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A STUDY ON STRONG (WEAK) VE-COVERING SETS AND VE-POSETS OF A GRAPH

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Abstract. The ve-degree of a vertex $v \in V$ denoted by $d_{ve}(v)$ is the number of edges in the subgraph $\langle N[v] \rangle$. A vertex v is said to ve-cover an edge e if e is an edge of the subgraph $\langle N[v] \rangle$. A set $S \subseteq V$ is called a ve-covering set of a graph G if every edge of G is ve-covered by some vertex in S. The ve-covering number denoted as $\alpha_{ve}(G)$ is the minimum cardinality of a ve-covering set of G. In this paper, we define new parameters such as strong (weak) ve-covering number and strong (weak) ve-separating number, and we establish a relationship between them. In addition, we define a partial order on the vertex set of a graph using ve-degree conditions, and study some of its properties.

Keywords: ve-covering number, strong ve-covering number, strong ve-separating number, ve-poset.

1. Introduction

By a graph $G = (V, E)$ we mean a finite, simple and undirected graph of order $|V| = p$ and size $|E| = q$, where V and E respectively denote the vertex set and the edge set of G. The terminologies and notations used here are as in [1, 2]. For any $v \in V$, the set $N(v) = \{u \in V : uv \in E\}$ is the open neighbourhood of the vertex v and the set $N[v] = N(v) \cup \{v\}$ is the closed neighbourhood of v. Then $\langle N[v] \rangle$ denotes the subgraph of G induced by the set $N[v]$.

A vertex v is said to cover an edge e if e is incident on v. A set $D \subseteq V$ is called a vertex cover of G if every edge in G is covered by some vertex in D. The vertex covering number $\alpha(G)$ is the minimum cardinality of a vertex cover of G. The strong (weak) vertex coverings of a graph was first introduced by S. S. Kamath and R. S. Bhat [5]. For an edge $e = uv$, v strongly covers the edge e if $d(v) \geq d(u)$. Then u weakly covers e. A set $S \subseteq V$ is a strong (weak) vertex cover of a graph G if every edge in G is strongly (weakly) covered by some vertex in S. The strong (weak) vertex covering number $s\alpha(G)$ $(w\alpha(G))$ is the minimum cardinality of a

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strong vertex cover (weak vertex cover) of G. These two parameters satisfy the following inequality: for any graph G, $s\alpha(G) \leq w\alpha(G) \leq \alpha(G)$.

In 1985, E. Sampathkumar and P. S. Neeralagi [4] initiated the study of neighbourhood set of a graph. A set $S \subseteq V$ is a neighbourhood set of G if $G = \bigcup_{v \in S} \langle N[v] \rangle$ where $\langle N[v] \rangle$ is the subgraph of G induced by $N[v]$. The neighbourhood number $n_0(G)$ is the minimum cardinality of a neighbourhood set of G. The concept of ve-degree of a vertex was introduced by S. S. Kamath and R. S. Bhat [3]. The ve-degree of a vertex $v \in V$ denoted by $d_{ve}(v)$ is the number of edges in the subgraph $\langle N[v] \rangle$. For any graph G without triangles, we have $d_{ve}(v) = d(v)$, for every vertex $v \in V$. The maximum ve-degree of a graph G is denoted by $\Delta_{ve}(G)$ and the minimum ve-degree of G is denoted by $\delta_{ve}(G)$. A graph G is said to be ve-regular if $d_{ve}(v) = d_{ve}(u)$, for every $v, u \in V$. A vertex v is said to ve-cover an edge e if e is an edge of the subgraph $\langle N[v] \rangle$. A set $S \subseteq V$ is called a ve-covering set of a graph G if every edge in G is ve-covered by some vertex in S. The ve-covering number denoted as $\alpha_{ve}(G)$ is the minimum cardinality of a ve-covering set of G. Note that, for any graph G without isolated vertices, any ve-covering set of G is also a neighbourhood set of G, and vice versa. Hence, $n_0(G) = \alpha_{ve}(G)$, for any graph G without isolated vertices. Further, if a graph G has k isolated vertices, then $n_0(G) = \alpha_{ve}(G) + k$.

2. Strong (weak) ve-covering sets and strong (weak) ve-separating sets of a graph

Definition. Let $G = (V, E)$ be a graph. We say that a vertex $v \in V$ strongly (weakly) ve-covers an edge $e = uw \in E$ if v ve-covers e and $d_{ve}(v) \geq d_{ve}(u)$ $(d_{ve}(v) \leq d_{ve}(u))$ and $d_{ve}(v) \geq d_{ve}(w)$ $(d_{ve}(v) \leq d_{ve}(w))$. A set $S \subseteq V$ is said to be a strong (weak) ve-covering set of G if elements of S strongly (weakly) ve-covers all the edges of G. The strong (weak) ve-covering number $s\alpha_{ve}(G)$ $(w\alpha_{ve}(G))$ of G is the minimum cardinality of a strong (weak) ve-covering set of G.

