ON NON-STANDARD QUASIVARIETIES OF DIFFERENTIAL GROUPOIDS AND UNARY ALGEBRAS

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ABSTRACT. We prove that there are continuum many non-standard quasivarieties of differential groupoids and unary algebras.

Keywords: quasivariety, differential groupoid, unary algebra, non-standard quasivariety.

1. INTRODUCTION

In a recent article of the authors and their co-author, the notion of a B-class was introduced which allows us to treat certain complexity problems for quasivarieties in a uniform way [13]. Namely, the existence of such a class with respect to a quasivariety implies extremely high level of complexity from many points of view. In addition to known properties of Q-universality and the undecidability of certain decision problems, the existence of a B-class leads to continuum many subquasivarieties lacking finitely partitionable quasi-equational bases, continuum many subquasivarieties whose quasi-equational theory is undecidable, and continuum many non-standard subquasivarieties [14, 18]. Yet another consequence of the existence of a B-class is a series of representation results for finite lattices and their ordinal sums [15, 16].

Although the variety $\mathbf{Dm}$ of differential groupoids and a certain quasivariety $\mathbf{V}$ of unary algebras with two unary operations do not admit B-classes, a series of complexity results holds for them. For example, the representation, undecidability, and independent axiomatization results can be proven for these two quasivarieties, see [11, 12]. Notice that the proof of the facts that the quasivarieties $\mathbf{Dm}$ and $\mathbf{V}$

are Q-universal in [8] and [7] required an individual approach and adjustment of known sufficient conditions for Q-universality.

As is mentioned in [18], quasivarieties with finite B-classes possess natural classes of Boolean topological structures (i.e., the structures carrying a compatible compact Hausdorff topology with a basis of clopen sets). Such structures and topological quasivarieties arise, for example, in the study of natural dualities [2, 6] and profinite structures [19].

Recall that a quasivariety $\mathbf{K}$ is said to be standard if every Boolean topological structure whose algebraic reduct belongs to $\mathbf{K}$ is profinite with respect to $\mathbf{K}$. As is proven in [18], the existence of a finite B-class with respect to a quasivariety $\mathbf{K}$ implies the existence of continuum many non-standard subquasivarieties with no independent quasi-equational bases and continuum many non-standard subquasivarieties with finitely patitionable quasi-equational bases. These results cannot be applied to the quasivarieties of differential groupoids and unary algebras mentioned above because they admit no B-classes. In the present article, we prove similar assertions for $\mathbf{Dm}$ and $\mathbf{V}$.

2. Basic definitions and auxiliary results

For all definitions and notation concerning (algebraic) structures and their quasivarieties, we refer to the monograph [4, Ch. 1] and the articles [11, 12, 13, 14, 17, 18].

2.1. **Quasivarieties and class operators.** *Quasi-identities* are universal Horn sentences of the form

$$\forall \overline{x} \; [\varphi_1(\overline{x}) \;\&\; \dots \;\&\; \varphi_k(\overline{x}) \longrightarrow \varphi_0(\overline{x})],$$

where $\varphi_i(\overline{x})$ is an atomic formula for each $i \leqslant k$. A class $\mathbf{K}$ of structures is a *quasivariety* if it coincides with the class of all models of some set Φ of quasi-identities. Then the set Φ is called a *quasi-equational basis* of $\mathbf{K}$.

We denote structures by calligraphic letters. The universe of a structure is denoted by the corresponding italic letter. For classes of structures, we use boldface letters. We assume that all classes are *abstract*, i.e., closed under isomorphism.

Let $\mathbf{K}(\sigma)$ denote the class of all structures of type σ. For a class $\mathbf{K} \subseteq \mathbf{K}(\sigma)$, let $\mathbf{Q}(\mathbf{K})$ denote the least quasivariety extending $\mathbf{K}$. Let $\mathbf{H}(\mathbf{K})$ denote the class of homomorphic images of structures from $\mathbf{K}$; let $\mathbf{P}(\mathbf{K})$ denote the class of structures that are isomorphic to the Cartesian product of a family of structures from $\mathbf{K}$; let $\mathbf{S}(\mathbf{K})$ denote the class of structures that are embeddable into structures from $\mathbf{K}$; see, for example, [4, Sec. 1.2.7]. Finally, let $\mathbf{T}$ denote the trivial (quasi)variety.

We recall the notion of the inverse (projective) limit, see, for example, [3, Sec. 1].

Definition 1. A triple $\Lambda = \langle I, \mathcal{A}_i, \pi_{ij} \rangle$ is an *inverse spectrum* if $\langle I; \leqslant \rangle$ is an up-directed partially ordered set, $\{\mathcal{A}_i \mid i \in I\}$ is a set of structures of the same similarity type σ, the mapping $\pi_{ij} \colon \mathcal{A}_j \to \mathcal{A}_i$ is a homomorphism for all $i, j \in I$ with $i \leqslant j$, and the following holds:

    (i) the mapping π_{ii} is the identity automorphism on $\mathcal{A}_i$ for each $i \in I$;
    (ii) we have $\pi_{ik} = \pi_{ij}\pi_{jk}$ for all $i, j, k \in I$ with $i \leqslant j \leqslant k$.

If each homomorphism π_{ij} is onto then the inverse spectrum is said to be *surjective*. A structure $\mathcal{A} \in \mathbf{K}(\sigma)$ is the *inverse limit* of Λ if its universe

$$A = \Big\{ (a_i \mid i \in I) \in \prod_{i \in I} A_i \mid \pi_{ij}(a_j) = a_i \text{ for all } i \leqslant j \text{ in } I \Big\}$$

is not empty.

We denote the inverse limit of Λ whenever it exists by $\varprojlim \Lambda$. It is clear that $\varprojlim \Lambda \in \mathbf{SP}(\mathcal{A}_i \mid i \in I)$ if Λ is as in Definition 1. For $i \in I$, we denote by π_i the canonical projection from $\varprojlim \Lambda$ to $\mathcal{A}_i$. Notice that π_i is surjective if so is Λ.

A subclass $\mathbf{K}'$ of $\mathbf{K}$ is a $\mathbf{K}$-*quasivariety* if $\mathbf{K}'$ is defined within $\mathbf{K}$ by a set of quasi-identities or, equivalently, $\mathbf{K}' = \mathbf{Q}(\mathbf{K}') \cap \mathbf{K}$.

2.2. **Finite B-classes.** As usual, we denote the least infinite cardinal by ω. For a set X, we denote by $\mathcal{P}_{\mathrm{fin}}(X)$ the set of all finite subsets of X and by $\mathcal{P}_{\mathrm{inf}}(X)$ the set of all infinite subsets $I \subseteq X$ such that the complement $X \backslash I$ is infinite too. Notice that $|\mathcal{P}_{\mathrm{inf}}(X)| = 2^\omega$ for each countable infinite set X.

The following notion is introduced in [13].

Definition 2. Let $\mathbf{M} \subseteq \mathbf{K}(\sigma)$ be a quasivariety of a finite type σ. A class $\mathbf{A} = \{\mathcal{A}_X \mid X \in \mathcal{P}_{\mathrm{fin}}(\omega)\} \subseteq \mathbf{M}$ of finite structures is a *finite* B-*class* with respect to $\mathbf{M}$ if $\mathbf{A}$ satisfies the following conditions:

(B_0) $\mathcal{A}_\varnothing$ is a trivial structure;
(B_1) if $X = Y \cup Z$ in $\mathcal{P}_{\mathrm{fin}}(\omega)$ then $\mathcal{A}_X \in \mathbf{Q}(\mathcal{A}_Y, \mathcal{A}_Z)$;
(B_2) if $\varnothing \neq X \in \mathcal{P}_{\mathrm{fin}}(\omega)$ and $\mathcal{A}_X \in \mathbf{Q}(\mathcal{A}_Y)$ then $X = Y$;
(B_3) if $F \in \mathcal{P}_{\mathrm{fin}}(\omega)$, $i \in \omega$, and $f\colon \mathcal{A}_F \to \mathcal{A}_{\{i\}}$ is a homomorphism then either $f(\mathcal{A}_F) \cong \mathcal{A}_\varnothing$ or $i \in F$;
(B_4) if $F \in \mathcal{P}_{\mathrm{fin}}(\omega)$ then $\mathbf{H}(\mathcal{A}_F) \cap \mathbf{M} \subseteq \mathbf{A}$.

As is mentioned in the introduction, the existence of a (finite) B-class with respect to a quasivariety $\mathbf{K}$ witnesses a very complicated structure of $\mathbf{K}$ as well as of its quasivariety lattice. However, there exists natural classes of structures that do not admit B-classes by obvious reasons (say, the existence of homomorphic images in "small" subvarieties, which spoils B_3 or B_4). In the following two subsections, we recall necessary information about two such classes.

