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RANKS AND APPROXIMATIONS FOR FAMILIES OF CUBIC THEORIES

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Abstract. In this paper, we study the rank characteristics of families of cubic theories, as well as new properties of cubic theories, such as pseudofiniteness and smooth approximability. It is proved that in the family of cubic theories, any theory is a theory of finite structure or is approximated by theories of finite structures. The property of pseudofiniteness or smoothly approximability allows one to investigate finite objects instead of complex infinite ones, or vice versa, to produce more complex ones from simple structures.

Keywords: approximation of theory, cube, cubic structure, cubic theory, pseudofinite theory, smoothly approximated structure.

1. Introduction

The ranks [17] and degrees for families of complete theories similar to the Morley rank and degree for a fixed theory, as well as the Cantor-Bendixson rank and degrees, were introduced by Sergei Vladimirovich Sudoplatov. The problem of describing ranks and degrees for natural families of theories is posed. The ranks and degrees for families of incomplete theories are studied in [13, 14], for families of permutation theories are described in [11] and also for families of all theories of arbitrary given languages are described in [12].

In work [16], approximations of theories are investigated both in the general context and with respect to some natural families of theories. The problem of describing the forms of approximation for natural families of theories is posed.

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This work is devoted to the description of the ranks and degrees of families of cubic theories, as well as approximation by theories of finite cubic structures.

1.1. **Preliminaries from cubic theories.** Cubic structures are defined in [20], model-theoretic properties are studied and included in monograph [18], applications in discrete mathematics are shown [19]. The following necessary terminology about cubic structures is taken without specifications from [19, 20].

Definition 1. A *n-dimensional cube*, or a *n-cube* (where $n \in \omega$) is a graph isomorphic to the graph $\mathcal{Q}_n$ with the universe $\{0,\ 1\}^n$ and such that any two vertices $(\delta_1, \ldots,\ \delta_n)$ and $(\delta'_1, \ldots,\ \delta'_n)$ are adjacent if and only if these vertices differ exactly in one coordinate. The described graph $\mathcal{Q}_n$ is called the *canonical representative* for the class of n-cubes.

Let λ be an infinite cardinal. A *λ-dimensional cube*, or a *λ-cube*, is a graph isomorphic to a graph $\Gamma = \langle X;\ R\rangle$ satisfying the following conditions:

(1) the universe $X \subseteq \{0;\ 1\}^\lambda$ is generated from an arbitrary function $f \in X$ by the operator $\langle f\rangle$ attaching, to the set $\{f\}$, all results of substitutions for any finite tuples $(f(i_1), \ldots,\ f(i_m))$ by tuples $(1 - f(i_1), \ldots, 1 - f(i_m))$;

(2) the relation R consists of edges connecting functions differing exactly in one coordinate.

Any graph $\Gamma = \langle X;\ R\rangle$, where any connected component is a cube, is called a *cubic structure*. A theory T of graph language $\{R^{(2)}\}$ is *cubic* if $T = Th(\mathcal{M})$ for some cubic structure $\mathcal{M}$. In this case, the structure $\mathcal{M}$ is called a *cubic model* of T.

The *invariant* of theory T is the function

$$Inv_T : \omega \cup \{\infty\} \to \omega \cup \{\infty\},$$

satisfying the following conditions:

(1) for any natural n; $Inv_T(n)$ is the number of connected components in any model of T, being n-cubes, if that number is finite, and $Inv_T(n) = \infty$ if that number is infinite;

(2) $Inv_T(\infty) = 0$ if models of T do not contain infinite-dimensional cubes (i. e., dimensions of cubes are totally bounded), otherwise we set $Inv_T(\infty) = 1$.

The *diameter* $d(T)$ of cubic theory T is the maximal distance between elements in models of T, if these distances are bounded, and we set $d(T) \rightleftharpoons \infty$ otherwise. The *support* (accordingly the *∞-support*) $Supp(T)(Supp_\infty(T))$ of theory T is the set $\{n \in \omega | Inv_T(n) \neq 0\}(\{n \in \omega | Inv_T(n) = \infty\})$.

