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AN ALGORITHMIC APPROACH TO FIND N-COVERING SETS AND AN INTRODUCTION TO VE-POSET OF A GRAPH

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Abstract. In this paper, we defined strong (weak) n-covering number and strong (weak) n'-covering number and obtained Gallai's type result. We obtained algorithms to find the above defined numbers and showed that it is possible to get arbitrarily large difference between n-covering and strong n-covering numbers. Further, we investigated a partial order on set of vertices using ve-degree conditions and studied some of its properties.

Keywords: n-covering number, strong n-covering number, weak n-covering number, partially ordered set, lattice.

1. Introduction

By a graph $G = (V, E)$ we mean a finite, undirected, simple graph of order $|V| = p$ and size $|E| = q$, where V and E respectively denote the vertex set and the edge set of G. The terminologies and notations used here are as in [2, 5]. For $v \in V$, the open neighbourhood of v is given by $N(v) = \{u \in V : uv \in E\}$ and the closed neighbourhood of v is $N[v] = N(v) \cup \{v\}$. In 1985, E. Sampathkumar et al. [4] initiated the study of neighbourhood set of a graph. A set $S \subseteq V$ is a neighbourhood set of G if $G = \bigcup_{v \in S} \langle N[v] \rangle$ where $\langle N[v] \rangle$ is the subgraph of G induced by $N[v]$. The neighbourhood number $n_0(G)$ is the minimum cardinality of a neighbourhood set of G. A vertex $v \in V$ is said to *n-cover* an edge $e = uw \in E$ if $u, w \in N[v]$. The concept of ve-degree of a vertex was introduced by S.S. Kamath and R.S. Bhat [3]. A vertex $v \in V$ is ve-adjacent to an edge $e \in E$ if e is an edge of the subgraph $\langle N[v] \rangle$. The ve-degree of a vertex $v \in V$ denoted by $d_{ve}(v)$ is the number of edges ve-adjacent to v. In other words, $d_{ve}(v)$ is the number of edges n-covered by v. A graph $G = (V, E)$ is said to be ve-regular if $d_{ve}(v) = d_{ve}(u), \forall v, u \in V$. A partial order on a set A is a relation on A which is reflexive, antisymmetric and transitive. A set on which a partial order is defined is called a partially ordered set or briefly

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poset. A lattice is a poset in which every pair of elements has greatest lower bound and least upper bound. For a survey refer [1].

2. Strong n-covering set and strong n'-covering set of a graph

Definition. A vertex $v \in V$ is strongly (weakly) n-covers an edge $e = uw \in E$ if v n-covers e and $d_{ve}(v) \geq d_{ve}(u)$ ($d_{ve}(v) \leq d_{ve}(u)$) and $d_{ve}(v) \geq d_{ve}(w)$ ($d_{ve}(v) \leq d_{ve}(w)$). A set $S \subseteq V$ is said to be a strong (weak) n-covering set of G if elements of S strongly (weakly) n-covers all the edges of G. The strong (weak) n-covering number $sn_0(G)$ ($wn_0(G)$) of G is the minimum cardinality of a strong (weak) n-covering set of G.

Definition. A set $S \subseteq V$ is said to be a strong (weak) n'-covering set of G if for every $v \in S$ and for any edge e in $\langle N[v] \rangle$ there exists a vertex $w \in V - S$ such that w weakly (strongly) n-covers e. The strong (weak) n'-covering number $sn'_0(G)$ ($wn'_0(G)$) of G is the maximum cardinality of a strong (weak) n'-covering set of G.

Example 2.1.