Definition. A set $S \subseteq V$ is said to be a strong (weak) ve-separating set of G if for every $v \in S$ and for any edge e in $\langle N[v] \rangle$ there exists a vertex $w \in V - S$ such that w weakly (strongly) ve-covers e. The strong (weak) ve-separating number $s\alpha'_{ve}(G)$ $(w\alpha'_{ve}(G))$ of G is the maximum cardinality of a strong (weak) ve-separating set of G.

Remark 2.1. *(i) For a null p-vertex graph $\overline{K_p}$, we assume that $s\alpha_{ve}(\overline{K_p}) = w\alpha_{ve}(\overline{K_p}) = 0$ and $s\alpha'_{ve}(\overline{K_p}) = w\alpha'_{ve}(\overline{K_p}) = p$.*

(ii) Let $G = (V, E)$ be a non-trivial and non-null graph and $u_{\Delta_{ve}}$ $(u_{\delta_{ve}})$ be a vertex of G of maximum (minimum) ve-degree. Then $V - \{u_{\delta_{ve}}\}$ $(V - \{u_{\Delta_{ve}}\})$ is a strong (weak) ve-covering set of G. Further, $\{u_{\Delta_{ve}}\}$ $(\{u_{\delta_{ve}}\})$ is a strong (weak) ve-separating set of G.

Example 2.1. *For the graph G given in the Figure 1, $d_{ve}(v_1) = 3$, $d_{ve}(v_2) = d_{ve}(v_3) = 5$, $d_{ve}(v_4) = 4$, $d_{ve}(v_5) = 2$ and $d_{ve}(v_6) = 1$. Thus, $\{v_3, v_5\}$ is a ve-covering set of G, $\{v_3, v_4, v_5\}$ is a strong ve-covering set of G and $\{v_1, v_4, v_5, v_6\}$ is a weak ve-covering set of G. Further, $\{v_2, v_3\}$ is a strong ve-separating set of G and $\{v_1, v_3, v_6\}$ is a weak ve-separating set of G. Hence, $\alpha_{ve}(G) = 2$, $s\alpha_{ve}(G) = 3$, $w\alpha_{ve}(G) = 4$, $s\alpha'_{ve}(G) = 2$ and $w\alpha'_{ve}(G) = 3$.*

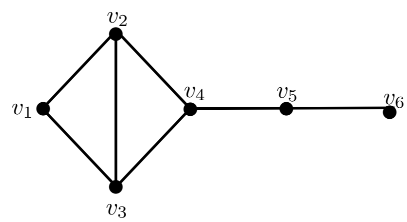

FIG. 1. Graph G

2.1. Preliminary results.

(i) For any graph G, $\alpha_{ve}(G) \leq min\{s\alpha_{ve}(G), w\alpha_{ve}(G)\}$ and, for any ve-regular graph G, $\alpha_{ve}(G) = s\alpha_{ve}(G) = w\alpha_{ve}(G)$

(ii) If a graph G has no triangles, then $s\alpha_{ve}(G) = s\alpha(G)$ and $w\alpha_{ve}(G) = w\alpha(G)$.

(iii) Let G be a connected graph of order p. Then $s\alpha_{ve}(G) = 1$ if and only if there exists $v \in V$ such that $d(v) = p - 1$.

(iv) Let G be a connected graph of order p. Then $w\alpha_{ve}(G) = 1$ if and only if $G = K_p$.

2.2. **Gallai-type results.** We now obtain Gallai-type results for the new parameters defined. We first prove the following.

Lemma 2.1. *Let $G = (V, E)$ be a graph. For any set $S \subseteq V$,*

(i) S is a strong ve-covering set of G if and only if $V - S$ is a weak ve-separating set of G.

(ii) S is a weak ve-covering set of G if and only if $V - S$ is a strong ve-separating set of G.