If the diameter $d(T)$ is finite then there exists an upper estimate for dimensions of cubes, being in models of T. It means that $Supp(T)$ is finite, i. e., $Inv_T(\infty) = 0$. In this case the ∞-support is non-empty.

If $d(T) = \infty$ then $Inv_T(\infty) = 1$. In this case the support $Supp(T)$ can be either finite or infinite.

1.2. **Preliminaries from model theory and approximations of theories.** Historically, pseudofinite fields were first introduced by J. Ax and S. Kochen [1] in the form of non-principal ultraproducts of finite fields. Later, J. Ax in [3] connect the notion of pseudofiniteness and the construction of ultraproducts. The class of *pseudofinite fields* was defined in the work of J. Ax [3] and regardless of him in the work of Yu.L. Ershov [7] with an axiom system indicating this class.

In 1965 James Ax [2] investigated fields F having the property that *every absolutely irreducible variety over F has a F-rational point*. It was shown that the non-principal ultraproduct of finite fields has such property. Yuri Leonidovich Ershov called such fields *regularly closed*. The notion of *pseudofiniteness* is credited to work in the 1968s by James Ax [3]. He introduced the notion of *pseudofiniteness* to show the decidability of the theory of all finite fields, i.e. there is an algorithm to decide whether a given statement is true for all finite fields. It was proved that pseudofinite fields are exactly those infinite fields that have every elementary property common to all finite fields, that is, pseudofinite fields are infinite models of the theory of finite fields.

In the early 1990s, E. Hrushovski resumed research in the field of pseudofinite structures in meeting on Finite and Infinite Combinatorics in Sets and Logic [8], as well as in the joint works of E. Hrushovski and G. Cherlin and the following definition first occurs in [5], subsequently in [6]:

Definition 2. Let Σ be a language and $\mathcal{M}$ be an Σ-structure. An Σ-structure $\mathcal{M}$ is *pseudofinite* if for all Σ-sentences φ, $\mathcal{M} \models \varphi$ implies that there is a finite $\mathcal{M}_0$ such that $\mathcal{M}_0 \models \varphi$. The theory $T = Th(\mathcal{M})$ of the pseudofinite structure $\mathcal{M}$ is called pseudofinite.

Definition 3. [16] Let $\mathcal{T}$ be a family of theories and T be a theory such that $T \notin \mathcal{T}$. The theory T is said to be *$\mathcal{T}$-approximated,* or *approximated by the family* $\mathcal{T}$, or a *pseudo-$\mathcal{T}$-theory,* if for any formula $\varphi \in T$ there exists $T' \in \mathcal{T}$ for which $\varphi \in T'$.

If the theory T is $\mathcal{T}$-approximated, then $\mathcal{T}$ is said to be an *approximating family* for T, and theories $T' \in \mathcal{T}$ are said to be *approximations* for T. We put $\mathcal{T}_\varphi = \{T \in \mathcal{T} \mid \varphi \in T\}$. Any set $\mathcal{T}_\varphi$ is called the *φ-neighbourhood*, or simply a *neighbourhood*, for $\mathcal{T}$. An approximating family $\mathcal{T}$ is called *e-minimal* if for any sentence $\varphi \in \Sigma(\mathcal{T})$, $\mathcal{T}_\varphi$ is finite or $\mathcal{T}_{\neg\varphi}$ is finite. It was shown in [16] that any *e-minimal* family $\mathcal{T}$ has unique accumulation point T with respect to neighbourhoods $\mathcal{T}_\varphi$, and $\mathcal{T} \cup \{T\}$ is also called *e-minimal*.

Recall that the *E-closure* [15] for the family $\mathcal{T}$ of complete theories is characterized by the following proposition.