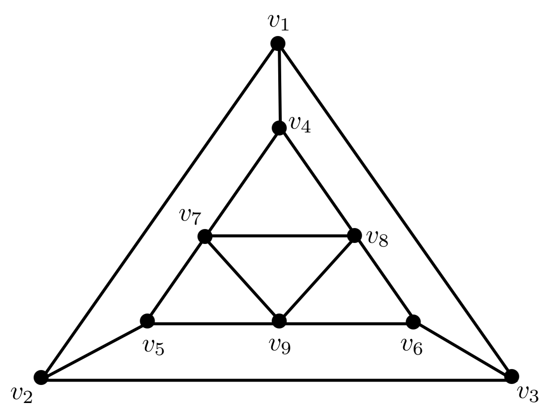

Fig. 1. Graph G_1

Consider the graph G_1 with vertex set $V = \{v_1, v_2, ..., v_9\}$ as given in the FIG. 1. Then $S_1 = \{v_1, v_4, v_5, v_6\}$ is both n-covering set and weak n-covering set of G_1 and $S_2 = \{v_1, v_2, v_3, v_7, v_8\}$ is a strong n-covering set of G_1. Also, $V - S_1$ is a strong n'-covering set of G_1 and $V - S_2$ is a weak n'-covering set of G_1. We observe that, $n_0(G_1) = 4$, $wn_0(G_1) = 4$, $sn_0(G_1) = 5$, $sn'_0(G_1) = 5$ and $wn'_0(G_1) = 4$.

2.1. Preliminary results.

(i) For any graph G, $n_0(G) \leq min\{sn_0(G), wn_0(G)\}$.
(ii) For any ve-regular graph G, $n_0(G) = sn_0(G) = wn_0(G)$
(iii) Let G be a connected graph of order p. Then $sn_0(G) = 1$ if and only if there exists $v \in V$ such that $d(v) = p - 1$.
(iv) Let G be a connected graph of order p. Then $wn_0(G) = 1$ if and only if $G = K_p$.
(v) For a path graph $G = P_n$ with $n > 2$, $n_0(G) = sn_0(G) = \lfloor \frac{n}{2} \rfloor$ and $wn_0(G) = \lfloor \frac{n}{2} \rfloor + 1$.

(vi) For a complete bipartite graph $G = K_{m,n}$, $n_0(G) = sn_0(G) = min\{m, n\}$ and $wn_0(G) = max\{m, n\}$.

Proof. (iv) If G is connected graph of order p and size q with $wn_0(G) = 1$, then there exists $v \in V$ which weakly n-covers all the edges of G. This implies that $q = d_{ve}(v) \leq d_{ve}(u), \forall u \in V$. Therefore, $d_{ve}(u) = q$, $\forall u \in V$. Thus $G = K_p$. Conversely, if $G = K_p$ then $d_{ve}(v) = q, \forall v \in V$. Hence, $wn_0(G) = 1$. □

Theorem 2.1. *For any graph $G = (V, E)$ of order p,*
$sn_0(G) + wn'_0(G) = p$
$wn_0(G) + sn'_0(G) = p.$

Proof. Let S be a strong n-covering set of G and $W = V - S$. Then for any $w \in W$, let e be an edge in the subgraph $\langle N[w]\rangle$. Since S is a strong n-covering set, there exists $s \in S$ such that s strongly n-covers e. Thus, W is a weak n'-covering set of G. Conversely, let W be a weak n'-covering set and $S = V - W$. Let $e = uv \in E$. Then we consider the following cases:
case 1: If either $u \in W$ or $v \in W$, then either $e \in \langle N[u]\rangle$ or $e \in \langle N[v]\rangle$. Since W is a weak n'-covering set, there exists $s \in S$ such that s strongly n-covers e.
case 2: If $u \notin W$ and $v \notin W$. Then either u or v strongly n-covers e.
Hence, S is a strong n-covering set.
Similarly, we can prove that the complement of a weak n-covering set of G is a strong n'-covering set of G. □

3. Algorithms

Let $G = (V, E)$ be a graph of order p. If e is an edge joining the vertices v and w, then we denote e by vw. For any $v \in V$, let $E(\langle N[v]\rangle)$ denote the edge set of the subgraph $\langle N[v]\rangle$.

3.1. **Algorithm to find N[v] and ve-degree of a vertex v ∈ V. Input:** The vertex set V and the edge set E of a graph G and a vertex $v \in V$.
Output: $N(v)$, $N[v]$, $d_{ve}(v)$.
Algorithm:

```
N(v) = φ
while V ≠ φ
  if u ∈ V AND uv ∈ E then
    N(v) = N(v) ∪ {u}
  end if
  V = V − {u}
end while
N[v] = N(v) ∪ {v}
d_ve(v) = 0
N = φ
while N(v) ≠ φ
  if u ∈ N(v) then
    N = N ∪ {uv}
    while N(u) ≠ φ
    if w ∈ N(u) ∩ N(v) AND uw ∈ E then
      N = N ∪ {uw}
    end if
```

$N(u) = N(u) - \{w\}$
end while
end if
$N(v) = N(v) - \{u\}$
end while
$d_{ve}(v) = |N|$

