\limits_\gamma v_{,11}^2,$$

where $B = B_L + B_R$,

$$B_L(w, \bar{w}) = \int\limits_{\Omega_L} a_{ijkl} w_{,kl} \bar{w}_{,ij},\ B_R(w, \bar{w}) = \int\limits_{\Omega_R} a_{ijkl} w_{,kl} \bar{w}_{,ij}.$$

Introduce a space (with arbitrary a_2)

$$S = \{(w, v) \in V \mid w(x) = a_2 x_2 \text{ in } \Omega_L;\ w = 0 \text{ in } \Omega_R;$$
$$v = 0 \text{ on } \gamma;\ a_2 \in \mathbb{R}\}$$

and consider the following inner product in V :

$$\langle (w, v), (\bar{w}, \bar{v})\rangle = \int\limits_\Omega w \int\limits_\Omega \bar{w} + \int\limits_\Omega \nabla w \nabla \bar{w} + \tag{11}$$
$$+ B(w, \bar{w}) + \int\limits_\gamma v_{,11} \bar{v}_{,11}.$$

It is easy to prove that the inner product (11) induces a norm equivalent to the standard one. We need to prove the following statement.

Proposition 1. *The space V can be presented as a direct sum of two orthogonal subspaces with respect to the inner product* (11),

$$V = S \oplus S^{\perp},$$

where

$$S^{\perp} = \{(w, v) \in V \mid \int\limits_{\Omega_L} w \int\limits_{\Omega_L} x_2 + \int\limits_{\Omega_L} w_{,2} = 0\}.$$

Proof. We take $(w, v) \in V$, $(\bar{w}, \bar{v}) \in S$, $\bar{w}(x) = a_2 x_2$ in Ω_L, $a_2 \in \mathbb{R}$. Then

$$\langle (w, v), (\bar{w}, \bar{v}) \rangle = \int\limits_{\Omega_L} w \int\limits_{\Omega_L} a_2 x_2 + \int\limits_{\Omega_L} a_2 w_{,2}.$$

From here, it follows that a necessary and sufficient condition for the identity $\langle (w, v), (\bar{w}, \bar{v}) \rangle = 0 \ \ \forall\, (\bar{w}, \bar{v}) \in S$ is of the form

$$\int\limits_{\Omega_L} w \int\limits_{\Omega_L} x_2 + \int\limits_{\Omega_L} w_{,2} = 0.$$

Proposition 1 is proved.

Consider the problem

$$\text{Find } (w, v) \in S^{\perp} \text{ such that } \Pi(w, v) = \inf_{S^{\perp}} \Pi. \tag{12}$$

To check a solvability of the problem (12) it suffices to establish a coercivity of the functional Π on the set $S^{\perp}$ since the weak lower semicontinuity of Π is obvious. To this end, we prove the following statement.

Proposition 2. *There exists a constant $c > 0$ such that*

$$B(w, w) + \int\limits_{\gamma} v_{,11}^2 \geq c \|(w, v)\|_V^2 \ \ \forall (w, v) \in S^{\perp}.$$

Proof. Assume that the statement is not true. In this case, there exists a sequence $(w^k, v^k) \in S^{\perp}$ such that

$$\|(w^k, v^k)\|_V^2 = 1, \tag{13}$$

$$B(w^k, w^k) + \int\limits_{\gamma} (v_{,11}^k)^2 \to 0. \tag{14}$$

Choosing a sequence (with the same notation), we assume that as $k \to \infty$,

$$(w^k, v^k) \to (w, v) \text{ weakly in } V.$$

By (14), it follows

$$B(w, w) + \int\limits_{\gamma} v_{,11}^2 = 0.$$

Then we conclude

$$w = 0 \ \text{ in } \Omega_R; \ v = 0 \text{ on } \gamma_2, \tag{15}$$

(16) $$w(x) = a_0 + a_1 x_1 + a_2 x_2 \text{ in } \Omega_L; \; v(x) = a_0 + a_1 x_1 \text{ on } \gamma_1.$$

Moreover, (15) implies $v(1) = v_{,1}(1) = 0$. Taking into account that $v(x) = c_0 + c_1 x_1$ on γ_0, one concludes that

$$v = 0 \text{ on } \gamma_0.$$

Analogously,

$$v = 0 \text{ on } \gamma_1.$$

Consequently, from (16) it follows

$$w(x) = a_2 x_2 \text{ in } \Omega_L,$$

i.e. the limiting function (w, v) belongs to S. The set $S^\perp$ is weakly closed, hence $(w, v) \in S^\perp$. Thus,

(17) $$(w, v) = (0, 0) \text{ in } V.$$

On the other hand, as $k \to \infty$,

$$\int_\Omega w^k \int_\Omega w^k + \int_\Omega |\nabla w^k|^2 + B(w^k, w^k) + \int_\gamma (v^k_{,11})^2 \to$$
$$\to \int_\Omega w \int_\Omega w + \int_\Omega |\nabla w|^2 + B(w, w) + \int_\gamma v^2_{,11} = \|(w, v)\|^2_V.$$

Consequently,

$$\|(w^k, v^k)\|_V \to \|(w, v)\|_V.$$

Taking into account the weak convergence of the sequence (w^k, v^k) in the space V, we conclude that

$$(w^k, v^k) \to (w, v) \text{ strongly in } V.$$

In this case, from (13) it follows

$$\|(w, v)\|_V = 1$$

what contradicts to (17). Proposition 2 is completely proved.