Proof. Let S be a strong ve-covering set of G and $W = V - S$. Then for any $w \in W$, let e be an edge in the subgraph $\langle N[w] \rangle$. Since S is a strong ve-covering set, there exists $s \in S$ such that s strongly ve-covers e. Thus, W is a weak ve-separating set of G. Conversely, let W be a weak ve-separating set and $S = V - W$. Let $e = uv \in E$. Then we consider the following cases:
case 1: If either $u \in W$ or $v \in W$, then either $e \in \langle N[u] \rangle$ or $e \in \langle N[v] \rangle$. Since W is a weak ve-separating set, there exists $s \in V - W = S$ such that s strongly ve-covers e.
case 2: If $u \notin W$ and $v \notin W$. Then either u or v strongly ve-covers e.
Hence, S is a strong ve-covering set.
Similarly, we can prove that the complement of a weak ve-covering set of G is a strong ve-separating set of G. □

Theorem 2.2. *For any graph $G = (V, E)$ of order p,*

(i) $s\alpha_{ve}(G) + w\alpha'_{ve}(G) = p$

(ii) $w\alpha_{ve}(G) + s\alpha'_{ve}(G) = p$.

Proof. Let S be a strong ve-covering set of G such that $|S| = s\alpha_{ve}(G)$. Then by Lemma 2.1, $V - S$ is a weak ve-separating set of G. Hence, $w\alpha'_{ve}(G) \geq |V - S|$. Therefore, $s\alpha_{ve}(G) + w\alpha'_{ve}(G) \geq p$. Again, if W is a weak ve-separating set of G such that $|W| = w\alpha'_{ve}(G)$. Then $V - W$ is a strong ve-covering set by Lemma 2.1. Hence, $s\alpha_{ve}(G) + w\alpha'_{ve}(G) \leq p$. Then from the above inequalities (i) follows. With similar arguments we can prove (ii). □

Remark 2.2. *(i) A minimum strong (weak) covering set of a graph G is a ve-covering set of G, but need not be a strong (weak) ve-covering set of G. For example, consider the graphs G and H given in the Figure 2. Note that $S = \{v_2, v_3, v_4, v_5\}$ is the minimum strong covering set of G; i.e., $|S| = s\alpha(G)$. But S is not a strong ve-covering set. Further, $W = \{u_1, u_3, u_4, u_6\}$ is the minimum weak covering set of H, which is not a weak ve-covering set of H.*

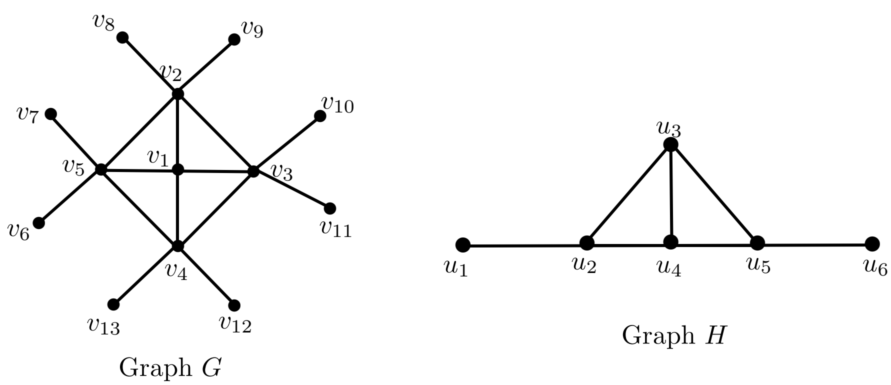

FIG. 2

(ii) The numbers $s\alpha_{ve}(G)$ and $w\alpha_{ve}(G)$ are incomparable in general. For example, in Figure 3, $\{v_1, v_4, v_5, v_6\}$ is a weak ve-covering set of G_1 and $\{v_1, v_2, v_3, v_7, v_8\}$ is a strong ve-covering set of G_1. Hence, $s\alpha_{ve}(G_1) = 5 > 4 = w\alpha_{ve}(G_1)$. On the other hand, let S_3 denote the star graph with 3-vertices. Then $s\alpha_{ve}(S_3) = 1 < 3 = w\alpha_{ve}(S_3)$.