Proposition 1. *Let $\mathcal{T}$ be a family of complete theories of the language Σ. Then $Cl_E(\mathcal{T}) = \mathcal{T}$ for a finite $\mathcal{T}$, and for an infinite $\mathcal{T}$, a theory T belongs to $Cl_E(\mathcal{T})$ if and only if T is a complete theory of the language Σ and $T \in \mathcal{T}$, or $T \neq \mathcal{T}$ and for any formula φ the set $\mathcal{T}_\varphi$ is infinite.*

We denote by $\overline{\mathcal{T}}$ the class of all complete elementary theories of relational languages, by $\overline{\mathcal{T}}_{fin}$ the subclass of $\overline{\mathcal{T}}$consisting of all theories with finite models, and by $\overline{\mathcal{T}}_{inf}$ the class $\overline{\mathcal{T}} \setminus \overline{\mathcal{T}}_{fin}$.

Proposition 2. [16] *For any theory T the following conditions are equivalent:*
(1) T is pseudofinite;
(2) T is $\overline{\mathcal{T}}_{fin}$-approximated;
(3) $T \in Cl_E(\overline{\mathcal{T}}_{fin}) \setminus \overline{\mathcal{T}}_{fin}$.

The structure of relational language Σ with universe $\bigsqcup_{i\in I} \mathcal{M}_i$ and interpretations of relation symbols from Σ represented as a combination of their interpretations in the structures $\mathcal{M}_i$, $i \in I$, is called a *disjoint union* $\bigsqcup_{i\in I} \mathcal{M}_i$ of disjoint structures $\mathcal{M}_i$, $i \in I$.

1.3. **Preliminaries from ranks for families of theories.** In paper [17], the *rank* $\mathrm{RS}(\cdot)$ for families of complete theories is inductively defined.

(1) For the empty family $\mathcal{T}$ we put the rank $\mathrm{RS}(\mathcal{T}) = -1$,

(2) For finite nonempty families $\mathcal{T}$ we put $\mathrm{RS}(\mathcal{T}) = 0$,

(3) For infinite families $\mathcal{T}$ we put $\mathrm{RS}(\mathcal{T}) \geq 1$.

(4) For a family $\mathcal{T}$ and an ordinal $\alpha = \beta + 1$ we put $\mathrm{RS}(\mathcal{T}) \geq \alpha$ if there are pairwise inconsistent $\Sigma(\mathcal{T})$-sentences φ_n, $n \in \omega$, such that $\mathrm{RS}(\mathcal{T}_{\varphi_n}) \geq \beta$, $n \in \omega$.

(5) If α is a limit ordinal then $\mathrm{RS}(\mathcal{T}) \geq \alpha$ if $\mathrm{RS}(\mathcal{T}) \geq \beta$ for any $\beta < \alpha$.

(6) We set $\mathrm{RS}(\mathcal{T}) = \alpha$ if $\mathrm{RS}(\mathcal{T}) \geq \alpha$ and $\mathrm{RS}(\mathcal{T}) \not\geq \alpha + 1$.

(7) If $\mathrm{RS}(\mathcal{T}) \geq \alpha$ for any α, we put $\mathrm{RS}(\mathcal{T}) = \infty$.

A family $\mathcal{T}$ is called *e-totally transcendental*, or *totally transcendental*, if $\mathrm{RS}(\mathcal{T})$ is an ordinal.

If $\mathcal{T}$ is e-totally transcendental, with $\mathrm{RS}(\mathcal{T}) = \alpha \geq 0$, we define the *degree* $\mathrm{ds}(\mathcal{T})$ of $\mathcal{T}$ as the maximal number of pairwise inconsistent sentences φ_i such that $\mathrm{RS}(\mathcal{T}_{\varphi_i}) = \alpha$.

Proposition 3. [17] *The family $\mathcal{T}$ is e-minimal if and only if $RS(\mathcal{T}) = 1$ and $ds(\mathcal{T}) = 1$*

Definition 4. [17] A family $\mathcal{T}$, with infinitely many accumulation points, is called *a-minimal* if for any sentence $\varphi \in \Sigma(T)$, $\mathcal{T}_\varphi$ or $\mathcal{T}_{\neg\varphi}$ has finitely many accumulation points.