Proposition 2 provides a coercivity of the functional Π on the set $S^\perp$ what implies a solvability of the problem (12). A solution of this problem satisfies the relations

(18) $$(w, v) \in S^\perp,$$

(19) $$B(w, \bar{w}) - \int_\Omega f \bar{w} + \int_\gamma v_{,11} \bar{v}_{,11} = 0 \;\; \forall (\bar{w}, \bar{v}) \in S^\perp.$$

Now assume that the following condition is satisfied

(20) $$\int_{\Omega_L} f x_2 = 0.$$

In this case, we can write (18)-(19) in the form

$$(w,v)\in S^{\perp}, \tag{21}$$

$$B(w,\bar w+\tilde w)-\int\limits_{\Omega} f(\bar w+\tilde w)+\int\limits_{\gamma} v_{,11}(\bar v_{,11}+\tilde v_{,11})=0 \tag{22}$$

$$\forall(\bar w,\bar v)\in S^{\perp},\ \forall(\tilde w,\tilde v)\in S.$$

By (20) and Proposition 1, relations (21)-(22) can be rewritten as

$$(w,v)\in S^{\perp}, \tag{23}$$

$$B(w,\hat w)-\int\limits_{\Omega} f\hat w+\int\limits_{\gamma} v_{,11}\hat v_{,11}=0\ \ \forall(\hat w,\hat v)\in V. \tag{24}$$

To conclude the above reasonings, we formulate the result obtained.

Theorem 1. *There exists a solution of the problem* (23)-(24) *provided that the condition* (20) *is satisfied.*

Note that the condition (20) is also necessary for a solvability of the problem (23)-(24). To prove this, it suffices to substitute $(\hat w,\hat v)\in S$ in (24).

The problem (23)-(24) is, in fact, a differential formulation of (3)-(10). The following statement takes place providing a connection between problem formulations (3)-(10) and (23)-(24).

Theorem 2. *Problem formulations* (3)-(10) *and* (23)-(24) *are equivalent for smooth solutions.*

Proof. Let (3)-(10) be fulfilled. Then we have for all $(\hat w,\hat v)\in V$,

$$-\int\limits_{\Omega_\gamma}(\nabla\nabla m+f)\hat w+\int\limits_{\gamma_0} v_{,1111}\hat v+\int\limits_{\gamma_1\cup\gamma_2}(v_{,1111}+[t^\nu])\hat v=0.$$

From here, by (2), it follows

$$\begin{aligned}
&-\int\limits_{\Omega_\gamma} m\nabla\nabla\hat w-\int\limits_{\Omega_\gamma} f\hat w+\int\limits_{\gamma_0} v_{,11}\hat v_{,11}+\int\limits_{\gamma_1\cup\gamma_2} v_{,11}\hat v_{,11}+\\
&+\int\limits_{\gamma_1\cup\gamma_2}[m_\nu\hat w_\nu]-\int\limits_{\gamma_1\cup\gamma_2}[t^\nu\hat w]-\int\limits_{\Gamma^1\cup\Gamma^2} m_n\hat w_n+\int\limits_{\Gamma^1\cup\Gamma^2} t^n\hat w+\\
&+\int\limits_{\gamma_1\cup\gamma_2}[t^\nu]\hat v+v_{,111}\hat v|_{-1}^{1}-v_{,11}\hat v_{,1}|_{-1}^{1}+\\
&+v_{,111}\hat v|_{-2}^{-1}-v_{,11}\hat v_{,1}|_{-2}^{-1}+v_{,111}\hat v|_{1}^{2}-v_{,11}\hat v_{,1}|_{1}^{2}=0.
\end{aligned} \tag{25}$$

By boundary conditions (6)-(9), the second, third and fourth lines give zero in (25), thus the identity (24) follows. In so doing, we change the integration over Ω_γ by integration over Ω in the first two integrals of (25). Hence, we proved that (3)-(10) imply (23)-(24).

Conversely, assume that (23)-(24) hold. Integrating by parts in (24) we obtain (with $m = -A\nabla\nabla w$)

$$-\int\limits_{\Omega_\gamma} \nabla\nabla m \cdot \hat{w} - \int\limits_{\Omega_\gamma} f\hat{w} + \int\limits_{\gamma_0} v_{,11}\hat{v}_{,11} + \int\limits_{\gamma_1\cup\gamma_2} v_{,11}\hat{v}_{,11} +$$

$$+ \int\limits_{\gamma_1\cup\gamma_2} [m_\nu \hat{w}_\nu] - \int\limits_{\gamma_1\cup\gamma_2} [t^\nu \hat{w}] - \int\limits_{\Gamma^1\cup\Gamma^2_N} m_n \hat{w}_n + \int\limits_{\Gamma^1\cup\Gamma^2_N} t^n \hat{w} +$$

$$+ \int\limits_{\gamma_1\cup\gamma_2} [t^\nu]\hat{v} - v_{,111}\hat{v}|_{-1}^{1} + v_{,11}\hat{v}_{,1}|_{-1}^{1} -$$

$$-v_{,111}\hat{v}|_{-2}^{-1} + v_{,11}\hat{v}_{,1}|_{-2}^{-1} - v_{,111}\hat{v}|_{1}^{2} + v_{,11}\hat{v}_{,1}|_{1}^{2} = 0.$$

This identity holds for all $(\hat{w}, \hat{v}) \in V$. Consequently, equations (3)-(5) and boundary conditions (6)-(9) follow. Notice that the second boundary condition of (8) and the first group of conditions (9) are consequences of the definition of the space V. Theorem 2 is proved.