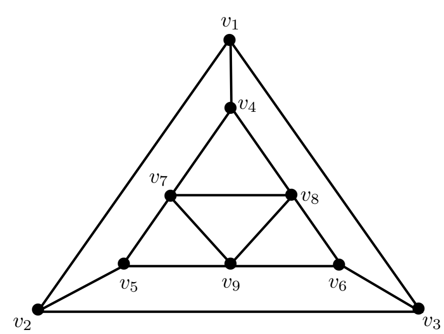

FIG. 3. Graph G_1

2.3. **Construction of a graph with arbitrarily large difference between $s\alpha_{ve}(G)$ and $\alpha_{ve}(G)$.** Consider the graph G_1 given in the Figure 3 and rename v_j by v_{1j},$\forall j$. We have $s\alpha_{ve}(G_1) - \alpha_{ve}(G_1) = 1$. Let $G_2' = G_1$ and we rename v_j by v_{2j},$\forall j$. Let G_2 be the graph obtained by joining the vertices v_{11} of G_1 and v_{22} of G_2' by an edge, as shown in the Figure 4. Note that $\{v_{11}, v_{12}, v_{13}, v_{17}, v_{18}, v_{21}, v_{22}, v_{23}, v_{27}, v_{28}\}$ is a strong ve-covering set of minimum cardinality and $\{v_{11}, v_{14}, v_{15}, v_{16}, v_{21}, v_{24}, v_{25}, v_{26}\}$ is a ve-covering set of minimum cardinality of G_2. Hence, $s\alpha_{ve}(G_2) - \alpha_{ve}(G_2) = 2$. Similarly, for $n \geq 3$, let $G_n' = G_1$ and rename v_j by v_{nj}, $\forall j$. Consider the graph

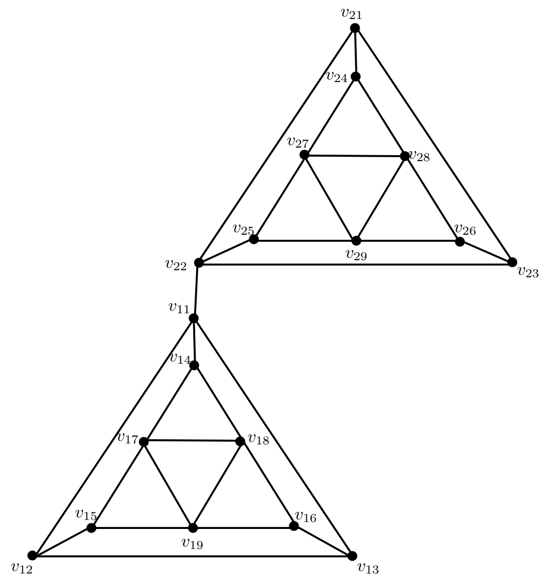

FIG. 4. Graph G_2

G_n obtained by joining the vertices v_{n-11} of G_{n-1} and v_{n2} of G'_n by an edge. Then $\bigcup_{i=1}^{n} \{v_{i1}, v_{i2}, v_{i3}, v_{i7}, v_{i8}\}$ is a strong ve-covering set of minimum cardinality of G_n and $\bigcup_{i=1}^{n} \{v_{i1}, v_{i4}, v_{i5}, v_{i6}\}$ is a ve-covering set of minimum cardinality of G_n. Therefore, $s\alpha_{ve}(G_n) - \alpha_{ve}(G_n) = n$. Thus, it is possible to find a graph with arbitrarily large difference between $s\alpha_{ve}$ and α_{ve}.

Remark 2.3. *Consider a star graph S_n with n vertices. We observe that the difference between $\alpha_{ve}(S_n)$ and $\alpha_{wve}(S_n)$ is $n-1$, which can be made arbitrarily large.*

2.4. **Algorithm to find the strong ve-covering number of a graph $G = (V, E)$. Input:** The vertex set $V = \{v_1, v_2, ..., v_p\}$ with $d_{ve}(v_1) \geq d_{ve}(v_2) \geq ... \geq d_{ve}(v_p)$, the edge set E and the edge set $E(\langle N[v_i]\rangle)$ of the subgraph $\langle N[v_i]\rangle$, for all $i, 1 \leq i \leq p$.

Output: $s\alpha_{ve}(G)$, $w\alpha'_{ve}(G)$

Algorithm:

$S = \{v_1\}$
$E = E - E(\langle N[v_1]\rangle)$
for $(i = 2;\ i \leq p;\ i = i + 1)$
  if $E \cap E(\langle N[v_i]\rangle) \neq \phi$ **then**
    for $(j = i + 1;\ j \leq p;\ j = j + 1)$
      if $d_{ve}(v_i) = d_{ve}(v_j)$ **AND** $|E(\langle N[v_i] - S\rangle)| < |E(\langle N[v_j] - S\rangle)|$ **then**
        $temp = v_i$ (Here *temp* is a temporary variable)
        $v_i = v_j$
        $v_j = temp$
      end if
    end for

$S = S \cup \{v_i\}$
$E = E - E(\langle N[v_i]\rangle)$
end if
end for
$s\alpha_{ve}(G) = |S|$
$W = V - S$
$w\alpha'_{ve}(G) = |W|$

Note: In a similar manner. we can construct an algorithm to find $w\alpha_{ve}(G)$ and $s\alpha'_{ve}(G)$ by considering the vertex set $V = \{v_1, v_2, ..., v_p\}$ with $d_{ve}(v_1) \leq d_{ve}(v_2) \leq ... \leq d_{ve}(v_p)$.