2.3. **Differential groupoids.** A *differential groupoid* is an algebra endowed with one binary operation $\cdot$ that satisfies the following identities:

$$\forall x \; [x \cdot x = x], \quad \forall x \, \forall y \; [x \cdot (x \cdot y) = x],$$
$$\forall x \, \forall y \, \forall z \, \forall t \; [(x \cdot y) \cdot (z \cdot t) = (x \cdot z) \cdot (y \cdot t)].$$

Let $\mathbf{Dm}$ denote the variety of all differential groupoids.

For brevity, we write $x_1 x_2 \ldots x_n$ for $(\ldots (x_1 \cdot x_2) \cdot \ldots) \cdot x_n$ and xy^n for $x \underbrace{y \ldots y}_{n}$.

We use the following representation of differential groupoids from [20]. A groupoid $\mathcal{G}$ is an $\mathbf{Lz}$-$\mathbf{Lz}$-*sum* (of orbits $\mathcal{G}_i$ over a groupoid $\mathcal{I}$) *satisfying the left normal law* if there is a partition $G = \bigcup_{i \in I} G_i$ such that, for every pair $(i, j) \in I^2$, there is a mapping $h_i^j\colon G_i \to G_i$ satisfying the following conditions:

(i) for every $i \in I$, h_i^i is the identity mapping;
(ii) we have $h_i^j(h_i^k(x)) = h_i^k(h_i^j(x))$ for all $i, j, k \in I$ and $x \in G_i$;
(iii) we have $a_i \cdot a_j = h_i^j(a_i)$ for all $i, j \in I$, $a_i \in G_i$ and $a_j \in G_j$.

According to [20, Theorem 2.2], a groupoid is differential if and only if it can be represented as an $\mathbf{Lz}$-$\mathbf{Lz}$-sum satisfying the left normal law. For more detailed information on differential groupoids, we refer to the monograph [21, Secs. 5.6 and 8.4].

Let $n > 0$. The structure defined in **Dm** by the generators $\{x, y\}$ and the defining relations $\{yx = y, xy^n = x\}$ is called the *cycle* of length n and is denoted by $\mathcal{D}_n$. It is convenient to regard D_n as $G_0 \cup G_1$, where G_1 is the singleton orbit $\{b\}$ and $G_0 = \{a, ab, ab^2, \ldots, ab^{n-1}\}$. We denote the trivial groupoid by $\mathcal{D}_0$.

Let $\mathbb{P}$ denote the set of all primes; we assume that $\mathbb{P} = \{p_i \mid i < \omega\}$, where $p_i \leqslant p_j$ if and only if $i \leqslant j$ for all $i, j < \omega$. For an arbitrary set $F \in \mathcal{P}_{\text{fin}}(\omega)$, we put $[F] = \prod_{i \in F} p_i$ if $F \neq \varnothing$ and $[F] = 1$ if $F = \varnothing$.

We will need the following basic properties of the cycles, see [8, 11] as well as [23].

Lemma 3. *Let $n > 0$.*

    (i) *The class $\{\mathcal{D}_m \mid m \text{ divides } n\}$ coincides with the class of nontrivial homomorphic images of $\mathcal{D}_n$.*

    (ii) *If $m \in \omega$ and $\varphi \colon \mathcal{D}_n \to \mathcal{D}_m$ is a homomorphism then either $\varphi(\mathcal{D}_n) \cong \mathcal{D}_0$ or m divides n and φ is onto.*

    (iii) *If $n > 0$ and $X, X_1, \ldots, X_n \in \mathcal{P}_{\text{fin}}(\omega)$ are such that the set $\{1, \ldots, n\}$ is minimal with respect to the property that $\mathcal{D}_{[X]} \in \mathbf{SP}(\mathcal{D}_{[X_1]}, \ldots, \mathcal{D}_{[X_n]})$, then $X = X_1 \cup \ldots \cup X_n$.*
    Conversely, if $X = X_1 \cup \ldots \cup X_n$ then $\mathcal{D}_{[X]} \in \mathbf{SP}(\mathcal{D}_{[X_1]}, \ldots, \mathcal{D}_{[X_n]})$.

The structure of the variety lattice of differential groupoids is explicitly described in [20], see also [21, Theorem 8.4.14]. In particular, each proper subvariety of **Dm** is defined by a single identity within **Dm** and is locally finite. In contrast to that description, the structure of the quasivariety lattice $\mathrm{Lq}(\mathbf{Dm})$ is much more complicated. Namely, the variety **Dm** is Q-universal [8], there exist 2^ω classes **K** of differential groupoids such that the set of (isomorphism types of) finite sublattices of $\mathrm{Lq}(\mathbf{K})$ is not computable [22, 23], and there exist continuum many quasivarieties of differential groupoids with no independent quasi-equational basis [1].

The following assertions are proven in [1, 8, 11, 12].

Theorem 4. *For each of the following properties, there exists continuum many quasivarieties of differential groupoids possessing this property:*

- *Q-universality;*
- *the undecidability of the set of (isomorphism types) of finite sublattices of the lattice of **K**-subquasivarieties for a suitable subclass **K**;*
- *the existence of an ω-independent quasi-equational basis and the lack of an independent quasi-equational basis within **Dm**;*
- *the existence of an independent quasi-equational basis;*
- *the undecidability of the quasi-equational theory;*
- *the undecidability of the finite membership problem and the membership problem for finitely presented differential groupoids.*

2.4. **Unary algebras.** As is proven in [7], the variety $\mathbf{K}_3$ of unary algebras of the type $\sigma = \{f, g\}$ defined by the identities

$$\forall x \, \forall y \; [f(f(x)) = f(f(y)) = f(g(y))],$$
$$\forall x \, \forall y \; [g(g(x)) = g(g(y)) = g(f(y))],$$
$$\forall x \; [f(f(x)) = g(g(x))]$$

is a minimal Q-universal variety. It follows from the proof that the proper subquasi-variety $\mathbf{V} \subseteq \mathbf{K}_3$ defined by the quasi-identities

$$\forall x \; [f(x) = f(f(x)) \longrightarrow f(x) = g(x)],$$
$$\forall x \; [g(x) = g(g(x)) \longrightarrow f(x) = g(x)],$$
$$\forall x \; [f(x) = g(x) \longrightarrow f(x) = f(f(x))],$$
$$\forall x \, \forall y \; [f(x) = f(y) \longrightarrow g(x) = g(y)],$$
$$\forall x \, \forall y \; [g(x) = g(y) \longrightarrow f(x) = f(y)]$$

is Q-universal; moreover, so is the lattice of $\mathbf{W}$-quasivarieties, where $\mathbf{W}$ denotes the subclass of $\mathbf{V}$ defined by the sentences

(1) $$\forall x \, \forall y \; [g(x) = g(y) \,\&\, x \neq y \longrightarrow g(x) = g(g(x))],$$

(2) $$\forall x \; [g(x) = g(g(x))] \longrightarrow \forall x \, \forall y \; [x = y].$$

We recall the notation for certain unary algebras, see [7, 9].

For $n > 1$, let $\mathcal{C}_n$ denote the algebra whose universe is

$$C_n = \{0\} \cup A_n \cup B_n \text{ with } A_n = \{a_0^n, \ldots, a_{n-1}^n\}, \;\; B_n = \{b_0^n, \ldots, b_{n-1}^n\}$$

and the operations are defined as follows: $f(0) = g(0) = f(a_i^n) = g(a_i^n) = 0$ and $g(b_i^n) = a_i^n$ for $0 \leqslant i \leqslant n - 1$, $f(b_i^n) = a_{i+1}^n$ for $0 \leqslant i \leqslant n - 2$, and $f(b_{n-1}^n) = a_0^n$. Let $\mathcal{C}_1$ denote the 2-element algebra with the universe $\{0, a\}$, where $f(0) = g(0) = f(a) = g(a) = 0$. Let $\mathcal{C}_0$ denote the trivial algebra.

It is clear that, for $n \geqslant 0$, we have $\mathcal{C}_n \in \mathbf{W}$.

We recall necessary properties of the algebras $\mathcal{C}_n$ from [7, 9, 11].