3. Rigidity parameter δ tends to infinity

In this section, we introduce a positive parameter δ into the model (18)-(19) and analyze a passage to the limit as $\delta \to \infty$. The parameter δ characterizes a rigidity of the inclusion γ. The condition (20) is assumed to be fulfilled. Instead of (18)-(19), for any δ we consider the following problem

$$(w^\delta, v^\delta) \in S^\perp, \tag{26}$$

$$B(w^\delta, \bar{w}) - \int\limits_{\Omega} f\bar{w} + \delta \int\limits_{\gamma} v^\delta_{,11}\bar{v}_{,11} = 0 \;\; \forall (\bar{w}, \bar{v}) \in S^\perp. \tag{27}$$

The solution of this problem is supplied with the index δ. Note that we can take test functions from V in (27) and write an equivalent differential formulation of the problem (26)-(27) similar to (3)-(10). In this case, instead of (5) we have the following equations

$$\delta v^\delta_{,1111} = 0 \text{ on } \gamma_0; \;\; \delta v^\delta_{,1111} = -[t^\nu] \text{ on } \gamma_1 \cup \gamma_2.$$

Our aim is to justify a passage to the limit as $\delta \to \infty$ in (26)-(27). To this end, we first obtain a priori estimates of the solutions. From (26)-(27) it follows

$$B(w^\delta, w^\delta) - \int\limits_{\Omega} f w^\delta + \delta \int\limits_{\gamma} (v^\delta_{,11})^2 = 0. \tag{28}$$

By the arguments used to prove the coercivity of the functional Π (see Section 2), the following estimate takes place being uniform in $\delta \geq \delta_0 > 0$,

$$\|(w^\delta, v^\delta)\|_V \leq c, \tag{29}$$

moreover, the relation (28) implies

$$\int\limits_{\gamma} (v^\delta_{,11})^2 \leq \frac{c}{\delta}. \tag{30}$$

By (29)-(30), we assume that as $\delta \to \infty$

$$(w^\delta, v^\delta) \to (w, v) \text{ weakly in } V, \tag{31}$$

$$v(x_1) = a_0 + a_1 x_1,\ x_1 \in (-2,2);\ a_0, a_1 \in \mathbb{R}. \tag{32}$$

Now define a space of infinitesimal rigid displacements for the limit problem

$$L(\gamma_1 \cup \gamma_2) = \{l \mid l(x_1) = b_0 + b_1 x_1,$$
$$x_1 \in (-2,-1) \cup (1,2);\ b_0, b_1 \in \mathbb{R}\}$$

and a space of suitable functions

$$V_r = \{\bar{w} \in H^2(\Omega) \,|\, \bar{w} = \bar{w}_n = 0 \text{ on } \Gamma_D^2;\ \bar{w}|_{\gamma_1 \cup \gamma_2} \in L(\gamma_1 \cup \gamma_2)\}.$$

Also, denote

$$V_{r0} = \{\bar{w} \in H^2(\Omega) \,|\, \bar{w}(x) = a x_2 \text{ in } \Omega_L; \bar{w} = 0 \text{ in } \Omega_R;\ a \in \mathbb{R}\}.$$

$$V_{r0}^{\perp} = \{\bar{w} \in V^\infty \,|\, \int\limits_{\Omega_L} \bar{w} \int\limits_{\Omega_L} x_2 + \int\limits_{\Omega_L} \bar{w}_{,2} = 0\}.$$

From the proof of Proposition 1 it follows

$$V_r = V_{r0} \oplus V_{r0}^{\perp} \tag{33}$$

with respect to the inner product in V^∞

$$\{w, \bar{w}\} = \int\limits_{\Omega} w \int\limits_{\Omega} \bar{w} + \int\limits_{\Omega} \nabla w \nabla \bar{w} + B(w, \bar{w}),$$

We take any element $\bar{w} \in V_1^\infty$ and denote $\bar{v} = \bar{w}|_{\gamma_1 \cup \gamma_2}$. Let $\bar{v}(x_1) = b_0 + B_L x_1$. It is clear that we can consider $\bar{v}(x_1)$ for $x_1 \in (-2,2)$. In this case $(\bar{w}, \bar{v}) \in S^\perp$. Substitute this function in (27) as a test one. By (31)-(32), it is possible to pass to the limit in (26)-(27) as $\delta \to \infty$. The limit relations are of the form

$$w \in V_{r0}^{\perp}, \tag{34}$$

$$B(w, \bar{w}) - \int\limits_{\Omega} f \bar{w} = 0 \quad \forall \bar{w} \in V_{r0}^{\perp}. \tag{35}$$