Let α be an ordinal. A family $\mathcal{T}$ of rank α is called *α-minimal* if for any sentence $\varphi \in \Sigma(T)$, $\mathrm{RS}(\mathcal{T}_\varphi) < \alpha$ or $\mathrm{RS}(\mathcal{T}_{\neg\varphi}) < \alpha$.

Proposition 4. [17] (1) *A family $\mathcal{T}$ is 0-minimal if and only if $\mathcal{T}$ is a singleton.*

(2) *A family $\mathcal{T}$ is 1-minimal if and only if $\mathcal{T}$ is e-minimal.*

(3) *A family $\mathcal{T}$ is 2-minimal if and only if $\mathcal{T}$ is a-minimal.*

(4) *For any ordinal α a family $\mathcal{T}$ is α-minimal if and only if* $\mathrm{RS}(\mathcal{T}) = \alpha$ *and* $\mathrm{ds}(\mathcal{T}) = 1$.

2. Ranks for families of cubic theories

Let a language Σ consist of $R^{(2)}$. Denote by $\mathcal{T}_{cub}$ family of all cubic theories of language Σ. Let T be a cubic theory and $\mathcal{Q} \models T$. For a given cubic theory T, we consider the above invariants and the following possibilities:

<table>
<tr><td rowspan="4">$\mathcal{T}_{cub}$</td><td></td><td>$d(T)$ is finite</td><td>$d(T)$ is infinite</td></tr>
<tr><td rowspan="2">the number of $Inv_T(n)$ is bounded for $T \in \mathcal{T}$</td><td>$\mathcal{T}$ with $Inv_T(\infty) = 0$ and $Supp(T)$ is finite</td><td>$\mathcal{T}$ with $Inv_T(\infty) = 1$ and $Supp(T)$ is finite</td></tr>
<tr><td>$\mathcal{T}$ with $Inv_T(\infty) = 0$ and $Supp(T)$ is infinite</td><td>$\mathcal{T}$ with $Inv_T(\infty) = 1$ and $Supp(T)$ is infinite</td></tr>
<tr><td>the number of $Inv_T(n)$ is unbounded for $T \in \mathcal{T}$</td><td>$\mathcal{T}$ with $Inv_T(\infty) = 0$ and $Supp(T)$ is infinite</td><td>$\mathcal{T}$ with $Inv_T(\infty) = 1$ and $Supp(T)$ is infinite</td></tr>
</table>

2.1. **Family of cubic theories with a bounded number of $Inv_T(n)$.** In the case when for each theory T from the subfamily $\mathcal{T} \subset \mathcal{T}_{cub}$ it is true that the diameter $d(T)$ and $Inv_T(n)$ are finite, as well as $Inv_T(\infty) = 0$ or $Supp(T)$ is finite, the subfamily $\mathcal{T}$ is finite, therefore $RS(\mathcal{T}) = 0$, and the degree $ds(\mathcal{T})$ is equal to the number of invariants. Let's illustrate how the degrees of families vary.

Example 1. Let us deal with the finite family $\mathcal{T} = \{T_1\}$. If we consider n_0-cubes with invariant $Inv_{T_1}(n_0) = m$, then $RS(\mathcal{T}) = 0$, $ds(\mathcal{T}) = 1$. And if we work with n_0-cubes and n_1-cubes with $Inv_{T_1}(n_0) = m$ and $Inv_{T_1}(n_1) = l$ for $m \neq l$, then $ds(\mathcal{T}) = 2$. For a finite number k, if we are dealing with n_k-cubes with the set of invariants $\{Inv_{T_1}(n_0), \ldots, Inv_{T_1}(n_k)\}$, $n_i \neq n_j$, we still have $RS(\mathcal{T}) = 0$ and degree $ds(\mathcal{T}) = k + 1$.