Remark 2.4. *The time complexity of the algorithm 2.4. in worst case scenario is $O(n^2)$.*

3. Strong and weak ve-degree of a vertex

Definition. The strong (weak) ve-degree of a vertex $v \in V$ denoted by $d_{sve}(v)$ ($d_{wve}(v)$) is the number of edges strongly (weakly) ve-covered by v.

Definition. The regular ve-degree of a vertex $v \in V$ denoted by $d_{rve}(v)$ is the number of edges which are both strongly and weakly ve-covered by v.

Definition. The balanced ve-degree of a vertex $v \in V$ denoted by $d_{bve}(v)$ is the number of edges which are neither strongly nor weakly ve-covered by v.

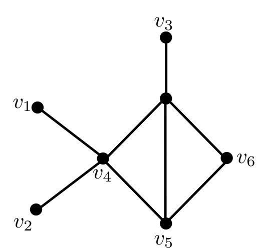

Fig. 5. Graph G

Example 3.1. *Consider the graph G given in the Figure 5. Then $d_{sve}(v_5) = 2$, $d_{wve}(v_5) = 3$, $d_{bve}(v_5) = 1$, $d_{rve}(v_5) = 1$.*

Theorem 3.1. *Let G=(V,E) be a graph. Then for any vertex $v \in V$, $d_{ve}(v) = d_{sve}(v) + d_{wve}(v) + d_{bve}(v) - d_{rve}(v)$.*

Proof. Consider any vertex $v \in V$ and let D be the set of all edges ve-covered by v, S be the set of all edges strongly ve-covered by v, W be the set of all edges weakly ve-covered by v, R be the set of all edges which are both strongly and weakly ve-covered by v and B be the set of all edges which are neither strongly nor weakly ve-covered by v. By definition, $S \cap W = R$, $S \cap B = \phi$, $W \cap B = \phi$ and $R \cap B = \phi$. Then, $d_{ve}(v) = |D| = |S \cup W \cup B| = |S| + |W| + |B| - |S \cap W| - |S \cap B| - |W \cap B| + |S \cap W \cap B| = d_{sve}(v) + d_{wve}(v) + d_{bve}(v) - d_{rve}(v)$. □

3.1. **Algorithm to find strong, weak, regular and balanced ve-degree of a vertex.** **Input:** A vertex $v \in V$ and the edge set of the subgraph $\langle N[v] \rangle$.
Output: $d_{sve}(v)$, $d_{wve}(v)$, $d_{rve}(v)$, $d_{bve}(v)$
Algorithm:
$S = \phi$
$W = \phi$
while $E(\langle N[v] \rangle) \neq \phi$
  if uw is any edge in $E(\langle N[v] \rangle)$ **then**
    if $d_{ve}(v) \geq d_{ve}(u)$ **AND** $d_{ve}(v) \geq d_{ve}(w)$ **then**
      $S = S \cup \{uw\}$
    end if
    if $d_{ve}(v) \leq d_{ve}(u)$ **AND** $d_{ve}(v) \leq d_{ve}(w)$ **then**
      $W = W \cup \{uw\}$
    end if
  end if
  $E(\langle N[v] \rangle) = E(\langle N[v] \rangle) - \{uw\}$
end while
$d_{sve}(v) = |S|$
$d_{wve}(v) = |W|$
$d_{rve}(v) = |S \cap W|$
$d_{bve}(v) = |E(\langle N[v] \rangle) - (S \cup W)|$

Remark 3.1. *The time complexity of the algorithm 3.1. in worst case scenario is $O(n)$.*

4. VE-STRONG NUMBER AND VE-WEAK NUMBER OF A GRAPH

Definition. A vertex $v \in V$ is called a ve-strong (ve-weak) vertex if $d_{ve}(v) \geq d_{ve}(u), \forall u \in N(v)$ $(d_{ve}(v) \leq d_{ve}(u), \forall u \in N(v))$.

Definition. A vertex $v \in V$ is called a ve-regular vertex if $d_{ve}(v) = d_{ve}(u), \forall u \in N(v)$.

Definition. A vertex $v \in V$ is called a ve-balanced vertex if there exists $u, w \in N(v)$ such that $d_{ve}(u) < d_{ve}(v) < d_{ve}(w)$.

Definition. A set $S \subseteq V$ is said to be a ve-strong set if every vertex in S is a ve-strong vertex. The ve-strong number of G denoted by $s_{ve}(G)$ is defined as $s_{ve}(G) = max\{|S|$:S is a ve-strong set of G$\}$.
Similarly, ve-weak set, ve-regular set, ve-balanced set, ve-weak number $(w_{ve}(G))$, ve-regular number $(r_{ve}(G))$ and ve-balanced number $(b_{ve}(G))$ of G are defined.