Lemma 5. *If $n > 1$ then the following assertions hold.*

(i) *If m divides n then there exists a homomorphism φ from $\mathcal{C}_n$ onto $\mathcal{C}_m$; moreover, the kernels of all such homomorphisms coincide and we have*

$$\ker \varphi = \{(x, x) : x \in C_n\} \cup \{(a_i, a_j), (b_i, b_j) \mid i \equiv j \pmod{m}\}.$$

(ii) *If $\mathcal{A} \in \mathbf{W}$ is nontrivial and there exists a homomorphism from $\mathcal{C}_n$ onto $\mathcal{A}$ then $\mathcal{A}$ is isomorphic to $\mathcal{C}_m$ for a suitable divisor m of n.*

(iii) *If $\mathcal{A} \in \mathbf{V}$ is nontrivial and there exists a homomorphism φ from $\mathcal{C}_n$ onto $\mathcal{A}$ then one of the following conditions holds:*

    (a) *$\mathcal{A} \in \mathbf{W}$ and assertion (ii) is valid;*

    (b) *the kernel $\ker \varphi$ contains a pair of one of the forms $(0, a_i^n)$, $(0, b_i^n)$, (a_i^n, b_j^n), (a_i^n, a_j^n), where the difference $i - j$ is a multiple of no divisor of n, and the structure $\mathcal{A}$ satisfies the premise of sentence (2);*

    (c) *the kernel $\ker \varphi$ contains a pair of the form (a_i^n, a_{i+k}^n) but $(b_i^n, b_{i+k}^n) \notin \ker \varphi$, where k is the least positive number with this property, we have $k > 1$, and k divides n (in this case, the algebra $\mathcal{A}$ satisfies the premise of sentence (1) but violates its conclusion, i.e., $\mathcal{C}_k$ is a substructure of $\mathcal{A}$ and $|g^{-1}(\varphi(a_i^n))| > 1$).*

If $X, Y, Z \in \mathcal{P}_{\text{fin}}(\omega)$ and $n > 1$ then the following assertions hold.

(iv) *There exists a homomorphism from $\mathcal{C}_{[X]}$ onto $\mathcal{C}_{[Y]}$ if and only if $Y \subseteq X$.*

(v) *We have $\mathcal{C}_{[X]} \leq \mathcal{C}_{[Y]} \times \mathcal{C}_{[Z]}$ if and only if $X = Y \cup Z$.*

If $n > 1$ and $m > 0$ then there exists a quasi-identity $q(n, m)$ such that, for every $k > 1$, the structure $\mathcal{C}_k$ satisfies $q(n, m)$ if and only if either k is not a divisor of mn or k divides n.

The following assertions are proven in [7, 9, 11, 12].

Theorem 6. *For each of the following properties, there exists continuum many* **W***-subquasivarieties possessing this property*:

- *Q-universality*;
- *the undecidability of the set of (isomorphism types) of finite sublattices of the lattice of* **K***-subquasivarieties for a suitable subclass* **K**;
- *the existence of an ω-independent quasi-equational basis and lack of an independent quasi-equational basis within* **V**;
- *the existence of an independent quasi-equational basis*;
- *the undecidability of the quasi-equational theory*;
- *the undecidability of the finite membership problem and the membership problem for finitely presented unary algebras of the type $\{f, g\}$*.

2.5. **Standard and non-standard quasivarieties.** For a structure $\mathcal{A} \in \mathbf{K}(\sigma)$, we say that $\mathbb{A} = \langle \mathcal{A}, \tau \rangle$ is a *topological structure* if τ is a topology on A and all the basic operations of $\mathcal{A}$ are continuous and all the basic relations on $\mathcal{A}$ are closed with respect to τ. For a topological structure $\mathbb{A}$, we denote its algebraic reduct by $\mathcal{A}$ and its topology by $\tau_{\mathbb{A}}$. A topology τ on a set A is *Boolean* if the topological space $\langle A, \tau \rangle$ is compact Hausdorff and has a base of clopen sets. A topological structure $\mathbb{A}$ is *Boolean* if $\tau_{\mathbb{A}}$ is Boolean. *Profinite structures (with respect to* **K***)* are exactly those that are isomorphic to the inverse limits of families of finite structures (from **K**). Each profinite structure is naturally equipped with a Boolean topology (in this case, this is the product topology). A quasivariety **K** is *standard* if every Boolean topological structure $\mathbb{A}$ with $\mathcal{A} \in \mathbf{K}$ is profinite with respect to **K**. For more information and results about natural dualities, topological quasivarieties, Boolean topological structures, and topology, the reader is referred to [2, 3, 6].

We will need the following property of the inverse limit, see [3, Lemma 3.2].

Lemma 7. *Let σ contain no relation symbols, let $\mathcal{A} = \varprojlim_{i \in I} \mathcal{A}_i$, where Λ is a surjective inverse spectrum and $\mathcal{A}_i$ is a finite structure for every $i \in I$, let $\mathcal{B}$ be a finite structure, and let $\varphi \colon \mathcal{A} \to \mathcal{B}$ be a (continuous) homomorphism. Then there exist $i \in I$ and a (continuous) homomorphism $\psi \colon \mathcal{A}_i \to \mathcal{B}$ such that $\varphi = \psi \pi_i$.*

The following notion from [3] and its connection with standard quasivarieties is an essential tool in the proof of our main results.

Definition 8. Let $\mathbf{M} \subseteq \mathbf{K}(\sigma)$, where σ contains no relation symbols. A structure $\mathcal{A}$ is *pointwise non-separable* with respect to **M** if there exist $a_1, a_2 \in A$ such that $a_1 \neq a_2$ and $\varphi(a_1) = \varphi(a_2)$ for every finite structure $\mathcal{B} \in \mathbf{M}$ and every homomorphism $\varphi \colon \mathcal{A} \to \mathcal{B}$.

The following assertion is immediate from [3, Lemmas 3.3 and 3.4].

Lemma 9. *Let $\mathcal{A} = \varprojlim_{i \in I} \mathcal{A}_i$, where Λ is a surjective inverse spectrum and $\mathcal{A}_i$ is a finite structure for every $i \in I$, and let* **K** *be a quasivariety. If $\mathcal{A}$ is pointwise non-separable with respect to* **K** *then $\mathcal{A}$ is not profinite with respect to* **K***.*

In particular, if $\mathcal{A} \in \mathbf{K}$ then **K** *is not standard.*

3. MAIN RESULTS

We formulate and prove similar results for **Dm** and **V**. Namely, we find

(a) continuum many non-standard subquasivarieties with no independent quasi-equational bases;

(b) and continuum many non-standard subquasivarieties having an independent quasi-equational basis.

3.1. Differential groupoids.

Theorem 10. *There exist continuum many subquasivarieties of* **Dm** *that are not standard and have no independent quasi-equational basis.*

Proof. Let $I \subseteq \omega$ and let $F \in \mathcal{P}_{\mathrm{fin}}(\omega)$. We denote by φ_F^I the quasi-identity

$$\forall x \, \forall y \; xy^{[F]} = x \; \& \; yx = y \; \rightarrow \; xy^{[F \cap I]} = x.$$

Let $\Phi_I = \{\varphi_F^I \mid F \in \mathcal{P}_{\mathrm{fin}}(\omega)\}$ and let $\mathbf{K}_I$ be the subquasivariety of **Dm** defined by the set of quasi-identities Φ_I.

Let $I \in \mathcal{P}_{\mathrm{inf}}(\omega)$, $F \in \mathcal{P}_{\mathrm{fin}}(\omega)$, and $p \in \omega \backslash F$. We denote by ψ_F^p the quasi-identity

$$\forall x \, \forall y \; xy^{[F \cup \{p\}]} = x \; \& \; yx = y \; \rightarrow \; xy^{[F]} = x.$$

We put $\Psi_p = \{\psi_F^p \mid F \in P_{\mathrm{fin}}(\omega), \; p \in \omega \backslash F\}$ and $\Psi_I = \bigcup_{p \in \omega \backslash I} \Psi_p$. We denote by $\mathbf{K}'_I$ the subquasivariety of **Dm** defined by the set of quasi-identities Ψ_I.

The following assertion is proven in [12, Theorem 4].

Proposition 11. *We have* $\mathbf{K}_I = \mathbf{K}'_I$ *for every* $I \in \mathcal{P}_{\mathrm{inf}}(\omega)$. *The set* Ψ_I *is an* ω-*independent quasi-equational basis of* **K**. *There is no independent quasi-equational basis of* **K**.

We fix $I \in \mathcal{P}_{\mathrm{inf}}(\omega)$ and $k \in \omega \backslash I$. Assume that the numbers in I are ordered in the natural way, i.e., we have $I = \{i_n \mid n < \omega\}$, where $i_n \leqslant i_m$ if and only if $n \leqslant m$ for all $m, n < \omega$. For every $n < \omega$, we put

$$F_n = \{i_0, \ldots, i_n\}, \quad \mathcal{A} = \mathcal{D}_{p_k}, \quad \mathcal{A}_n = \mathcal{D}_{[F_n]}, \quad \mathcal{B}_n = \mathcal{D}_{[\{k\} \cup F_n]}.$$

By the definition of $\mathbf{K}_I$, we immediately obtain the following assertion.

Claim 1. *For every* $F \in \mathcal{P}_{\mathrm{fin}}(\omega)$, *we have* $\mathcal{D}_{[F]} \in \mathbf{K}_I$ *if and only if* $F \subseteq I$.

In particular, we have $\mathcal{A} \notin \mathbf{K}_I$ and $\mathcal{A}_n \in \mathbf{K}_I$ but $\mathcal{B}_n \notin \mathbf{K}_I$ for every $n < \omega$.