Example 2. Let us deal with the finite family $\mathcal{T} \subset \mathcal{T}_{cub}$ consisting of theories $T_1, \ldots, T_n$. If the number of m_i-cubes in each theory T_i is equal to k, in other words, each theory has the same number of m_i-cubes, that is, $Inv_{T_i}(m_i) = k$ with $Inv_{T_i}(m_i) \neq Inv_{T_j}(m_j)$, $i \neq j$, then $RS(\mathcal{T}) = 0$, $ds(\mathcal{T}) = n$, since $\mathcal{T}$ is represented as a disjoint union of finite subfamilies $\mathcal{T}_{\varphi_i} = \{T_i \in \mathcal{T} | \varphi_i \in T_i$ is a sentence describing m_i-cubes $\}$.

RS($\mathcal{T}$) = 0		$\mathcal{T}_{\varphi_{n1}} \sqcup \ldots \sqcup \mathcal{T}_{\varphi_{nk}}$
	T_1	φ_{11} φ_{12} $\cdots$ φ_{1k}
	$\cdots$	$\cdots$ $\cdots$ $\cdots$ $\cdots$
	T_n	φ_{n1} φ_{n2} $\cdots$ φ_{nk}

In the examples above, one can notice that the degree of the family depends on the number of invariants. If for the theories considered in Example 2 we complicate the conditions that each theory has the same number of invariants, let, for example, s, then the degree of the family $\mathcal{T}$ is $ds(\mathcal{T}) = n \cdot s$. And if for different $s_1, \ldots, s_n$, in each theory T_i there are s_i invariants, then the family $\mathcal{T}$ has degree $ds(\mathcal{T}) = \sum_{i=1}^{n} s_i$.

For a family $\mathcal{T} \subset \mathcal{T}_{cub}$ with the property that $Inv_T(\infty) = 0$ and $Supp(T)$ is finite for every theory $T \in \mathcal{T}$, the degree varies in a similar way.

Let us now consider infinite subfamilies $\mathcal{T} \subset \mathcal{T}_{cub}$ of all cubic theories with a bounded number of $Inv_T(n) = \infty$ and $Inv_T(\infty) = 0$ for every $T \in \mathcal{T}$. In this case, $Supp(T)$ is infinite and the rank of the family increases, and for the degree of the family, we consider the number of accumulation points.

For natural numbers $n, m \in \omega$, with $n \neq m$, we denote by $\mathcal{T}_n$ the family of cubic theories from $\mathcal{T}_{cub}$ with one arbitrary value $Inv_T(n)$, where $T \in \mathcal{T}_n$ and $Inv_T(m) = 0$.

Proposition 5. *Each subfamily $\mathcal{T}_n$ of the family $\mathcal{T}_{cub}$ of all cubic theories is e-minimal.*

Proof. By Proposition 3, we need to prove that $RS(\mathcal{T}_n) = 1$ and $ds(\mathcal{T}_n) = 1$. The family $\mathcal{T}_n$ consists of the theories $T_1, \ldots, T_s$ with $Inv_{T_i}(n) = k_i$, $k_i > 0$ $1 \leq i \leq s$, and the only theory T_∞ with $Inv_{T_\infty}(n) = \infty$. The theory T_∞ is the unique accumulation point for $\mathcal{T}_n$ and the number of accumulation points is equal to the degree of the family. We get $RS(\mathcal{T}_n) = 1$ and $ds(\mathcal{T}_n) = 1$, which implies e-minimality of $\mathcal{T}_n$. □

Example 3. We are dealing with cubes of different dimensions n_0 and n_1. Then we get a countable number of variants $(Inv_T(n_0), Inv_T(n_1))$. Thus, there are countably many theories with n_0-cubes and n_1-cubes, forming the family $\mathcal{T}'$. Here, each family with one infinite $Inv_T(n_0)$ or $Inv_T(n_1)$ has $RS = 1$, and the only accumulation point with $Inv_T(n_0) = Inv_T(n_1) = \infty$, has infinitely many n_0-cubes, infinitely many n_1-cubes and $RS(\mathcal{T}') = 2$. Thus, for a given family $\mathcal{T}'$, we obtain $RS(\mathcal{T}') = 2$ and $ds(\mathcal{T}') = 1$. Therefore, the family is a-minimal.