We can rewrite (35) in the form

$$B(w, \bar{w} + \tilde{w}) - \int\limits_{\Omega} f(\bar{w} + \tilde{w}) = 0 \quad \forall \bar{w} \in V_{r0}^{\perp},\ \forall \tilde{w} \in V_{r0},$$

and, by (33), it implies

$$w \in V_{r0}^{\perp}, \tag{36}$$

$$B(w, \hat{w}) - \int\limits_{\Omega} f \hat{w} = 0 \quad \forall \hat{w} \in V_r. \tag{37}$$

Now we can formulate the main result of this section.

Theorem 3. *As $\delta \to \infty$, the solutions of the problem* (26)-(27) *converge in the sense* (31)-(32) *to the solution of* (36)-(37) *provided that the condition* (20) *is fulfilled.*

We note that the condition (20) is necessary for a solvability of the problem (36)-(37) .

Along with the formulation (36)-(37), a differential formulation of this problem can be provided: find functions w, $m = \{m_{ij}\}, i,j = 1,2$, defined in Ω, Ω_γ, respectively, and a function $l^0 \in L(\gamma_1 \cup \gamma_2)$ such that

$$-\nabla\nabla m = f \text{ in } \Omega_\gamma, \tag{38}$$

$$m + A\nabla\nabla w = 0 \text{ in } \Omega, \tag{39}$$

$$w = w_n = 0 \text{ on } \Gamma_D^2;\ m_n = t^n = 0 \text{ on } \Gamma^1 \cup \Gamma_N^2, \tag{40}$$

$$w = l^0,\ [w_\nu] = 0,\ [m_\nu] = 0 \text{ on } \gamma_1 \cup \gamma_2, \tag{41}$$

$$\int\limits_{\gamma_1 \cup \gamma_2} [t^\nu] l = 0 \ \ \forall l \in L(\gamma_1 \cup \gamma_2), \tag{42}$$

$$\int\limits_{\Omega_L} w \int\limits_{\Omega_L} x_2 + \int\limits_{\Omega_L} w_{,2} = 0. \tag{43}$$

Since the function l^0 can be considered on γ_0 as well, the problem (38)-(43) describes an equilibrium state of the elastic plates Ω_L, Ω_R connected by the thin rigid junction γ.

Notice that the identity (42) shows that a principal vector of forces and a principal vector of moments acting on $\gamma_1 \cup \gamma_2$ (as well as acting on γ) are equal to zero.

The following statement takes place providing a connection between problems (36)-(37) and (38)-(43).

Theorem 4. *Problem formulations* (36)-(37) *and* (38)-(43) *are equivalent provided that the solutions are smooth.*

We omit a proof of this statement since it basically reminds that of Theorem 2.

4. Rigidity parameter of γ_0 tends to zero

In this section, we analyze a convergence to zero of the rigidity parameter δ assuming that this parameter is changing at γ_0. In this case, the rigidity parameter at γ_1, γ_2 is fixed and is equal to 1. Assume that the function f satisfies the following condition

$$\int\limits_{\Omega_L} fl = 0 \ \ \forall l \in L(\Omega_L), \tag{44}$$

where

$$L(\Omega_L) = \{l \mid l(x) = a_0 + a_1x_1 + a_2x_2,\ x \in \Omega_L;\ a_i \in \mathbb{R}, i = 0,1,2\}.$$

We first provide a formulation of the equilibrium problem for this case. It is necessary to find functions w^δ, $m = \{m_{ij}\}, i,j = 1,2$, defined in Ω, Ω_γ, respectively, and a function v^δ defined on γ such that

$$-\nabla\nabla m = f \ \text{ in } \ \Omega_\gamma, \tag{45}$$

$$m + A\nabla\nabla w^\delta = 0 \ \text{ in } \Omega, \tag{46}$$

$$\delta v^\delta_{,1111} = 0 \text{ on } \gamma_0; \ v^\delta_{,1111} = -[t^\nu] \text{ on } \gamma_1 \cup \gamma_2 \ , \tag{47}$$

$$w^\delta = w^\delta_n = 0 \text{ on } \Gamma^2_D; \ m_n = t^n = 0 \text{ on } \Gamma^1 \cup \Gamma^2_N, \tag{48}$$

$$v^\delta_{,11} = v^\delta_{,111} = 0 \text{ as } x_1 = -2, 2, \tag{49}$$

$$w^\delta = v^\delta, \ [w^\delta_\nu] = 0, \ [m_\nu] = 0 \ \text{ on } \ \gamma_1 \cup \gamma_2, \tag{50}$$

$$[v^\delta(\pm 1)] = [v^\delta_{,1}(\pm 1)] = 0, \ v^\delta_{,11}(\pm 1 \pm 0) = \delta v^\delta_{,11}(\pm 1 \mp 0), \tag{51}$$

$$v^\delta_{,111}(\pm 1 \pm 0) = \delta v^\delta_{,111}(\pm 1 \mp 0), \tag{52}$$

$$\int\limits_{\Omega_L} w^\delta = 0, \ \int\limits_{\Omega_L} w^\delta_{,i} = 0, \ i = 1, 2. \tag{53}$$

In relations (51)-(53), we should simultaneously take upper or lower signs.