3.2. Algorithm to find a neighbourhood set of a graph G.

3.2.1. *Algorithm to find the edge set $E(\langle \boldsymbol{S} \rangle)$ of the subgraph induced by a subset S of V.* **Input:** The vertex set $V = \{v_1, v_2, ..., v_p\}$ and the edge set E of of a graph G. A set $S = \{u_1, u_2, ..., u_k\} \subseteq V$.
Output: $E(\langle S \rangle)$
Algorithm:
$E(\langle S \rangle) = \phi$
for $(i = 1,\ i \leq k,\ i = i + 1)$
for $(j = i + 1,\ j \leq k,\ j = j + 1)$
if $u_i u_j \in E$ **then**
$E(\langle S \rangle) = E(\langle S \rangle) \cup \{u_i u_j\}$
end if
end for
end for

3.2.2. *Algorithm to sort the elements of the vertex set V of a graph G in the descending order of the ve-degrees.* **Input:** The vertex set $V = \{v_1, v_2, ..., v_p\}$ of a graph G and the ve-degree of each vertex $v_i \in V, \forall i$.
Output: The vertex set V with elements arranged in the descending order of the ve-degrees.
Algorithm:
initialize $v[\ ] = \{v_1, v_2, ..., v_p\}$
$temp = 0$
for $(i = 1;\ i \leq p;\ i = i + 1)$
for $(j = i + 1;\ j \leq p;\ j = j + 1)$
if $d_{ve}(v[i]) < d_{ve}(v[j])$ **then**
$temp = v[i]$
$v[i] = v[j]$
$v[j] = temp$
end if
end for
end for

Note: We name the above algorithm by the function $sort(V, G)$.

3.2.3. *Algorithm to find a minimal neighbourhood set of a graph G.* **Input:** The vertex set $V = \{v_1, v_2, ..., v_p\}$ and the edge set E of a graph G. The ve-degree and $E(\langle N[v_i] \rangle)$ of each vertex $v_i \in V, \forall i$.
Output: A minimal neighbourhood set S of G.
Algorithm:
$S = \phi$
while $E \neq \phi$

$v[\,] = sort(V - S, \langle V - S\rangle)$
if $E \cap E(\langle N[v[1]]\rangle) \neq \phi$ **then**
$S = S \cup \{v[1]\}$
$E = E - E(\langle N[v[1]]\rangle)$
end if
end while

3.3. Algorithm to find the strong n-covering number of a graph G.

Input: The vertex set $V = \{v_1, v_2, ..., v_p\}$ with $d_{ve}(v_1) \geq d_{ve}(v_2) \geq ... \geq d_{ve}(v_p)$ and the edge set E of a graph G. $E(\langle N[v_i]\rangle)$ of each vertex $v_i \in V$.
Output: $sn_0(G)$, $wn'_0(G)$
Algorithm:
$S = \{v_1\}$
$temp = 0$
for $(i = 2;\ i \leq p;\ i = i + 1)$
if $E \cap E(\langle N[v_i]\rangle) \neq \phi$ **then**
for $(j = i + 1;\ j \leq p;\ j = j + 1)$
if $d_{ve}(v_i) = d_{ve}(v_j)$ **AND** $|E(\langle N[v_i] - S\rangle)| < |E(\langle N[v_j] - S\rangle)|$ **then**
$temp = v_i$
$v_i = v_j$
$v_j = temp$
end if
end for
$S = S \cup \{v_i\}$
$E = E - E\left(\langle N[v_i]\rangle\right)$
end if
end for
$sn_0(G) = |S|$
$W = V - S$
$wn'_0(G) = |W|$

Note: In the similar manner we can construct an algorithm to find $wn_0(G)$ and $sn'_0(G)$ by considering the vertex set $V = \{v_1, v_2, ..., v_p\}$ with $d_{ve}(v_1) \leq d_{ve}(v_2) \leq ... \leq d_{ve}(v_p)$.

3.4. Construction of a graph with arbitrarily large difference between $sn_0(G)$ and $n_0(G)$.

Consider the graph G_1 given in the FIG. 1 and rename v_j by v_{1j},$\forall j$. We have $sn_0(G_1) - n_0(G_1) = 1$. Let $G'_2 = G_1$ and we rename v_j by v_{2j},$\forall j$. Let G_2 be the graph obtained by joining the vertices v_{11} of G_1 and v_{22} of G'_2 by an edge, as shown in the FIG. 2.