Example 4.1. *Consider the graph G given in the Figure 6. Here v_1 and v_4 are the*

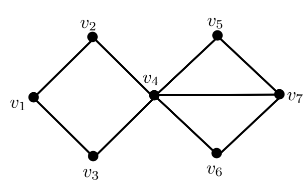

FIG. 6. Graph G

ve-strong vertices of G*,* v_1, v_2, v_3, v_5*, and* v_6 *are the ve-weak vertices of* G*,* v_1 *is the ve-regular vertex of* G *and* v_7 *is the ve-balanced vertex of* G*. Hence,* $s_{ve}(G) = 2$*,* $w_{ve}(G) = 5$*,* $r_{ve}(G) = 1$ *and* $b_{ve}(G) = 1$*.*

Theorem 4.1. *Let* $G = (V, E)$ *be a connected graph of order* p*. Then* $s_{ve}(G) + w_{ve}(G) - r_{ve}(G) + b_{ve}(G) = p$*.*

Proof. Let S, W, R and B respectively denotes the strong ve-set, weak ve-set, regular ve-set and balanced ve-set of G. By definition, $S \cap W = R$. Also, observe that any ve-balanced vertex is neither ve-strong vertex nor ve-weak vertex. Therefore, $(S \cup W) \cap B = \phi$. We have $V = (S \cup W) \cup B$. Hence, $p = |V| = |S \cup W| + |B| = |S| + |W| - |S \cap W| + |B| = s_{ve}(G) + w_{ve}(G) - r_{ve}(G) + b_{ve}(G)$. □

4.1. **Algorithm to find ve-strong number, ve-weak number and ve-balanced number of a graph G. Input:** The vertex set $V = \{v_1, v_2, ..., v_p\}$ of a graph G and $N(v_i) = \{u_{i_1}, u_{i_2}, ..., u_{i_{k_i}}\}$ of each vertex $v_i \in V$, where k_i is the usual degree of v_i, $\forall i$.

Output: $s_{ve}(G)$, $w_{ve}(G)$, $r_{ve}(G)$, $b_{ve}(G)$

Algorithm:

$S = \phi$
$W = \phi$
for $(i = 1;\ i \leq p;\ i = i + 1)$
  $flag1 = 0$
  $flag2 = 0$
  for $(j = 1;\ j \leq k_i;\ j = j + 1)$
    if $d_{ve}(u_{i_j}) > d_{ve}(v_i)$ **then**
      $flag1 = 1$
    else if $d_{ve}(u_{i_j}) < d_{ve}(v_i)$ **then**
      $flag2 = 1$
    end if
  end for
  if $flag1 = 0$ **then**
    $S = S \cup \{v_i\}$
  else if $flag2 = 0$ **then**
    $W = W \cup \{v_i\}$
  end if
end for
$s_{ve}(G) = |S|$
$w_{ve}(G) = |W|$
$r_{ve}(G) = |S \cap W|$
$b_{ve}(G) = p - |S \cup W|$

Remark 4.1. *The time complexity of the algorithm 4.1. in worst case scenario is* $O(n^2)$*.*

5. VE-POSET OF A GRAPH

A partial order on a set A is a relation on A which is reflexive, antisymmetric and transitive. A set on which a partial order is defined is called a partially ordered set or briefly poset. A poset P with a partial order $\leq$ is usually denoted by $(P, \leq)$. An element m of a poset $(P, \leq)$ is called a minimal (maximal) element of P if there is no element $x \in P$ such that $x < m$ $(x > m)$. A lattice is a poset in which every pair of elements has greatest lower bound and least upper bound. For a survey on posets refer [6].

Definition. Let $G = (V, E)$ be a graph. Define a relation $\geq$ on the vertex set V by, for any u, $v \in V$, $u \geq v$ if either $u = v$ or there exists a $u - v$ path in G say $u = v_1, v_2, ..., v_n = v$ such that $d_{ve}(v_1) > d_{ve}(v_2 > ... > d_{ve}(v_n)$. Then $\geq$ is a partial order on V. Hence, $(V, \geq)$ is a poset, called as the ve-poset of a graph G.

Example 5.1.