The problem (45)-(53) can be formulated in a variational form. Indeed, consider the energy functional $\pi_\delta : \ V \to \mathbb{R}$,

$$\pi_\delta(w, v) = \frac{1}{2} B(w, w) - \int\limits_\Omega f w + \frac{\delta}{2} \int\limits_{\gamma_0} v^2_{,11} + \frac{1}{2} \int\limits_{\gamma_1 \cup \gamma_2} v^2_{,11}$$

and a subspace $V_1 \subset V$,

$$V_1 = \{(w, v) \in V \mid \int\limits_{\Omega_L} w = 0, \ \int\limits_{\Omega_L} w_{,i} = 0, \ i = 1, 2\}.$$

Since $V_1 \subset S^\perp$, the functional π_δ is coercive on V_1. Hence, the problem

$$\text{Find } (w, v) \in V_1 \text{ such that } \pi_\delta(w, v) = \inf_{V_1} \pi_\delta$$

has a solution satisfying the relations

$$(w^\delta, v^\delta) \in V_1, \ \ B(w^\delta, \hat{w}) - \int\limits_\Omega f \hat{w} + \tag{54}$$

$$+\delta \int\limits_{\gamma_0} v^\delta_{,11} \hat{v}_{,11} + \int\limits_{\gamma_1 \cup \gamma_2} v^\delta_{,11} \hat{v}_{,11} = 0 \ \ \forall (\hat{w}, \hat{v}) \in V_1. \tag{55}$$

In what follows, we aim to justify a passage to the limit in (54)-(55) as $\delta \to 0$. Introduce two spaces

$$W_L = \{(w, v) \in H^2(\Omega_L) \times H^2(\gamma_1) \mid w = v \text{ on } \gamma_1; \\ \int\limits_{\Omega_L} w = 0, \ \int\limits_{\Omega_L} w_{,i} = 0, \ i = 1, 2\},$$

$$W_R = \{(w, v) \in H^2(\Omega_R) \times H^2(\gamma_2) \mid w = v \text{ on } \gamma_2; \\ w = w_n = 0 \text{ on } \Gamma^2_D\}.$$

We need to prove the following statement.

Proposition 3. *There exists a constant $c_1 > 0$ such that*

$$B_L(w,w) + \int\limits_{\gamma_1} v_{,11}^2 \geq c_1(\|w\|^2_{H^2(\Omega_L)} + \|v\|^2_{H^2(\gamma_1)}) \ \ \forall (w,v) \in W_L. \tag{56}$$

Proof. Assume that the statement is not true. Then, there exists a sequence $(w^k, v^k) \in W_L$ such that as $k \to \infty$

$$\|w^k\|^2_{H^2(\Omega_L)} + \|v^k\|^2_{H^2(\gamma_1)} = 1, \tag{57}$$

$$B_L(w^k, w^k) + \int\limits_{\gamma_1} (v^k_{,11})^2 \to 0. \tag{58}$$

We can choose a subsequence (with the same notations) such that

$$w^k \to w \text{ weakly in } H^2(\Omega_L), \text{ strongly in } H^1(\Omega_L), \tag{59}$$

$$v^k \to v \text{ weakly in } H^2(\gamma_1). \tag{60}$$

Limit functions w, v satisfy the conditions

$$B_L(w,w) = 0, \ \int\limits_{\gamma_1} v_{,11}^2 = 0.$$

This means that

$$w(x) = a_0 + a_1x_1 + a_2x_2, \ x \in \Omega_L; \ a_i \in \mathbb{R}, \ i = 0,1,2;$$
$$v(x_1) = a_0 + a_1x_1, \ x_1 \in (-2,-1).$$

On the other hand, $(w,v) \in W_L$, thus

$$(w,v) = (0,0). \tag{61}$$

In addition to this, we have

$$\|w^k\|^2_{H^1(\Omega_L)} + B_L(w^k,w^k) + \int\limits_{\gamma_1} (v^k_{,11})^2 \to$$
$$\to \|w\|^2_{H^1(\Omega_L)} + B_L(w,w) + \int\limits_{\gamma_1} v_{,11}^2.$$

Consequently, as $k \to \infty$,

$$\|(w^k, v^k)\|_{W_L} \to \|(w,v)\|_{W_L}.$$

Taking into account (59)-(60), we conclude that

$$(w^k, v^k) \to (w,v) \text{ strongly in } W_L.$$

Then from (57) it follows

$$\|w\|^2_{H^2(\Omega_L)} + \|v\|^2_{H^2(\gamma_1)} = 1$$

what contradicts to (61). Proposition 3 is proved.