Note that $\{v_{11}, v_{12}, v_{13}, v_{17}, v_{18}, v_{21}, v_{22}, v_{23}, v_{27}, v_{28}\}$ is a strong n-covering set of minimum cardinality and $\{v_{11}, v_{14}, v_{15}, v_{16}, v_{21}, v_{24}, v_{25}, v_{26}\}$ is a n-covering set of minimum cardinality of G_2. Hence, $sn_0(G_2) - n_0(G_2) = 2$. Similarly, for $n \geq 3$, let $G'_n = G_1$ and rename v_j by v_{nj}, $\forall j$. Consider the graph G_n obtained by joining the vertices v_{n-11} of G_{n-1} and v_{n2} of G'_n by an edge. Then $\bigcup\limits_{i=1}^{n}\{v_{i1}, v_{i2}, v_{i3}, v_{i7}, v_{i8}\}$ is

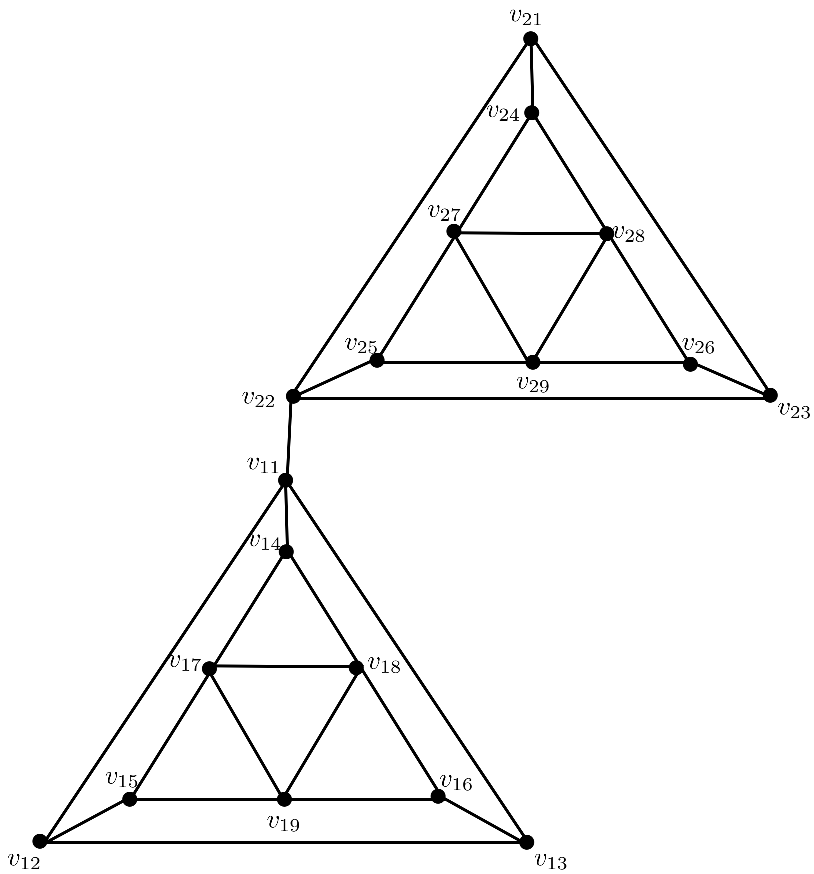

FIG. 2. Graph G_2

a strong n-covering set of minimum cardinality of G_n and $\bigcup_{i=1}^{n}\{v_{i1}, v_{i4}, v_{i5}, v_{i6}\}$ is a n-covering set of minimum cardinality of G_n. Therefore, $sn_0(G_n) - n_0(G_n) = n$. Thus, it is possible to find a graph with arbitrarily large difference between sn_0 and n_0.

4. Strong and weak ve-degree of a vertex

Definition. The strong (weak) ve-degree of a vertex $v \in V$ denoted by $d_{sve}(v)$ ($d_{wve}(v)$) is the number of edges strongly (weakly) n-covered by v.

Definition. The regular ve-degree of a vertex $v \in V$ denoted by $d_{rve}(v)$ is the number of edges which are both strongly and weakly n-covered by v.