We construct an inverse spectrum Λ. By Lemma 3, if $i < j < \omega$ then there is a surjective homomorphism $\pi_{ij} \colon \mathcal{B}_j \to \mathcal{B}_i$; moreover, we may assume that $\pi_{ij}(b) = b$ and $\pi_{ij}(a) = a$, where $\{b\}$ is the singleton orbit and a is the second generator of $\mathcal{D}_n$ for every $n < \omega$. According to the definition of the inverse limit, we denote by π_{ii} the identity automorphism on $\mathcal{B}_i$ for every $i < \omega$.

We immediately obtain the following assertion.

Claim 2. *The triple* $\Lambda = \langle \omega, \mathcal{B}_j, \pi_{ij} \rangle$ *is a surjective inverse spectrum.*

We put $\mathcal{B} = \varprojlim \Lambda$. Since each π_{ij} fixes a and b, we conclude that $B \neq \varnothing$. Moreover, we have $\mathcal{B} \in \mathbf{SP}(\mathbf{Dm}) \subseteq \mathbf{Dm}$. For every $n < \omega$, we denote by π_n the restriction of the nth projection $\prod_{n < \omega} B_n \to B_n$ to B.

Claim 3. *The differential groupoid* $\mathcal{B}$ *is infinite and belongs to* $\mathbf{K}_I$.

Proof of Claim. We show first that $\mathcal{B} \models \Phi_I$. Indeed, consider an arbitrary quasi-identity $\varphi_F^I \in \Phi_I$. Suppose that the premise of φ_F^I holds in $\mathcal{B}$ under an interpretation $\gamma \colon \{x, y\} \to B$. Then there exists a homomorphism $f \colon \mathcal{D}_{[F]} \to \mathcal{B}$ such that $f(a) = \gamma(x)$ and $f(b) = \gamma(y)$. For all $n < \omega$ and $u \in D_{[F]}$, we put $f_n(u) = \pi_n(f(u))$.

We find that f_n is a homomorphism from $\mathcal{D}_{[F]}$ to $\mathcal{B}_n$. By Lemma 3, we have either $f_n(\mathcal{D}_{[F]}) \cong \mathcal{D}_{[G]} \leqslant \mathcal{B}_n$ for some nonempty $G \subseteq F$ or $f_n(\mathcal{D}_{[F]}) \leqslant \mathcal{D}_1$. Since all proper subgroupoids of $\mathcal{B}_n$ are left-zero modes, we conclude that either $f_n(\mathcal{D}_{[F]}) \cong \mathcal{B}_n$, or $f_n(\mathcal{D}_{[F]}) \cong \mathcal{D}_1$, or $f_n(\mathcal{D}_{[F]}) \cong \mathcal{D}_0$. In the first case, we have $|B_n| \leqslant [F] + 1$. Since $\mathcal{D}_{[F]}$ is a finite groupoid and $|B_n| < |B_{n+1}|$ for each $n < \omega$, there is $s < \omega$ such that $[F] + 1 < |B_n|$ for all $n \geqslant s$. This implies that $f_n(\mathcal{D}_{[F]})$ is isomorphic to a subgroupoid of $\mathcal{D}_1$ if $n \geqslant s$. We denote $J_s = \{n < \omega \mid n \geqslant s\}$.

We summarize the above conclusions as follows. The differential groupoid $\mathcal{B}$ is isomorphic to the inverse limit $\varprojlim\langle J_s, \mathcal{B}_j, \pi_{ij}\rangle$ and is a subdirect product of the family of differential groupoids $\{\mathcal{B}_n \mid n \in J_s\}$. The differential groupoid $f(\mathcal{D}_{[F]})$ is a subdirect product of the family $\{f_n(\mathcal{D}_{[F]}) \mid n \in J_s\}$ of subgroupoids of $\mathcal{D}_1$; hence, it is a left-zero mode.

We conclude that the subgroupoid $f(\mathcal{D}_{[F]})$ of $\mathcal{B}$ generated by the set $\{a, b\}$ is a left-zero mode. But then the conclusion of φ_F^I holds in $\mathcal{B}$.

Finally, if the universe B were finite then, by Lemma 7, we would obtain $|B_{n+1}| \leqslant |B| \leqslant |B_n|$ for some $n < \omega$. Since $|B_{n+1}| > |B_n|$, we arrive at a contradiction. $\square$

We remind that $\mathcal{A} \notin \mathbf{K}_I$, $\mathcal{A}_n \in \mathbf{K}_I$, and $\mathcal{B}_n \notin \mathbf{K}_I$ for every $n < \omega$. By Lemma 3, the differential groupoid $\mathcal{B}_n$ is a subdirect product of the differential groupoids $\mathcal{A}$ and $\mathcal{A}_n$. We denote the corresponding projections (which are onto homomorphisms) by $\alpha_n \colon \mathcal{B}_n \to \mathcal{A}$ and $\beta_n \colon \mathcal{B}_n \to \mathcal{A}_n$. As $\mathcal{B}_n \notin \mathbf{K}_I$ and $\mathcal{A}_n \in \mathbf{K}_I$, the kernel $\ker \beta_n$ is the least congruence on $\mathcal{B}_n$ for no $n < \omega$. For each $n < \omega$, we can find distinct elements $b_1^n, b_2^n \in B_n$ such that $\beta_n(b_1^n) = \beta_n(b_2^n)$. We conclude that $\alpha_n(b_1^n) \neq \alpha_n(b_2^n)$.

By Lemma 3, if $i < j < \omega$ then there is a surjective homomorphism $\varphi_{ij} \colon \mathcal{A}_j \to \mathcal{A}_i$; moreover, we may assume that $\varphi_{ij}(b) = b$ and $\varphi_{ij}(a) = a$, where $\{b\}$ is the singleton orbit and a is the second generator of $\mathcal{D}_n$ for every $n < \omega$. It is easy to see that, for all $i, j < \omega$ with $i < j$ and all $u \in B_j$, we have $\beta_i(\pi_{ij}(u)) = \varphi_{ij}(\beta_j(u))$.

The proof of the following assertion repeats the proof of Claim 4 from the proof of [18, Theorem 4].

Claim 4. *If $(b_1, b_2) \in \ker \beta_j \setminus \ker \alpha_j$ for some $j < \omega$ then, for $i \leqslant j$, we have*

$$\big(\pi_{ij}(b_1), \pi_{ij}(b_2)\big) \in \ker \beta_i \setminus \ker \alpha_i.$$

Claim 5. *The differential groupoid $\mathcal{B}$ is pointwise non-separable with respect to $\mathbf{K}_I$.*

Proof of Claim. We consider the set

$$H_{-1} = \bigcup_{n < \omega} \big\{(h_1^n, h_2^n) \in B_n^2 \mid (h_1^n, h_2^n) \in \ker \beta_n \setminus \ker \alpha_n\big\}.$$

Since $(b_1^n, b_2^n) \in H_{-1} \cap B_n^2 \neq \varnothing$ for every $n < \omega$, the set H_{-1} is infinite. Since the set B_0 is finite, there is a pair $\bar{c}_0$ of elements of B_0 such that the set

$$H_0 = \bigcup_{n < \omega} \big\{(h_1^n, h_2^n) \in H_{-1} \mid \big(\pi_{0n}(h_1^n), \pi_{0n}(h_2^n)\big) = \bar{c}_0\big\}$$

is infinite. By Claim 4, we have $\bar{c}_0 \in \ker \beta_0 \setminus \ker \alpha_0$.

We use induction on i and construct a chain of infinite sets H_i, $i < \omega$. Assume that, for some $i < \omega$, we have already found pairs $\bar{c}_j \in \ker \beta_j \setminus \ker \alpha_j$ for $0 \leqslant j \leqslant i$ such that the set

$$H_j = \bigcup_{n \geqslant j} \big\{(h_1^n, h_2^n) \in H_{j-1} \mid \big(\pi_{jn}(h_1^n), \pi_{jn}(h_2^n)\big) = \bar{c}_j\big\}$$

is infinite and $\pi_{jj'}$ takes the pair $\bar{c}_{j'}$ into the pair $\bar{c}_j$ whenever $0 \leqslant j \leqslant j' \leqslant i$. Since H_i is an infinite set and B_i and B_{i+1} are finite sets, there is a pair $\bar{c}_{i+1}$ of elements of B_{i+1}^m such that the set

$$H_{i+1} = \bigcup_{n \geqslant i+1} \left\{ (h_1^n, h_2^n) \in H_i \mid \left(\pi_{i+1,n}(h_1^n), \pi_{i+1,n}(h_2^n) \right) = \bar{c}_{i+1} \right\}$$

is infinite. By Claim 4, we have $\bar{c}_{i+1} \in \ker \beta_{i+1} \setminus \ker \alpha_{i+1}$. Moreover, $\bar{c}_{i+1} \in H_{i+1} \subseteq H_i$. We conclude therefore that $\pi_{i,i+1}$ takes the pair $\bar{c}_{i+1}$ into the pair $\bar{c}_i$. By the definition of an inverse spectrum and the induction hypothesis, we deduce that $\pi_{j,i+1}(\bar{c}_{i+1}) = \pi_{ji}(\pi_{i,i+1}(\bar{c}_{i+1})) = \pi_{ji}(\bar{c}_i) = \bar{c}_j$ whenever $j < i + 1$.