By (54)-(55), we have

$$B(w^\delta, w^\delta) - \int_\Omega f w^\delta + \tag{62}$$

$$+\delta \int_{\gamma_0} (v^\delta_{,11})^2 + \int_{\gamma_1 \cup \gamma_2} (v^\delta_{,11})^2 = 0.$$

Consequently, taking into account Proposition 3, the following uniform in δ estimates are obtained,

$$\|(w^\delta, v^\delta)\|_{W_L} \le c, \tag{63}$$

$$\|(w^\delta, v^\delta)\|_{W_R} \le c. \tag{64}$$

In view of (63)-(64), we can assume that as $\delta \to 0$,

$$(w^\delta, v^\delta) \to (w, v) \text{ weakly in } W_L, \tag{65}$$

$$(w^\delta, v^\delta) \to (w, v) \text{ weakly in } W_R. \tag{66}$$

Here and below, we use the same notations for functions (w^δ, v^δ), (w, v) and for suitable restrictions of these functions to Ω_i, γ_i. Moreover, from (62) we obtain

$$\delta \int_{\gamma_0} (v^\delta_{,11})^2 \le c. \tag{67}$$

By (63)-(64), it follows that $v^\delta(\pm 1 \pm 0)$, $v^\delta_{,1}(\pm 1 \pm 0)$ are bounded uniformly in δ. Consequently, by the first group of boundary conditions (51), we conclude that

$$v^\delta(\pm 1),\ v^\delta_{,1}(\pm 1) \text{ are bounded.} \tag{68}$$

Now, for a given function $h:\ [-1,1] \to \mathbb{R}$, introduce the notation

$$h_\pm(1) = h^2(1) + h^2(-1) + h^2_{,1}(1) + h^2_{,1}(-1).$$

To justify a passage to limit as $\delta \to 0$ in (54)-(55), we need to prove one more statement.

Proposition 4. *There exists a constant $c_2 > 0$ such that*

$$\int_{\gamma_0} h^2_{,11} + h_\pm(1) \ge c_2 \|h\|^2_{H^2(\gamma_0)} \quad \forall h \in H^2(\gamma_0). \tag{69}$$

Proof. Assume that the inequality (69) does not hold. In this case, there exists a sequence $h^k \in H^2(\gamma_0)$ such that as $k \to \infty$,

$$\|h^k\|^2_{H^2(\gamma_0)} = 1,\ \int_{\gamma_0} (h^k_{,11})^2 + h^k_\pm(1) \to 0. \tag{70}$$

Choosing a subsequence (with the same notations) we assume that

$$h^k \to h \text{ weakly in } H^2(\gamma_0), \text{ strongly in } H^1(\gamma_0). \tag{71}$$

From (70) it follows that $h(x_1) = a_0 + a_1 x_1$, $h_\pm(1) = 0$. Consequently, $h = 0$ on γ_0. On the other hand,

$$\|h^k\|^2_{H^1(\gamma_0)} + \int_{\gamma_0} (h^k_{,11})^2 \to \|h\|^2_{H^1(\gamma_0)} + \int_{\gamma_0} h^2_{,11},$$

i.e. $\|h^k\|_{H^2(\gamma_0)} \to \|h\|_{H^2(\gamma_0)}$. In view of (71), it follows

$$h^k \to h \text{ strongly in } H^2(\gamma_0). \tag{72}$$

In this case, (70) implies

$$\|h\|^2_{H^2(\gamma_0)} = 1,$$

and we obtain a contradiction since $h = 0$. Proposition 4 is proved.

By Proposition 4, (67), (68), we conclude that for a small δ,

$$\sqrt{\delta} v^\delta \text{ are bounded in } H^2(\gamma_0).$$

By this, we assume that as $\delta \to 0$,

$$\sqrt{\delta} v^\delta \to \bar{v} \text{ weakly in } H^2(\gamma_0). \tag{73}$$

In view of (65), (66), (73), we can pass to the limit in (54)-(55) what implies

$$(w, v) \in W_L, \ (w, v) \in W_R, \tag{74}$$

$$B_L(w, \hat{w}) + B_R(w, \hat{w}) - \int_\Omega f \hat{w} + \tag{75}$$

$$+ \int_{\gamma_1 \cup \gamma_2} v_{,11} \hat{v}_{,11} = 0 \ \ \forall (\hat{w}, \hat{v}) \in V_1.$$

It is seen that (74)-(75) can be rewritten as two independent relations

$$(w, v) \in W_L, \ \ B_L(w, \bar{w}) - \int_{\Omega_L} f \bar{w} + \tag{76}$$

$$+ \int_{\gamma_1} v_{,11} \bar{v}_{,11} = 0 \ \ \forall (\bar{w}, \bar{v}) \in W_L,$$

$$(w, v) \in W_R, \ \ B_R(w, \bar{w}) - \int_{\Omega_R} f \bar{w} + \tag{77}$$

$$+ \int_{\gamma_2} v_{,11} \bar{v}_{,11} = 0 \ \ \forall (\bar{w}, \bar{v}) \in W_R.$$

Define the space

$$W_L^0 = \{(\bar{w}, \bar{v}) \in H^2(\Omega_L) \times H^2(\gamma_1) \mid \bar{w} = \bar{v} \text{ on } \gamma_1\}$$

Consider next an inner product in the space W_L^0

$$\langle (w, v), (\bar{w}, \bar{v}) \rangle_1 = \int_{\Omega_L} w \int_{\Omega_L} \bar{w} + \int_{\Omega_L} \nabla w \nabla \bar{w} + \tag{78}$$

$$+ B_L(w, \bar{w}) + \int_{\gamma_1} v_{,11} \bar{v}_{,11}$$

and prove the following statement.