Definition. The balanced ve-degree of a vertex $v \in V$ denoted by $d_{bve}(v)$ is the number of edges which are neither strongly nor weakly n-covered by v.

Example 4.1.

Consider the graph G given in the FIG. 3. Then $d_{sve}(v_5) = 2$, $d_{wve}(v_5) = 3$, $d_{bve}(v_5) = 1$, $d_{rve}(v_5) = 1$.

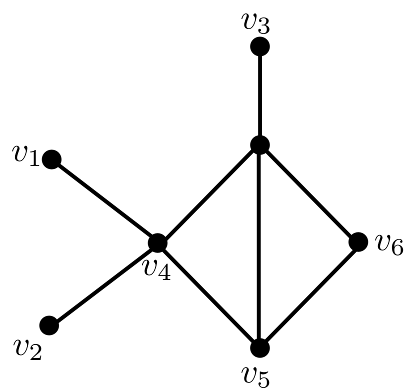

FIG. 3. Graph G

Theorem 4.1. *Let G=(V,E) be a graph. Then for any vertex $v \in V$,*
$d_{ve}(v) = d_{sve}(v) + d_{wve}(v) + d_{bve}(v) - d_{rve}(v)$.

Proof. Consider any vertex $v \in V$ and let D be the set of all edges n-covered by v, S be the set of all edges strongly n-covered by v, W be the set of all edges weakly n-covered by v, R be the set of all edges which are both strongly and weakly n-covered by v and B be the set of all edges which are neither strongly nor weakly n-covered by v. By definition, $S \cap W = R$, $S \cap B = \phi$, $W \cap B = \phi$ and $R \cap B = \phi$. Then, $d_{ve}(v) = |D| = |S \cup W \cup B| = |S| + |W| + |B| - |S \cap W| - |S \cap B| - |W \cap B| + |S \cap W \cap B|$ $=d_{sve}(v) + d_{wve}(v) + d_{bve}(v) - d_{rve}(v)$. □

4.1. **Algorithm to find strong, weak, regular and balanced ve-degree of a vertex. Input:** A vertex $v \in V$ and the edge set of the subgraph $\langle N[v] \rangle$.
Output: $d_{sve}(v)$, $d_{wve}(v)$, $d_{rve}(v)$, $d_{bve}(v)$
Algorithm:
$d_{sve}(v) = 0$
$d_{wve}(v) = 0$
$d_{rve}(v) = 0$
$d_{bve}(v) = 0$
$S = \phi$
$W = \phi$
while $E(\langle N[v] \rangle) \neq \phi$
  if $uw \in E(\langle N[v] \rangle)$ **then**
    if $d_{ve}(v) \geq d_{ve}(u)$ **AND** $d_{ve}(v) \geq d_{ve}(w)$ **then**
      $S = S \cup \{uw\}$
    end if
    if $d_{ve}(v) \leq d_{ve}(u)$ **AND** $d_{ve}(v) \leq d_{ve}(w)$ **then**
      $W = W \cup \{uw\}$
    end if
  end if
  $E(\langle N[v] \rangle) = E(\langle N[v] \rangle) - \{uw\}$
end while
$d_{sve}(v) = |S|$
$d_{wve}(v) = |W|$

$d_{rve}(v) = |S \cap W|$
$d_{bve}(v) = |E(\langle N[v] \rangle) - (S \cup W)|$

5. VE-STRONG NUMBER AND VE-WEAK NUMBER OF A GRAPH

Definition. A vertex $v \in V$ is called a ve-strong (ve-weak) vertex if $d_{ve}(v) \geq d_{ve}(u), \forall u \in N(v)$ $(d_{ve}(v) \leq d_{ve}(u), \forall u \in N(v))$.

Definition. A vertex $v \in V$ is called a ve-regular vertex if $d_{ve}(v) = d_{ve}(u), \forall u \in N(v)$.

Definition. A vertex $v \in V$ is called a ve-balanced vertex if there exists $u, w \in N(v)$ such that $d_{ve}(u) < d_{ve}(v) < d_{ve}(w)$.

Definition. A set $S \subseteq V$ is said to be a ve-strong set if every vertex in S is a ve-strong vertex. The ve-strong number of G denoted by $S_{ve}(G)$ is defined as $S_{ve}(G) = max\{|S|$:S is a ve-strong set of G$\}$.
Similarly, ve-weak set, ve-regular set, ve-balanced set, ve-weak number $(W_{ve}(G))$, ve-regular number $(R_{ve}(G))$ and ve-balanced number $(B_{ve}(G))$ of G are defined.