Example 4. If there is a countable number of n_i-cubes, $i \in \omega$, having a countable number of options $(Inv_T(n_0), Inv_T(n_1), \ldots, Inv_T(n_i))$ one can construct an α-minimal family $\mathcal{T}$ consisting of a countable number of e-minimal subfamilies $\mathcal{T}_i$, $i \in \omega$. By the definition of α-minimality, the family $\mathcal{T}$ has $RS(\mathcal{T}) = \alpha$, $ds(\mathcal{T}) = 1$ and is represented as $\mathcal{T}_{\bigwedge_{i\in\omega}\varphi_i}$.

Thus, by increasing the number of invariants $Inv_T(n)$ and the dimension of the cubes, one can unboundedly increase the rank to any pre-given natural number. If the set of $Inv_T(n)$ is countable, then the family $\mathcal{T} \subset \mathcal{T}_{cub}$ of cubic theories is e-totally transcendental and may contain e-minimal, a-minimal, α-minimal subfamilies.

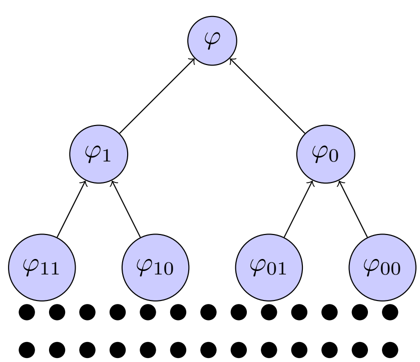
Realizations of e-minimal, a-minimal, α-minimal subfamilies of the $\mathcal{T}_{cub}$ family of all cubic theories allow one to construct a subfamily $\mathcal{T}$ having a given countable rank and degree. By the definition of α-minimality, the family $\mathcal{T}$ of cubic theories with $RS(\mathcal{T}) = \alpha$ and $ds(\mathcal{T}) = n$ can be represented as a disjoint union of subfamilies $\mathcal{T}_{Inv_T(k_0)}, \ldots, \mathcal{T}_{Inv_T(k_{n-1})}$, for some different $Inv_T(k_0), \ldots, Inv_T(k_{n-1})$, such that each $\mathcal{T}_{Inv_T(k_i)}$ is α-minimal.

RS($\mathcal{T}$) = α		$\mathcal{T}_{Inv_T(k_0)} \sqcup \ldots\ldots\ldots\ldots \sqcup \mathcal{T}_{Inv_T(k_{n-1})}$
	$\mathcal{T}_1$	$\mathcal{T}_{(1,1)} = \{T_{(1,1)} \ldots T_{(1,\infty)}\} \ldots \mathcal{T}_{(1,n)} = \{T_{(n,1)} \ldots T_{(n,\infty)}\}$
	$\cdots$	$\cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots$
	$\mathcal{T}_k$	$\mathcal{T}_{(k,1)} = \{T_{(1,1)} \ldots T_{(1,\infty)}\} \ldots \mathcal{T}_{(k,n)} = \{T_{(n,1)} \ldots T_{(n,\infty)}\}$
	$\cdots$	$\cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots$
	$\mathcal{T}_\alpha$	$\mathcal{T}_{(\alpha,1)} = \{T_{(1,1)} \ldots T_{(1,\infty)}\} \ldots \mathcal{T}_{(\alpha,n)} = \{T_{(n,1)} \ldots T_{(n,\infty)}\}$

2.2. **Family of cubic theories with an unbounded number of $Inv_T(n)$.** The next result shows that the family $\mathcal{T}_{cub}$ of all cubic theories is not e-totally transcendental.

Theorem 1. $RS(\mathcal{T}_{cub}) = \infty$.