Proposition 5. *The space W_L^0 can be presented as a direct sum of two orthogonal subspaces with respect to the inner product* (78),

$$W_L^0 = W_L \oplus W_L^\perp,$$

where (with arbitrary a_i)

$$W_L^{\perp} = \{(l,q) \in W_L \mid l(x) = a_0 + a_1x_1 + a_2x_2,\ x \in \Omega_L;\\ q(x_1) = a_0 + a_1x_1;\ a_i \in \mathbb{R},\ i = 0,1,2\}.$$

Proof. Take $(w,v) \in W_L^0$, $(l,q) \in W_L^{\perp}$. Let $l(x) = a_0 + a_1x_1 + a_2x_2,\ q(x_1) = a_0 + a_1x_1$. Then we have

$$\langle (w,v),(l,q)\rangle_1 = \int\limits_{\Omega_L} w \int\limits_{\Omega_L} (a_0 + a_1x_1 + a_2x_2) + \int\limits_{\Omega_L} a_i w_{,i}.$$

We conclude that a necessary and sufficient condition for $\langle (w,v),(l,q)\rangle_1 = 0\ \ \forall (l,q) \in W_L^{\perp}$ has the form

$$\int\limits_{\Omega_L} w = 0,\ \int\limits_{\Omega_L} w_{,i} = 0,\ i = 1,2,$$

i.e. $(w,v) \in W_L$. Proposition 5 is proved.

Condition (44) allows us to rewrite (76) in another form. Indeed, we have

$$(w,v) \in W_L,\\ B_L(w, \bar{w} + l) - \int\limits_{\Omega_L} f(\bar{w} + l) +\\ + \int\limits_{\gamma_1} v_{,11}(\bar{v}_{,11} + q_{,11}) = 0\ \ \forall (\bar{w},\bar{v}) \in W_L,\ \forall (l,q) \in W_L^{\perp},$$

and, according to Proposition 5,

$$(w,v) \in W_L,\ \ B_L(w,\hat{w}) - \int\limits_{\Omega_L} f\hat{w} +\\ + \int\limits_{\gamma_1} v_{,11}\hat{v}_{,11} = 0\ \ \forall (\hat{w},\hat{v}) \in W_L^0. \tag{79}$$

Thus, we have proved the following statement.

Theorem 5. *As $\delta \to 0$, the solutions of the problem* (54)-(55) *converge in the sense* (65),(66),(73) *to the solutions of* (77) *and* (79) *provided that the condition* (44) *is fulfilled.*

To complete the proof of Theorem 5, we need to justify the passage from (76) to (79). To this end, we have to represent the space W as a direct sum of two subspaces.

To conclude the section, we provide differential formulations of the problems (77) and (79). First, consider the problem (77). It is necessary to find a plate displacement w, a moment tensor $m = \{m_{ij}\}, i,j = 1,2$, defined in $\Omega_R,\ \Omega_R \setminus \bar{\gamma}_2$, respectively, and a thin inclusion displacement v defined on γ_2 such that

$$-\nabla\nabla m = f \text{ in } \Omega_R \setminus \bar{\gamma}_2, \tag{80}$$

$$m + A\nabla\nabla w = 0 \text{ in } \Omega_R, \tag{81}$$

$$v_{,1111} = -[t^\nu] \text{ on } \gamma_2, \tag{82}$$

$$v = w, \ [w_\nu] = [m_\nu] = 0 \text{ on } \gamma_2, \tag{83}$$

$$w = w_n = 0 \text{ on } \Gamma_D^2; \ m_n = t^n = 0 \text{ on } \Gamma_N^2, \tag{84}$$

$$v_{,11} = v_{,111} = 0 \text{ as } x_1 = 1, 2. \tag{85}$$

A differential formulation of the problem (79) is as follows. We have to find a plate displacement w, a moment tensor $m = \{m_{ij}\}, i, j = 1, 2$, defined in $\Omega_L, \Omega_L \setminus \bar{\gamma}_1$, respectively, and a thin inclusion displacement v defined on γ_1 such that

$$-\nabla\nabla m = f \text{ in } \Omega_L \setminus \bar{\gamma}_1, \tag{86}$$

$$m + A\nabla\nabla w = 0 \text{ in } \Omega_L, \tag{87}$$

$$v_{,1111} = -[t^\nu] \text{ on } \gamma_1, \tag{88}$$

$$v = w, \ [w_\nu] = [m_\nu] = 0 \text{ on } \gamma_1, \tag{89}$$

$$m_n = t^n = 0 \text{ on } \Gamma^1, \tag{90}$$

$$v_{,11} = v_{,111} = 0 \text{ as } x_1 = -2, -1, \tag{91}$$

$$\int\limits_{\Omega_L} w = 0, \ \int\limits_{\Omega_L} w_{,i} = 0, \ i = 1, 2. \tag{92}$$

The following statements take place.