Example 5.1.

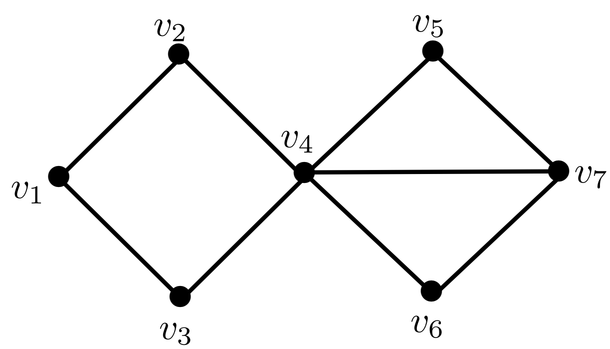

FIG. 4. Graph G

Consider the graph G given in the FIG. 4. Then v_1 and v_4 are the ve-strong vertices of G, v_1, v_2, v_3, v_5, and v_6 are the ve-weak vertices of G, v_1 is the ve-regular vertex of G and v_7 is the ve-balanced vertex of G. Hence, $S_{ve}(G) = 2$, $W_{ve}(G) = 5$, $R_{ve}(G) = 1$ and $B_{ve}(G) = 1$.

Theorem 5.1. *Let $G = (V, E)$ be a connected graph of order p. Then $S_{ve}(G) + W_{ve}(G) - R_{ve}(G) + B_{ve}(G) = p$.*

Proof. Let S, W, R and B respectively denotes the strong ve-set, weak ve-set, regular ve-set and balanced ve-set of G. By definition, $S \cap W = R$. Also, observe that any ve-balanced vertex is neither ve-strong vertex nor ve-weak vertex. Therefore, $(S \cup W) \cap B = \phi$. We have $V = (S \cup W) \cup B$. Hence, $p = |V| = |S \cup W| + |B| = |S| + |W| - |S \cap W| + |B| = S_{ve}(G) + W_{ve}(G) - R_{ve}(G) + B_{ve}(G)$. □

5.1. **Algorithm to find ve-strong number, ve-weak number and ve-balanced number of a graph G. Input:** The vertex set $V = \{v_1, v_2, ..., v_p\}$ of a graph G and $N(v_i) = \{u_{i_1}, u_{i_2}, ..., u_{i_{k_i}}\}$ of each vertex $v_i \in V$, where k_i is the degree of v_i, $\forall i$.

Output: $S_{ve}(G)$, $W_{ve}(G)$, $R_{ve}(G)$, $B_{ve}(G)$

Algorithm:

$S = \phi$
$W = \phi$
for $(i = 1;\ i \leq p;\ i = i + 1)$
  $flag1 = 0$
  $flag2 = 0$
  for $(j = 1;\ j \leq k_i;\ j = j + 1)$
    if $d_{ve}(u_{i_j}) > d_{ve}(v_i)$ **then**
      $flag1 = 1$
    else if $d_{ve}(u_{i_j}) < d_{ve}(v_i)$ **then**
      $flag2 = 1$
    end if
  end for
  if $flag1 = 0$ **then**
    $S = S \cup \{v_i\}$
  else if $flag2 = 0$ **then**
    $W = W \cup \{v_i\}$
  end if
end for
$S_{ve}(G) = |S|$
$W_{ve}(G) = |W|$
$R_{ve}(G) = |S \cap W|$
$B_{ve}(G) = p - |S \cup W|$

6. VE-POSET OF A GRAPH

Definition. Let $G = (V, E)$ be a graph. Define a relation $\geq$ on the vertex set V by, for any u, $v \in V$, $u \geq v$ if either $u = v$ or there exists a $u - v$ path in G say $u = v_1, v_2, ..., v_n = v$ such that $d_{ve}(v_1) > d_{ve}(v_2) > ... > d_{ve}(v_n)$. Then $\geq$ is a partial order on V. Hence, $(V, \geq)$ is a poset, called as the ve-poset of a graph G.

Example 6.1.