Proof. Repeating the arguments of [16, Proposition 4.4] and [12, Proposition 2.5] we can construct a 2-tree of sentences φ, φ_0, φ_1, φ_{01}, ... indicating an infinite rank.



□

3. Approximations of cubic theories

The following theorem shows that any cubic theory is approximated by theories of finite cubic structures.

Theorem 2. *Any cubic theory T with an infinite set is pseudofinite.*

Proof. Let $\mathcal{Q}$ be an infinite model of the cubic theory T. Since for finite k and n, $Inv_T(n) = k$ and $Inv_T(\infty) = 0$ the cubic model $\mathcal{Q}$ is finite and consists of a finite number of finite connected components (n-cubes), we will consider only the following cases:

Case 1. If $Inv_T(n) = \infty$ and $Inv_T(\infty) = 0$ (that is, ∞-support is a singleton), then the model $\mathcal{Q}$ consists of an infinite number of connected components of finite diameters. The $\mathcal{Q}$ model is approximated by the disjoint union $\bigsqcup_{i\in\omega} \mathcal{Q}_i$ of models $\mathcal{Q}_i$, $i \in \omega$ which the connected components are n-cubes. Each such n-cubes are pairwise isomorphic that implies the pseudofiniteness of T.

Case 2. If for finite k and $n \in \omega$, $Inv_T(n) = k$ and $Inv_T(\infty) = 1$, then the theory T has models $\mathcal{Q} = \mathcal{Q}_0 \bigsqcup \mathcal{Q}_1$, where $\mathcal{Q}_0$ is a finite cubic model consisting of $m \leq k$ connected components (n-cubes) of finite diameters, $\mathcal{Q}_1$ is an infinite cubic model consisting of $k - m$ connected components of infinite diameters. Since the components of the model $\mathcal{Q}_0$ do not affect the pseudofiniteness, $\mathcal{Q}_1$ is approximated by increasing the dimension, as well as the diameters of the connected components. Let $\mathcal{Q}'_n$ be a finite model with $k - m$ connected components which are n-cubes. Using $\mathcal{Q}'_i = \mathcal{Q}'_2 \cup \mathcal{Q}'_{i-1}, i > 2$ in the limit, we obtain the desired model $\mathcal{Q}_1$. The set of theories $\{Th(\mathcal{Q}'_i)|i \in \omega\}$ approximate the theory $Th(\mathcal{Q}_1)$ and theories $\{Th(\mathcal{Q}_0 \bigsqcup \mathcal{Q}'_i)|i \in \omega\}$ approximate the T theory.

We can also grow connected components to get a pseudofinite model $\mathcal{Q}'$ with $Inv_T(n) = \infty$ and $Inv_T(\infty) = 1$, having components of both finite and infinite diameters.

Case 3. Let $Inv_T(n) = \infty$ and $Inv_T(\infty) = 1$. Let the cubic model $\mathcal{Q}$ have only an infinite number of connected components of infinite diameters. For the cubic model $\mathcal{Q}$, it is true that $\mathcal{Q} = \bigsqcup_{i\in\omega} \mathcal{Q}'_i$, where $\mathcal{Q}'_i = \mathcal{Q}'_2 \cup \mathcal{Q}'_{i-1}, i > 2$. That is, first we take the finite model and increase the diameters of the connected components, we get a model with a finite number of connected components, each of which is infinite-dimensional cubes, then, increasing the number of the connected components, we get the desired model $\mathcal{Q}$. □

4. Discussion

Pseudofinite structures in an explicit form after J. Ax were not studied for a long time. Until the 1990s, only a few results on this topic were obtained, and the very first result is the result of B.I. Zilber [23] asserting that ω-categorical theory is not finitely axiomatizable. At the time, the property of being pseudofinite was not considered particularly important or interesting, but the proof is based on pseudofiniteness.

One of the first results in the theory of classification of pseudofinite structures is the famous theorem of G. Cherlin, L. Harrington and A. Lachlan [4], which generalizes Zilber's theorem to the class of ω-stable ω-categorical structures, stating that totally categorical theories (and in more generally, ω-categorical ω-stable theories) are pseudofinite. They also proved that such structures are smoothly approximated by finite structures.