Theorem 6. *Problem formulations* (77) *and* (80)-(85) *are equivalent for smooth solutions.*

Theorem 7. *Problem formulations* (79) *and* (86)-(92) *are equivalent for smooth solutions.*

We do not provide proofs of Theorems 6, 7 since the arguments remind those of Theorem 2.

A solution existence of the problems (80)-(85) and (86)-(92) is obtained by passing to the limit as $\delta \to 0$ in (45)-(53). On the other hand, it is possible to prove a solution existence directly by the minimization of suitable functionals. Note that each problem has a unique solution.

We should underline that the problems (80)-(85) and (86)-(92) describe equilibrium states of the two elastic plates Ω_R and Ω_L, respectively. The plates are not connected to each other. In fact, we have two independent problems for elastic plates Ω_L, Ω_R with thin elastic inclusions γ_1, γ_2.

References

[1] H. Attouch, G. Buttazzo, G. Michaille, *Variational Analysis in Sobolev and BV Spaces: Applications to PDEs and Optimization*, second edition, SIAM, 2014.

[2] R. Ballarini, *Elastic stress diffusion around a thin corrugated inclusion,* IMA J. Appl. Math., **76** (2011) 633–641.

[3] M. Goudarzi, F. Dal Corso, D. Bigoni, A. Simone, *Dispersion of rigid line inclusions as stiffeners and shear band instability triggers,* Int. J. Solids Structures, **210-211** (2021) 255–272.

[4] L. Freddi, R. Paroni, T. Roubicek, C. Zanini, *Quasistatic delamination models for Kirchhoff-Love plates,* Z. Angew. Math. Mech., **91** (2011) 845–865.
[5] L. Freddi, T. Roubicek, C. Zanini, *Quasistatic delamination of sandwich-like Kirchhoff-Love plates,* J. Elasticity, **113** (2013) 219–250.
[6] A.I. Furtsev, *On contact between a thin obstacle and a plate containing a thin inclusion,* J. Math. Sciences, **237**(2019) 530-–545.
[7] A. Furtsev, E. Rudoy, H. Itou. *Modeling of bonded elastic structures by a variational method: theretical and numerical simulation,* Int. J. Solids Structures, **182-183** (2020) 100–111.
[8] A. Gaudiello, G. Panasenko, A. Piatnitski, *Asymptotic analysis and domain decomposition for a biharmonic problem in a thin multistructure,* Commun. Contemp. Math., **18** (2016) 1550057.
[9] A. M. Khludnev, V.A. Kovtunenko, *Analysis of cracks in solids*, WIT Press, Southampton, Boston, 2000.
[10] A.M. Khludnev, *Elasticity problems in non-smooth domains*, Fizmatlit, Moscow, 2010.
[11] A. M. Khludnev, *On bending an elastic plate with a delaminated thin rigid inclusion,* J. Appl. Industr. Math., **5** (2011) 582—594.
[12] A.M. Khludnev, *Thin rigid inclusions with delaminations in elastic plates*, Europ. J. Mech. A/Solids, **32**(2012) 69–75
[13] A. M. Khludnev, G. Leugering, *Delaminated thin elastic inclusion inside elastic bodies*, Mathematics and Mechanics of Complex Systems, **2** (2014) 1–21.
[14] A. M. Khludnev, G. Leugering, *On elastic bodies with thin rigid inclusions and cracks*, Mathematics and Mechanics of Solids, **20** (2015) 495–511.
[15] A.M. Khludnev, T.S. Popova, *On crack propagations in elastic bodies with thin inclusions*, Siberian Electronic Mathematical Reports, **14** (2017) 586–599.
[16] A.M. Khludnev, *On modeling elastic bodies with defects*, Siberian Electronic Mathematical Reports, **15** (2018) 153–166.
[17] V.A. Kovtunenko, G. Leugering, *A shape-topological control problem for nonlinear crack - defect interaction: the anti-plane variational model,* SIAM J.Control Optim. **54** (2016) 1329–1351.?
[18] N.P. Lazarev, V.A. Kovtunenko, *Signorini-type problems over non-convex sets for composite bodies contacting by sharp edges of rigid inclusions,* MDPI Mathematics, **10** (2022), 2, 250.
[19] V.A. Kozlov, V.G. Maz'ya, A.B. Movchan, *Asymptotic analysis of fields in a multi-structure*, Oxford Math. Monogr., Oxford University Press, New York, 1999.
[20] P. Mallick, *Fiber-reinforced Composites. Materials, Manufacturing, and Design*, Marcel Dekker, New York, 1993.
[21] A. Morassi, E. Rosset, S. Vessella, *Stable determination of a rigid inclusion in an anisotropic elastic plate,* SIAM J. Math. Anal., **44** (2012) 2204–2235.
[22] A. Morassi, E. Rosset, S. Vessella, *Optimal stability in the identifcation of a rigid inclusion in an isotropic Kirchhoff-Love plate,* SIAM J. Math. Anal., **51** (2019) 731–747.
[23] G. Saccomandi, M.F. Beatty, *Universal relations for fiber-reinforced elastic materials*, Mathematics and Mechanics of Solids, **7**(2002) 99–110.

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