We introduce a pair $\bar{c} = (c_1, c_2)$ of elements of B such that for all $n < \omega$ and $i \in \{1, 2\}$, $\pi_i(c_i)$ is the ith component of $\bar{c}_n$.

We consider a homomorphism $f \colon \mathcal{B} \to \mathcal{D}$, where $\mathcal{D} \in \mathbf{K}_I$ and $\mathcal{D}$ is a finite differential groupoid. By Lemma 7, there exist $n < \omega$ and a homomorphism $g \colon \mathcal{B}_n \to \mathcal{D}$ such that $f = g\pi_n$. By Lemma 3 and Claim 1, we obtain $g(\mathcal{B}_n) \cong \mathcal{D}_{[G]} \in \mathbf{K}_I$ for some $G \subseteq (\{k\} \cup F_n) \cap I = F_n$ or $g(\mathcal{B}_n) \cong \mathcal{D}_0$. We can find therefore a homomorphism $h \colon \mathcal{A}_n \to \mathcal{D}$ with $g = h\beta_n$. We conclude that $f = g\pi_n = h\beta_n\pi_n$ whence $\ker \beta_n \pi_n \subseteq \ker f$. As $\bar{c}_n \in \ker \beta_n$, we obtain $(c_1, c_2) \in \ker \beta_n \pi_n \subseteq \ker f$. $\quad\square$

By Claims 3 and 5 and Lemma 9, the quasivariety $\mathbf{K}_I$ is not standard.

It remains to notice that, in view of Claim 1, the quasivarieties of the form $\mathbf{K}_I$ are pairwise distinct. $\quad\square$

Following [5], a set Φ of first-order sentences is said to be *directed* if, for arbitrary $\varphi_0, \varphi_1 \in \Phi$, there is $\varphi \in \Phi$ such that both φ_0 and φ_1 are consequences of φ. Directed sets of first-order sentences are, in a sense, antagonists of independent sets of sentences.

Theorem 12. *There exist continuum many subquasivarieties in $\mathbf{Dm}$ that are not standard and have an independent quasi-equational basis [a directed quasi-equational basis, respectively].*

Proof. This proof is similar to the one of Theorem 10 but less complicated.

We consider an infinite set $I = \{i_n \mid n < \omega\}$ and assume that its members are ordered in the natural way. That is, we have $i_n \leqslant i_m$ if and only if $n \leqslant m$ for all $m, n < \omega$. We put $F_{-1} = \varnothing$, $F_n = \{i_0, \ldots, i_n\}$, and $\mathcal{B}_n = \mathcal{D}_{[F_n]}$ for every $n < \omega$.

For each $m < \omega$, let ξ_m denote the quasi-identity

$$\forall x \, \forall y \; xy^{[F_m]} = x \; \& \; yx = y \; \to \; xy^{[F_{m-1}]} = x.$$

We denote

$$\Xi_I = \{\xi_m \mid m < \omega\} \quad \text{and} \quad \mathbf{M}_I = \mathrm{Mod}\,\Xi_I \cap \mathbf{Dm}.$$

Claim 1. *The set Ξ_I is an independent quasi-equational basis of $\mathbf{M}_I$ within $\mathbf{Dm}$. The quasivariety $\mathbf{M}_I$ consists of all structures $\mathcal{A} \in \mathbf{Dm}$ with the following property:*

if $F \in \mathcal{P}_{\mathrm{fin}}(\omega)$ is nonempty and $\mathcal{D}_{[F]} \in \mathbf{S}(\mathcal{A})$ then $F \not\subseteq I$.

Proof of Claim. It is not difficult to see that $\mathcal{B}_n$ satisfies ξ_m if $m \neq n$ and violates ξ_n. Hence, Ξ_I is an independent set of quasi-identities.

If $\mathcal{D}_{[F]}$ is embeddable into $\mathcal{A}$ for no $F \in \mathcal{P}_{\mathrm{fin}}(\omega)$ with $\varnothing \neq F \subseteq I$ then, for every $m < \omega$, the premise of the quasi-identity ξ_m can hold under no interpretation $\gamma \colon \{x, y\} \to A$ in view of Lemma 3. Therefore, $\mathcal{A} \models \xi_m$ for all $m < \omega$ and $\mathcal{A} \in \mathbf{M}_I$. Conversely, let $\mathcal{A} \in \mathbf{M}_I$ and let $\mathcal{D}_{[F]}$ embed into $\mathcal{A}$ for some nonempty set $F \in$

$\mathcal{P}_{\text{fin}}(\omega)$ with $F \subseteq I$. Let i_n be the greatest element of F; then $F \subseteq F_n$. Therefore, the differential groupoid $\mathcal{D}_{[F]}$ satisfies the premise of ξ_n under the interpretation γ with $\gamma(x) = a$, $\gamma(y) = b$ but violates the conclusion of ξ_n under the same interpretation γ in view of Lemma 3. Hence, $\mathcal{A} \not\models \xi_n$ which contradicts our assumption $\mathcal{A} \in \mathbf{M}_I$. $\square$

Notice that Claim 1 also provides us with a directed quasi-equational basis of $\mathbf{M}_I$ within $\mathbf{Dm}$. Namely, for each nonempty set $F \in \mathcal{P}_{\text{fin}}(\omega)$ with $F \subseteq I$, we introduce the following quasi-identity χ_F:

$$\forall x \, \forall y \; xy^{[F]} = x \; \& \; yx = y \; \rightarrow \; xy = x.$$

We consider the set $X = \{\chi_F \mid F \in \mathcal{P}_{\text{fin}}(\omega) \text{ and } \varnothing \neq F \subseteq I\}$.

Claim 2. *The set X forms a directed quasi-equational basis of $\mathbf{M}_I$ within $\mathbf{Dm}$.*

Proof of Claim. It is clear that X is a quasi-equational basis of $\mathbf{M}_I$ within $\mathbf{Dm}$. Moreover, the quasi-identities $\chi_F, \chi_G \in X$ are consequences of the quasi-identity $\chi_{F \cup G}$ for all nonempty sets $F, G \in \mathcal{P}_{\text{fin}}(\omega)$. $\square$

We construct an inverse spectrum $\Lambda = \langle \omega, \mathcal{B}_j, \pi_{ij} \rangle$ as follows. According to Lemma 3, for all $i, j < \omega$ with $i < j$, there is a surjective homomorphism $\pi_{ij} \colon \mathcal{B}_j \to \mathcal{B}_i$; moreover, we may assume that $\pi_{ij}(b) = b$ and $\pi_{ij}(a) = a$, where $\{b\}$ is the singleton orbit and a is the second generator of $\mathcal{D}_n$ for each $n < \omega$. According to the definition of an inverse spectrum, let π_{ii} be the identity automorphism of $\mathcal{B}_i$ for each $i < \omega$.

We immediately obtain the following assertion.

Claim 3. *The triple $\Lambda = \langle \omega, \mathcal{B}_j, \pi_{ij} \rangle$ is a surjective inverse spectrum.*

We put $\mathcal{B} = \varprojlim \Lambda$. Since each π_{ij} fixes a and b, we conclude that $B \neq \varnothing$. Moreover, we have $\mathcal{B} \in \mathbf{SP}(\mathbf{Dm}) \subseteq \mathbf{Dm}$. For every $n < \omega$, we denote by π_n the restriction of the nth projection $\prod_{n<\omega} B_n \to B_n$ to B.

Claim 4. *The differential groupoid $\mathcal{B}$ is infinite and belongs to $\mathbf{M}_I$.*

Proof of Claim. The same arguments as in the proof of Theorem 10 (see Claim 3 there) show that $\mathcal{B}$ is infinite.

We prove that $\mathcal{B} \in \mathbf{M}_I$. In view of Claim 2, it suffices to prove that $\mathcal{B} \models X$. Assume that $F \in \mathcal{P}_{\text{fin}}(\omega)$ and $\varnothing \neq F \subseteq I$. If the premise of χ_F holds in $\mathcal{B}$ under an interpretation $\gamma \colon \{x, y\} \to B$ then there exists a homomorphism $f \colon \mathcal{D}_{[F]} \to \mathcal{B}$ such that $f(a) = \gamma(x)$ and $f(b) = \gamma(y)$.