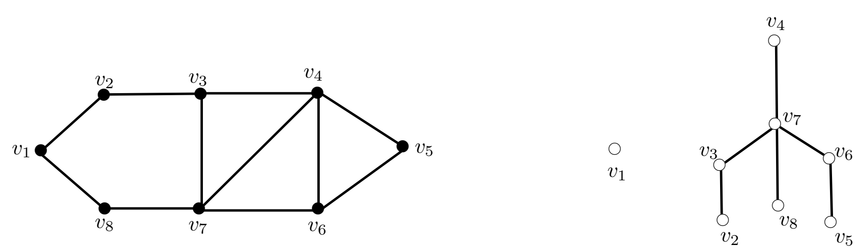

FIG. 7. Graph G and ve-poset $(V, \geq)$

Consider the graph $G = (V, E)$ given in the Figure 7. Then $d_{ve}(v_1) = 2$, $d_{ve}(v_2) = 2$, $d_{ve}(v_3) = 4$, $d_{ve}(v_4) = 7$, $d_{ve}(v_5) = 3$, $d_{ve}(v_6) = 5$, $d_{ve}(v_7) = 6$, $d_{ve}(v_8) = 2$. Then $v_4 > v_7 > v_6 > v_5$, $v_4 > v_7 > v_8$, $v_4 > v_3 > v_2$ and v_1 is not related to any element of $V - \{v_1\}$ with respect to $\geq$. The Hasse diagram of the ve-poset $(V, \geq)$ is given in the Figure 7.

Remark 5.1. *(i) The relation $\leq$ defined on V by, for any u, $v \in V$, $u \leq v$ if either $u = v$ or there exists a $u - v$ path in G say $u = v_1, v_2, ..., v_n = v$ such that $d_{ve}(v_1) < d_{ve}(v_2) < ... < d_{ve}(v_n)$ is a partial order on V. Moreover, $(V, \leq)$ is the dual of the ve-poset $(V, \geq)$.*

(ii) If two graphs G_1 and G_2 are isomorphic then their ve-posets are order isomorphic. But the converse need not be true. For example, consider the graphs $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ given in the Figure 8. Note that the graphs G_1

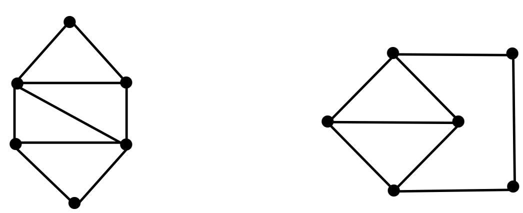

FIG. 8. Graphs G_1 and G_2

and G_2 are not isomorphic, but their ve-posets are order isomorphic. The Hasse diagram of the ve-posets $(V_1, \geq)$ and $(V_2, \geq)$ is given in the Figure 9.

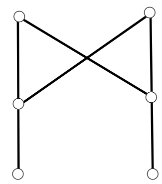

FIG. 9. Hasse diagram of the ve-posets $(V_1, \geq)$ and $(V_2, \geq)$

(iii) If a graph G is ve-regular, then its ve-poset is an antichain.

Theorem 5.1. *Let $G = (V, E)$ be a graph and $(V, \geq)$ be the ve-poset of G. Then*

- *(i) u is a ve-strong (ve-weak) vertex of G if and only if u is a maximal (minimal) element of V.*
- *(ii) u is a ve-balanced vertex of G if and only if u is neither a minimal nor a maximal element of V.*
- *(iii) u is a ve-regular vertex of G if and only if u is not related to any element of $V - \{u\}$ with respect to $\geq$.*

Proof. (i) Let u be a ve-strong vertex of G. Suppose there exists $v \in V$ such that $v > u$. Then there is a $v - u$ path in G say $v = v_1, v_2, ..., v_n = u$ such that $d_{ve}(v) > d_{ve}(v_2) > ... > d_{ve}(v_{n-1}) > d_{ve}(u)$. We observe that $v_{n-1} \in N(u)$. Since u is a ve-strong vertex, we have $d_{ve}(u) \geq d_{ve}(v_{n-1})$, which is a contradiction. Therefore, u is a maximal element of V. Conversely, assume that u is a maximal element of V. Suppose there exists $w \in N(u)$ such that $d_{ve}(w) > d_{ve}(u)$. Then, w, u is a $w - u$ path in G with the property $d_{ve}(w) > d_{ve}(u)$. This implies, $w > u$ in V, which is a contradiction to our assumption. Hence, u is a ve-strong vertex of G.

Similarly, we can prove that u is a ve-weak vertex of G if and only if u is a minimal element of V.

(ii) Let u be a ve-balanced vertex of G. Then there exists $v, w \in N(u)$ such that $d_{ve}(v) > d_{ve}(u) > d_{ve}(w)$. Then v, u is a $v - u$ path and u, w is a $u - w$ path in G such that $d_{ve}(v) > d_{ve}(u) > d_{ve}(w)$. This implies, $v > u > w$ in V. Hence, u is neither a minimal nor a maximal element of V. Conversely, let $u \in V$ such that u is neither a minimal nor a maximal element. Then there exists $v, w \in V$ such that $v > u > w$. Then there is a $v - u$ path in G say $v = v_1, v_2, ..., v_n = u$ such that $d_{ve}(v) > d_{ve}(v_2) > ... > d_{ve}(v_{n-1}) > d_{ve}(u)$ and a $u - w$ path in G say $u = u_1, u_2, ..., u_k = w$ such that $d_{ve}(u) > d_{ve}(u_2) > ... > d_{ve}(w)$. We observe that $v_{n-1}, u_2 \in N(u)$ such that $d_{ve}(v_{n-1}) > d_{ve}(u) > d_{ve}(u_2)$. Hence, u is a ve-balanced vertex of G.