Recall that the ω-categorical structure $\mathcal{M}$ is said to be *smoothly approximated* if it is the union of an ω-chain of finite homogeneous substructures; or equivalently, if any sentence $\varphi \in Th(\mathcal{M})$ is true of some finite homogeneous substructure $\mathcal{N}$ of $\mathcal{M}$.

A. Lachlan introduced the concept of *smoothly approximable* structures to change the direction of analysis from finite to infinite, that is, to classify large finite structures that appear to be *smooth approximations* to an infinite limit.

Smoothly approximated structures were first examined in generality in [9], subsequently in [10]. The model theory of smoothly approximable structures has been developed very much further by G. Cherlin and E. Hrushovski [6]. The class of smoothly approximable structures is a class of ω-categorical supersimple structures of finite rank which properly contains the class of ω-categorical ω-stable structures (so in particular the totally categorical structures).

Remark 1. Note, that the above shows that the stable structures which are homogeneous in a finite relational language are ω-categorical and ω-stable, and ω-categorical ω-stable structures are smoothly approximated and pseudofinite. The converses of these statements are false.

Recall [21, 22] that a countable model $\mathcal{Q}$ of a theory T is called a *limit model* if $\mathcal{Q}$ is represented as the union of a countable elementary chain of models of the theory T that are simple over tuples, and the model $\mathcal{Q}$ itself is not simple over any tuple. A theory T is called *l-categorical* if T has a unique (up to isomorphism) limit model.

Homogeneity and l-categoricity, as well as the Morley rank for a fixed cubic theory, are studied in [20, 18].

Theorem 3. *Any model $\mathcal{Q}$ of the l-categorical cubic theory T is smoothly approximable by finite cubic structures.*

Proof. The limit model $\mathcal{Q}$ of l-categorical cubic theories T is represented as an ascending chain of finite prime substructures $\mathcal{Q}'_0 \subseteq \mathcal{Q}'_1 \subseteq \ldots \subseteq \mathcal{Q}$ such that $\mathcal{Q} = \bigcup_{i\in\omega} \mathcal{Q}'_i$ and there is an automorphism σ of $\mathcal{Q}$ such that $\sigma(\mathcal{Q}'_i) = \mathcal{Q}'_i$. □

5. Conclusion

In the paper the ranks and degrees for families of cubic theories are described. Several examples of families of finite rank cubic theories are given. It is proved that any cubic theory on an infinite set is pseudofinite.

<table>
<tr><td rowspan="4">$RS(\mathcal{T}_{cub}) = \infty$</td><td></td><td>$d(T)$ is finite</td><td>$d(T)$ is infinite</td></tr>
<tr><td rowspan="2">the number of $Inv_T(n)$ is bounded</td><td>for $\mathcal{T}$ with $Inv_T(\infty) = 0$ and $Supp(T)$ is finite, $RS(\mathcal{T}) = 0$</td><td>for $\mathcal{T}$ with $Inv_T(\infty) = 1$ and $Supp(T)$ is finite, $RS(\mathcal{T}) = 0$</td></tr>
<tr><td>for $\mathcal{T}$ with $Inv_T(\infty) = 0$ and $Supp(T)$ is infinite, $RS(\mathcal{T}) = \alpha$</td><td>for $\mathcal{T}$ with $Inv_T(\infty) = 1$ and $Supp(T)$ is infinite, $RS(\mathcal{T}) = \alpha$</td></tr>
<tr><td>the number of $Inv_T(n)$ is unbounded</td><td>for $\mathcal{T}$ with $Inv_T(\infty) = 0$ and $Supp(T)$ is infinite, $RS(\mathcal{T}) = \infty$</td><td>for $\mathcal{T}$ with $Inv_T(\infty) = 1$ and $Supp(T)$ is infinite, $RS(\mathcal{T}) = \infty$</td></tr>
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