For all $n < \omega$ and $u \in D_{[F]}$, we put $f_n(u) = \pi_n f(u)$. We find that f_n is a homomorphism from $\mathcal{D}_{[F]}$ to $\mathcal{B}_n$. By Lemma 3, we have either $f_n(\mathcal{D}_{[F]}) \cong \mathcal{D}_{[G]} \leq \mathcal{B}_n$ for some $G \subseteq F$ or $f_n(\mathcal{D}_{[F]}) \cong \mathcal{D}_0$. This means that $f_n(\mathcal{D}_{[F]})$ is either a subgroupoid of $\mathcal{D}_1$ or isomorphic to $\mathcal{B}_n$. In the second case, we have $|B_n| \leqslant [F] + 1$. Since $\mathcal{D}_{[F]}$ is a finite structure and $|B_n| < |B_{n+1}|$ for each $n < \omega$, there is $s < \omega$ such that $[F] + 1 < |B_n|$ whenever $n \geqslant s$. Therefore, the differential groupoid $f_n(\mathcal{D}_{[F]})$ with $n \geqslant s$ is a subgroupoid of $\mathcal{D}_1$.

The same arguments as in the proof of Theorem 10 (see Claim 3 there) show that the subgroupoid $f(\mathcal{D}_{[F]})$ of $\mathcal{B}$ generated by the set $\{a, b\}$ is a left-zero mode, i.e., the conclusion of χ_F holds in $\mathcal{B}$. $\square$

Claim 5. *The differential groupoid $\mathcal{B}$ is pointwise non-separable with respect to $\mathbf{M}_I$.*

Proof of Claim. As π_n maps $\mathcal{B}$ onto $\mathcal{B}_n$ for all $n < \omega$, we have $\mathcal{B} \notin \mathbf{Q}(\mathcal{D}_1)$.

Consider an arbitrary homomorphism $f\colon \mathcal{B} \to \mathcal{D}$, where $\mathcal{D} \in \mathbf{M}_I$ is a finite differential groupoid. By Lemma 7, there are $n < \omega$ and $g\colon \mathcal{B}_n \to \mathcal{D}$ such that $f = g\pi_n$ and g is a homomorphism. By Lemma 3, we have either $g(\mathcal{B}_n) \cong \mathcal{D}_{[G]} \in \mathbf{M}_I$ for some $G \subseteq F_n \subseteq I$ or $g(\mathcal{B}_n) \cong \mathcal{D}_0$. Since $\mathcal{D}_{[G]} \in \mathbf{M}_I$, we have $G = \varnothing$ by Claim 2 in the first case. This means that $f(\mathcal{B}) \cong g(\mathcal{B}_n) \leq \mathcal{D}_1$ in any case, which contradicts the fact that $\mathcal{B} \notin \mathbf{Q}(\mathcal{D}_1)$. $\qquad\square$

By Claims 4 and 5 and Lemma 9, the quasivariety $\mathbf{M}_I$ is not standard.

It remains to notice that, in view of Claim 1, the quasivarieties of the form $\mathbf{M}_I$ are pairwise distinct. $\qquad\square$

3.2. Unary algebras.

Theorem 13. *There exist continuum many subquasivarieties of* $\mathbf{V}$ *that are not standard and have no independent quasi-equational basis.*

Proof. Let $I \subseteq \omega$ and let $F \in \mathcal{P}_{\mathrm{fin}}(\omega)$. We denote by φ_F^I the quasi-identity

$$\forall x\, \forall y_0 \ldots \forall y_{[F]-1}\, \forall z_0 \ldots \forall z_{[F]-1}\, \big[f(x) = x \,\wedge\, g(x) = x\, \wedge$$
$$\wedge \bigwedge_{0 \leqslant i < [F]} f(y_i) = x \,\wedge \bigwedge_{0 \leqslant i < [F]} g(y_i) = x \,\wedge \bigwedge_{0 \leqslant i < [F]} g(z_i) = y_i\, \wedge$$
$$\wedge \bigwedge_{0 \leqslant i < [F]-1} f(z_i) = y_{i+1} \,\wedge\, f(z_{[F]-1}) = y_0 \big] \longrightarrow g(z_0) = y_{[F \cap I]}.$$

This is the quasi-identity $q(n, m)$ from Lemma 5 with $n = [F \cap I]$, $m = [F \setminus I]$.

Let $\Phi_I = \{ \varphi_F^I \mid \varnothing \neq F \in \mathcal{P}_{\mathrm{fin}}(\omega) \}$ and let $\mathbf{K}_I$ be the subquasivariety of $\mathbf{V}$ defined by the set of quasi-identities Φ_I.

The following assertion is proven in [12, Theorem 8].

Proposition 14. *Each of the quasivarieties* $\mathbf{K}_I$ *admits an* ω-*independent quasi-equational basis but lacks an independent quasi-equational basis.*

We fix $I \in \mathcal{P}_{\mathrm{inf}}(\omega)$ and $k \in \omega \backslash I$. Assume again that $I = \{ i_n \mid n < \omega \}$ and $i_n \leqslant i_m$ if and only if $n \leqslant m$ for all $m, n < \omega$. For every $n < \omega$, we put

$$F_n = \{ i_0, \ldots, i_n \}, \ \mathcal{A} = \mathcal{C}_{p_k}, \ \mathcal{A}_n = \mathcal{C}_{[F_n]}, \ \mathcal{B}_n = \mathcal{C}_{[\{k\} \cup F_n]} = \mathcal{C}_{p_k \cdot [F_n]}.$$

By the definition of $\mathbf{K}_I$ and Lemma 5, we immediately obtain the following assertion.

Claim 1. *For every* $F \in \mathcal{P}_{\mathrm{fin}}(\omega)$, *we have* $\mathcal{C}_{[F]} \in \mathbf{K}_I$ *if and only if* $F \subseteq I$.

In particular, we have $\mathcal{A} \notin \mathbf{K}_I$ and $\mathcal{A}_n \in \mathbf{K}_I$ but $\mathcal{B}_n \notin \mathbf{K}_I$ for every $n < \omega$.

We construct an inverse spectrum Λ. By Lemma 5, if $i < j < \omega$ then there is a surjective homomorphism $\pi_{ij}\colon \mathcal{B}_j \to \mathcal{B}_i$; moreover, we may assume that $\pi_{ij}(0) = 0$, $\pi_{ij}(a_u^p) = a_v^q$, and $\pi_{ij}(b_u^p) = b_v^q$, where $p = p_k \cdot [F_j]$, $q = p_k \cdot [F_i]$, and $u \equiv v$ (mod q). According to the definition of an inverse spectrum, let π_{ii} be the identity automorphism on $\mathcal{B}_i$ for every $i < \omega$.

The following assertion holds.

Claim 2. *The triple* $\Lambda = \langle \omega, \mathcal{B}_j, \pi_{ij} \rangle$ *is a surjective inverse spectrum.*

We put $\mathcal{B} = \varprojlim \Lambda$. Since each homomorphism φ_{ij} fixes the element 0, we conclude that $B \neq \varnothing$. Moreover, we have $\mathcal{B} \in \mathbf{SP}(\mathbf{V}) \subseteq \mathbf{V}$. For each $n < \omega$, we denote by π_n the restriction of the nth projection $\prod_{n<\omega} \mathcal{B}_n \to \mathcal{B}_n$ to $\mathcal{B}$. As there π_n is onto for all $n < \omega$, the following statement is true.

Claim 3. *We have* $\mathcal{B} \not\models \forall x \, g(g(x)) = g(x)$.

Claim 4. *The algebra* $\mathcal{B}$ *is infinite and belongs to* $\mathbf{K}_I$.

Proof of Claim. The same arguments as in the proof of Theorem 10 (see the proof of Claim 3 there) show that $\mathcal{B}$ is infinite. We prove that $\mathcal{B} \models \Phi_I$.

We consider a quasi-identity $\varphi_F^I \in \Phi_I$ with $F \neq \varnothing$ and assume that the premise of φ_F^I holds in $\mathcal{B}$ under an interpretation $\gamma \colon \{x, y_0, \ldots, y_{[F]-1}, z_0, \ldots, z_{[F]-1}\} \to B$. Then there exists a homomorphism $\varphi \colon \mathcal{C}_{[F]} \to \mathcal{B}$ such that $\varphi(0) = \gamma(x)$, $\varphi(a_i^{[F]}) = \gamma(y_i)$, and $\varphi(b_i^{[F]}) = \gamma(z_i)$ for $i < [F]$. For all $n < \omega$ and $u \in C_{[F]}$, we put $f_n(u) = \pi_n \varphi(u)$, i.e., we again consider the composition of the projection π_n and the homomorphism φ. Then f_n is a homomorphism from $\mathcal{C}_{[F]}$ to $\mathcal{B}_n$. By Lemma 5, one of the following cases is possible:

    (a) we have $\mathcal{C}_{[G]} \leq f_n(\mathcal{C}_{[F]}) \leq \mathcal{B}_n$ for some nonempty set $G \subseteq F_n \cup \{k\}$,

    (b) we have $f_n(\mathcal{C}_{[F]}) \models \forall x \, g(g(x)) = g(x)$.