(iii) A vertex u of G is ve-regular if and only if u is both ve-strong and ve-weak vertex of G if and only if u is both maximal and minimal element of V if and only if u is not related to any element of $V - \{u\}$ with respect to $\geq$. □

Theorem 5.2. *Let $G = (V, E)$ be a connected graph. Then the Hasse diagram of the ve-poset $(V, \geq)$ is a connected graph if and only if for any two vertices u and v of G, there exists a $u - v$ path in G having adjacent vertices of distinct ve-degrees.*

Proof. Assume that the Hasse diagram of the ve-poset $(V, \geq)$ is a connected graph. Suppose there exists two vertices u and v such that every $u - v$ path in G contains some adjacent vertices of same ve-degree. Then u and v are not related to each other in $(V, \geq)$. Now, suppose a lowerbound of u and v exists in V. We choose a lowerbound l of u and v such that l is a maximal element of the set of all lowerbounds of u and v. Then $u > l$ and $v > l$ in V. This implies, there exists a $u - l$ path in G say $u = u_1, u_2, ..., u_k = l$ such that $d_{ve}(u) > d_{ve}(u_2) > ... > d_{ve}(l)$ and a $v - l$ path in G say $v = v_1, v_2, ..., v_n = l$ such that $d_{ve}(v) > d_{ve}(v_2) > ... > d_{ve}(l)$. Then $u, u_2, ..., u_{k-1}, l, v_{n-1}, ..., v_2, v$ is a $u - v$ path in G such that adjacent vertices has the distinct ve-degrees, which is a contradiction. Hence, a lowerbound u and v does not exists in V. Similarly, we can prove that a upperbound u and v does not exists in V. This implies, u and v does not lie in the same component of the Hasse diagram graph of ve-poset $(V, \geq)$, which is a contradiction to our assumption. Thus, for any two vertices u and v of G, there exists a $u-v$ path in G having adjacent vertices of distinct ve-degrees. Conversely, assume that for any two vertices u and v of G, there exists a $u-v$ path in G having adjacent vertices of distinct ve-degrees. Let $u, v \in V$. Then by our assumption there exists a $u - v$ path say $u = u_1, u_2, ..., u_n = v$ in G having adjacent vertices of distinct ve-degrees. For any i, $1 \leq i \leq n - 1$, consider u_i and u_{i+1}. Then either $d_{ve}(u_i) > d_v e(u_{i+1})$ or $d_{ve}(u_i) < d_v e(u_{i+1})$. This implies, either $u_i > u_{i+1}$ or $u_i < u_{i+1}$ in V. Then there is a line between u_i and u_{i+1} in the Hasse diagram of $(V, \geq)$. Therefore, u and v lies in the same component of the Hasse diagram graph of the ve-poset $(V, \geq)$. Hence, the Hasse diagram of the ve-poset $(V, \geq)$ is a connected graph. □

Theorem 5.3. *Let $G = (V, E)$ be a graph of order $p > 1$. If $(V, \geq)$ is a lattice, then $s_{ve}(G) = 1$, $w_{ve}(G) = 1$, $r_{ve}(G) = 0$ and $b_{ve}(G) = p - 2$.*

Proof. Assume that the ve-poset $(V, \geq)$ is a lattice. Then V has unique maximal and minimal elements. By the Theorem 6.1, we have $s_{ve}(G) = 1$ and $w_{ve}(G) = 1$. Also, if G has a ve-regular vertex say u, then by the Theorem 6.1. we observe that the greatest lower bound of u with any element of the set $V - \{u\}$ does not exists in V, which is a contradiction to our assumption. This implies, $r_{ve}(G) = 0$. Hence, by Theorem 5.1., $b_{ve}(G) = p - 2$. □

Conclusion

The notion of mixed degrees namely, ve-degree motivated us to define strong (weak) ve-covering sets and their compliment sets. Further, we defined a partial order on the vertex set using ve-degree which relates the special type of vertices with poset related parameters. In a similar manner, strong (weak) neighbourhood sets can be defined using ve-degree conditions for graphs with isolated vertices, which yields the similar results.

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