In case (a), we have $G = F_n \cup \{k\}$ whence $|B_n| = 2[G] + 1 \leqslant 2[F] + 1$. Since $\mathcal{C}_{[F]}$ is a finite algebra and $|B_n| < |B_{n+1}|$ for each $n < \omega$, there is $s < \omega$ such that $2[F] + 1 < |B_n|$ for all $n \geqslant s$. Therefore, $f_n(\mathcal{C}_{[F]}) \models \forall x \, g(g(x)) = g(x)$ for each $n \geqslant s$. As $\mathcal{B} \cong \varprojlim_{j \geqslant s} \mathcal{B}_j$, we conclude that $\varphi(\mathcal{C}_F) \models \forall x \, g(g(x)) = g(x)$. Therefore, the conclusion of φ_F^I also holds in $\mathcal{B}$ under γ and $\mathcal{B} \models \varphi_F^I$. $\qquad\square$

Claim 5. *There are elements* $c_1, c_2 \in B$ *such that* $\big(\pi_n(c_1), \pi_n(c_2)\big) \in \ker \beta_n \setminus \ker \alpha_n$ *and* $\pi_n(c_1), \pi_n(c_2) \in A_n$ *for all* $n < \omega$.

Proof of Claim. As in the proof of Claim 5 (see the proof of Theorem 10), one can establish the existence of elements $c_1, c_2 \in B$ such that $\big(\pi_n(c_1), \pi_n(c_2)\big) \in \ker \beta_n \setminus \ker \alpha_n$ for all $n < \omega$. It follows from Lemma 5(i) that one can choose $c_1, c_2 \in B$ so that $\pi_n(c_1), \pi_n(c_2) \in A_n$ for all $n < \omega$. $\qquad\square$

Claim 6. *The algebra* $\mathcal{B}$ *is not profinite with respect to* $\mathbf{K}_I$.

Proof of Claim. By Lemma 5, the algebra $\mathcal{B}_n$ is a subdirect product of the algebras $\mathcal{A}$ and $\mathcal{A}_n$. As above, we denote the corresponding projections (which are onto homomorphisms) by $\alpha_n \colon \mathcal{B}_n \to \mathcal{A}$ and $\beta_n \colon \mathcal{B}_n \to \mathcal{A}_n$. According to Claim 5, there is a pair $\bar{c} = (c_1, c_2) \in B^2$ such that $\big(\pi_n(c_1), \pi_n(c_2)\big) \in \ker \beta_n \setminus \ker \alpha_n$ and $\pi_n(c_1), \pi_n(c_2) \in A_n$ for all $n < \omega$.

Suppose that $\mathcal{B} \cong \varprojlim_{t \in T} \mathcal{U}_t$ and $\mathcal{U}_t \in \mathbf{K}_I$ is a finite unary algebra for every $t \in T$. Then $\mathcal{B} \leqslant_s \prod_{t \in T} \mathcal{U}_t$. Let $\pi_t' \colon \mathcal{B} \to \mathcal{U}_t$ denote the canonical projection for every $t \in T$. It follows from Lemma 7 that, for each $t \in T$, there exist $n(t) < \omega$ and a homomorphism $\psi_t \colon \mathcal{B}_{n(t)} \to \mathcal{U}_t$ such that $\pi_t' = \psi_t \pi_{n(t)}$. Since π_t' is an onto homomorphism, we conclude that ψ_t is also onto. It follows from Lemma 5 that the following two cases are possible:

    (a) there is a nonempty set $G_t \subseteq \{k\} \cup F_{n(t)}$ such that $\mathcal{C}_{[G_t]} \leq \psi_t(\mathcal{B}_{n(t)}) = \mathcal{U}_t \in \mathbf{K}_I$ and $U_t = \psi_t(B_{n(t)}) = \{0\} \cup A_{[G_t]} \cup \psi_t(B_{n(t)})$, where $A_{[G_t]} = \psi_t(A_{n(t)})$;

    (b) we have $\psi_t(\mathcal{B}_{n(t)}) \models \forall x \, g(g(x)) = g(x)$.

Since $\mathcal{C}_{[G_t]} \in \mathbf{K}_I$ in case (a), we conclude that $G_t \subseteq (\{k\} \cup F_{n(t)}) \cap I = F_{n(t)}$ by Claim 1. Since $\mathcal{B} \not\models \forall x \, g(g(x)) = g(x)$ by Claim 3, the set

$$T_0 = \big\{ t \in T \mid \mathcal{C}_{[G_t]} \leq \psi_t(\mathcal{B}_{n(t)}) \text{ for some nonempty } G_t \subseteq F_{n(t)} \big\}$$

is coinitial in T. This implies that $\mathcal{B} \cong \varprojlim_{t \in T_0} \mathcal{U}_t$ whence $\mathcal{B} \leqslant_s \prod_{t \in T_0} \mathcal{U}_t$.

For each $t \in T_0$, we can find a homomorphism $\vartheta_t \colon \mathcal{A}_{n(t)} \to \mathcal{C}_{[G_t]} \leq \mathcal{U}_t$ such that $\vartheta_t \beta_{n(t)}(a) = \psi_t(a)$ for every $a \in A_{n(t)}$. This yields for each $i \in \{1, 2\}$:

$$\pi'_t(c_i) = \psi_t \pi_{n(t)}(c_i) = \vartheta_t \beta_{n(t)} \pi_{n(t)}(c_i).$$

Inclusion $\big(\pi_{n(t)}(c_1), \pi_{n(t)}(c_2)\big) \in \ker \beta_{n(t)}$ implies that $(c_1, c_2) \in \ker \pi'_t$ for every $t \in T_0$. As $\mathcal{B} \leq_s \prod_{t \in T_0} \mathcal{U}_t$, we have $c_1 = c_2$ in $\mathcal{B}$, which contradicts Claim 5. $\square$

By Claims 4 and 6, the quasivariety $\mathbf{K}_I$ is not standard.

In view of Claim 1, the quasivarieties of the form $\mathbf{K}_I$ are pairwise distinct. It remains to refer to Proposition 14. $\square$

Theorem 15. *There exist continuum many subquasivarieties of $\mathbf{V}$ that are not standard and have an independent quasi-equational basis [a directed quasi-equational basis, respectively].*

Proof. These arguments are similar to the proofs of Theorems 12 and 13.

We consider an infinite set $I = \{i_n \mid n < \omega\}$ and assume that its members are ordered in the natural way, i.e., $i_n \leqslant i_m$ if and only if $n \leqslant m$ for all $m, n < \omega$. We put $F_{-1} = \varnothing$, $F_n = \{i_0, \ldots, i_n\}$, and $\mathcal{B}_n = \mathcal{C}_{[F_n]}$ for every $n < \omega$.

For each $m < \omega$, let ξ_m denote the quasi-identity $q([F_m], [F_{m-1}])$, i.e.,

$$\forall x \, \forall y_0 \ldots \forall y_{[F_m]-1} \, \forall z_0 \ldots \forall z_{[F_m]-1} \, \big[f(x) = x \, \wedge \, g(x) = x \, \wedge$$
$$\wedge \bigwedge_{0 \leqslant i < [F_m]} f(y_i) = x \, \wedge \bigwedge_{0 \leqslant i < [F_m]} g(y_i) = x \, \wedge \bigwedge_{0 \leqslant i < [F_m]} g(z_i) = y_i \, \wedge$$
$$\wedge \bigwedge_{0 \leqslant i < [F_m]-1} f(z_i) = y_{i+1} \, \wedge \, f(z_{[F_m]-1}) = y_0 \big] \longrightarrow g(z_0) = y_{[F_{m-1}]}.$$

We denote

$$\Xi_I = \{\xi_m \mid m < \omega\} \quad \text{and} \quad \mathbf{M}_I = \operatorname{Mod} \Xi_I \cap \mathbf{V}.$$

Claim 1. *The set Ξ_I forms an independent quasi-equational basis of $\mathbf{M}_I$ within $\mathbf{V}$. The quasivariety $\mathbf{M}_I$ consists of all structures $\mathcal{A} \in \mathbf{V}$ with the following property:*

if $F \in \mathcal{P}_{\mathrm{fin}}(\omega)$ is nonempty and $\mathcal{C}_{[F]} \in \mathbf{S}(\mathcal{A})$ then $F \nsubseteq I$.

Proof of Claim. It is not difficult to see that $\mathcal{B}_n$ satisfies ξ_m if $m \neq n$ and violates ξ_n. Hence, Ξ_I is an independent set of quasi-identities.

If $\mathcal{C}_{[F]}$ is embeddable into $\mathcal{A}$ for no $F \in \mathcal{P}_{\mathrm{fin}}(\omega)$ with $\varnothing \neq F \subseteq I$ then, for every $m < \omega$, the premise of the quasi-identity ξ_m holds in $\mathcal{A}$ under no interpretation $\gamma \colon \{x, y_0, \ldots, y_{[F]-1}, z_0, \ldots, z_{[F]-1}\} \to A$ in view of Lemma 5. Therefore, $\mathcal{A} \models \xi_m$ for all $m < \omega$ and $\mathcal{A} \in \mathbf{M}_I$. Conversely, let $\mathcal{A} \in \mathbf{M}_I$ and let $\mathcal{C}_{[F]}$ embed into $\mathcal{A}$ for some nonempty set $F \in \mathcal{P}_{\mathrm{fin}}(\omega)$ with $F \subseteq I$. Let i_n be the greatest element of F; then $F \subseteq F_n$. Therefore, the algebra $\mathcal{C}_{[F]}$ satisfies the premise of ξ_n under the interpretation γ with

$$\gamma(x) = 0, \quad \gamma(y_i) = a^{[F]}_{i(\mathrm{mod}\,[F])}, \quad \gamma(z_i) = b^{[F]}_{i(\mathrm{mod}\,[F])}, \quad 0 \leqslant i < [F_n],$$

but violates the conclusion of ξ_n under the same interpretation γ, see Lemma 5. Hence, $\mathcal{A} \not\models \xi_n$ which contradicts our assumption $\mathcal{A} \in \mathbf{M}_I$. $\square$

As above, Claim 1 also provides us with a directed quasi-equational basis of $\mathbf{M}_I$ within $\mathbf{V}$. Namely, for each nonempty set $F \in \mathcal{P}_{\mathrm{fin}}(\omega)$ with $F \subseteq I$, we introduce the following quasi-identity χ_F:

$$\forall x \,\forall y_0 \ldots \forall y_{[F]-1} \,\forall z_0 \ldots \forall z_{[F]-1} \,\forall y \,\big[f(x) = x \,\wedge\, g(x) = x \,\wedge$$

$$\wedge \bigwedge_{0 \leqslant i < [F]} f(y_i) = x \,\wedge\, \bigwedge_{0 \leqslant i < [F]} g(y_i) = x \,\wedge\, \bigwedge_{0 \leqslant i < [F]} g(z_i) = y_i \,\wedge$$

$$\wedge \bigwedge_{0 \leqslant i < [F]-1} f(z_i) = y_{i+1} \,\wedge\, f(z_{[F]-1}) = y_0 \big] \longrightarrow x = y.$$

We consider the set $X = \{ \chi_F \mid F \in \mathcal{P}_{\mathrm{fin}}(\omega) \text{ and } \varnothing \neq F \subseteq I \}$.

Claim 2. *The set X forms a directed quasi-equational basis of $\mathbf{M}_I$ within $\mathbf{V}$.*

We construct an inverse spectrum $\Lambda = \langle \omega, \mathcal{B}_j, \pi_{ij} \rangle$ as follows. According to Lemma 5, for all $i, j < \omega$ with $i < j$, there is a surjective homomorphism $\pi_{ij} \colon \mathcal{B}_j \to \mathcal{B}_i$; moreover, we may assume that $\pi_{ij}(0) = 0$, $\pi_{ij}(a_k^{[F_j]}) = a_l^{[F_i]}$, and $\pi_{ij}(b_k^{[F_j]}) = b_l^{[F_i]}$, where $k \equiv l \pmod{[F_i]}$. According to the definition of an inverse spectrum, let π_{ii} be the identity automorphism of $\mathcal{B}_i$ for each $i < \omega$.

We immediately obtain the following assertion.

Claim 3. *The triple $\Lambda = \langle \omega, \mathcal{B}_j, \pi_{ij} \rangle$ is a surjective inverse spectrum.*

We put $\mathcal{B} = \varprojlim \Lambda$. Since each π_{ij} fixes 0, we conclude that $B \neq \varnothing$. Moreover, we have $\mathcal{B} \in \mathbf{SP}(\mathbf{V}) \subseteq \mathbf{V}$. For every $n < \omega$, we denote by π_n the restriction of the nth projection $\prod_{n < \omega} \mathcal{B}_n \to \mathcal{B}_n$ to $\mathcal{B}$. As π_n is onto for every $n < \omega$, the following statement is true.

Claim 4. $\mathcal{B} \not\models \forall x \, g(g(x)) = g(x)$.

Claim 5. *The algebra $\mathcal{B}$ is infinite and belongs to $\mathbf{M}_I$.*

Proof of Claim. The same arguments as in the proof of Theorem 12 (see Claim 4 there) show that B is infinite.

We prove that $\mathcal{B} \in \mathbf{M}_I$. In view of Claim 2, it suffices to prove that $\mathcal{B} \models X$. Assume that $F \in \mathcal{P}_{\mathrm{fin}}(\omega)$ and $\varnothing \neq F \subseteq I$. If the premise of χ_F holds in $\mathcal{B}$ under an interpretation $\gamma \colon \{x, y_0, \ldots, y_{[F]-1}, z_0, \ldots, z_{[F]-1}\} \to B$ then there exists a homomorphism $\varphi \colon \mathcal{C}_{[F]} \to \mathcal{B}$ such that $\varphi(0) = 0$, $\varphi(a_k^{[F]-1}) = \gamma(y_k^{[F]-1})$, and $\varphi(b_k^{[F]-1}) = \gamma(z_k^{[F]-1})$ for $0 \leqslant k < [F]$.

For all $n < \omega$ and $u \in C_{[F]}$, we put $f_n(u) = \pi_n \varphi(u)$. We find that f_n is a homomorphism from $\mathcal{C}_{[F]}$ to $\mathcal{B}_n$. By Lemma 5, one of the following cases occurs:

(a) we have $\mathcal{C}_{[G]} \leq f_n(\mathcal{C}_{[F]}) \leq \mathcal{B}_n$ for some nonempty set $G \subseteq F$,
(b) we have $f_n(\mathcal{C}_{[F]}) \models \forall x \, g(g(x)) = g(x)$.

In case (a), we have $F_n = G \subseteq F$ whence $|B_n| = 2[G] + 1 \leqslant 2[F] + 1$. Since $\mathcal{C}_{[F]}$ is a finite algebra and $|B_n| < |B_{n+1}|$ for each $n < \omega$, there is $s < \omega$ such that $2[F] + 1 < |B_n|$ for all $n \geqslant s$. Therefore, $f_n(\mathcal{C}_{[F]}) \models \forall x \, g(g(x)) = g(x)$ for all $n \geqslant s$. As $\mathcal{B} \cong \varprojlim_{j \geqslant s} \mathcal{B}_j$, we conclude that $\mathcal{B} \models \forall x \, g(g(x)) = g(x)$, which contradicts Claim 4. Therefore, the premise of χ_F holds in $\mathcal{B}$ under no interpretation γ and $\mathcal{B} \models \chi_F$. $\qquad\square$

Claim 6. *The algebra $\mathcal{B}$ is not profinite with respect to $\mathbf{M}_I$.*

Proof of Claim. Consider an arbitrary homomorphism $\varphi \colon \mathcal{B} \to \mathcal{D}$, where $\mathcal{D} \in \mathbf{M}_I$ is a finite structure. By Lemma 7, there is $n < \omega$ and a homomorphism $\psi \colon \mathcal{B}_n \to \mathcal{D}$ such that $\varphi = \psi \pi_n$. By Lemma 5, one of the following cases occurs:

(a) we have $\mathcal{C}_{[G]} \leq \psi(\mathcal{B}_n) \in \mathbf{M}_I$ for some nonempty set $G \subseteq F_n$;
(b) we have $\psi(\mathcal{B}_n) \models \forall x \, g(g(x)) = g(x)$.

Since $\mathcal{C}_{[G]} \in \mathbf{M}_I$ in case (a) and $\varnothing \neq G \subseteq F_n \subseteq I$, we arrive at a contradiction with Claim 2. Therefore, case (a) is impossible. Thus, we conclude that case (b) takes place and $\varphi(\mathcal{B}) = \psi(\mathcal{B}_n) \models \forall x \, g(g(x)) = g(x)$. Hence if the unary algebra $\mathcal{B}$ were profinite with respect to $\mathbf{M}_I$, it would satisfy the identity $\forall x \, g(g(x)) = g(x)$ which contradicts Claim 4. $\square$

By Claims 5 and 6, the quasivariety $\mathbf{M}_I$ is not standard.

It remains to notice that, in view of Claim 1, the quasivarieties of the form $\mathbf{M}_I$ are pairwise distinct. $\square$

For quasivarieties possessing finite B-classes, similar results on the existence of directed quasi-equational bases (cf. Claims 2 in the proofs of Theorems 12 and 15) were established in [10].

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