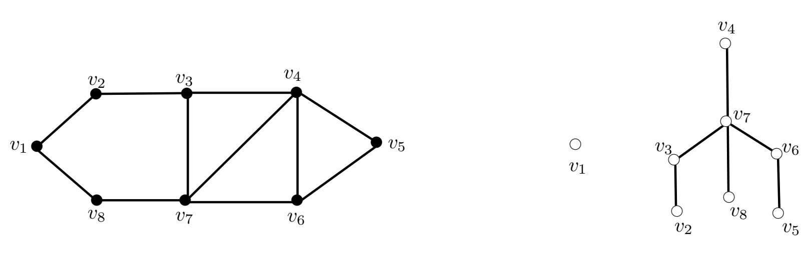

FIG. 5. Graph G and ve-poset $(V, \geq)$

Consider the graph $G = (V, E)$ given in the FIG. 5. Then $d_{ve}(v_1) = 2$, $d_{ve}(v_2) = 2$, $d_{ve}(v_3) = 4$, $d_{ve}(v_4) = 7$, $d_{ve}(v_5) = 3$, $d_{ve}(v_6) = 5$, $d_{ve}(v_7) = 6$, $d_{ve}(v_8) = 2$.

Then $v_4 > v_7 > v_6 > v_5$, $v_4 > v_7 > v_8$, $v_4 > v_3 > v_2$ and v_1 is not related to any element of $V - \{v_1\}$ with respect to $\geq$. The Hasse diagram of the ve-poset $(V, \geq)$ is given in the FIG. 5.

6.1. **Preliminary results.**

(i) The relation $\leq$ defined on V by, for any $u, v \in V$, $u \leq v$ if either $u = v$ or there exists a $u - v$ path in G say $u = v_1, v_2, ..., v_n = v$ such that $d_{ve}(v_1) < d_{ve}(v_2) < ... < d_{ve}(v_n)$ is a partial order on V. Moreover, $(V, \leq)$ is the dual of the ve-poset $(V, \geq)$.

(ii) If two graphs G_1 and G_2 are isomorphic then their ve-posets are order isomorphic. But the converse need not be true. For example, consider the graphs $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ given in the FIG. 6.

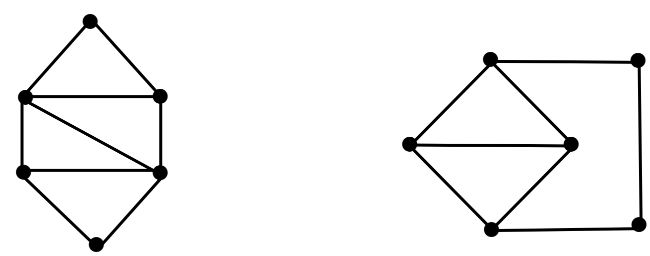

FIG. 6. Graphs G_1 and G_2

Note that the graphs G_1 and G_2 are not isomorphic, but their ve-posets are order isomorphic. The Hasse diagram of the ve-posets $(V_1, \geq)$ and $(V_2, \geq)$ is given in the FIG. 7.

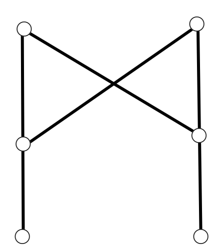

FIG. 7. Hasse diagram of the ve-posets $(V_1, \geq)$ and $(V_2, \geq)$

(iii) If a graph G is ve-regular, then its ve-poset is an antichain.

Theorem 6.1. *Let $G = (V, E)$ be a graph and $(V, \geq)$ be the ve-poset of G. Then*

(i) u is a ve-strong (ve-weak) vertex of G if and only if u is a maximal (minimal) element of V.

(ii) u is a ve-balanced vertex of G if and only if u is neither a minimal nor a maximal element of V.

(iii) u is a ve-regular vertex of G if and only if u is not related to any element of $V - \{u\}$ with respect to $\geq$.

Proof. *(i)* Let u be a ve-strong vertex of G. Suppose there exists $v \in V$ such that $v > u$. Then there is a $v - u$ path in G say $v = v_1, v_2, ..., v_n = u$ such that $d_{ve}(v) > d_{ve}(v_2) > ... > d_{ve}(v_{n-1}) > d_{ve}(u)$. We observe that $v_{n-1} \in N(u)$. Since u is a ve-strong vertex, we have $d_{ve}(u) \geq d_{ve}(v_{n-1})$, which is a contradiction. Therefore, u is a maximal element of V. Conversely, assume that u is a maximal element of V. Suppose there exists $w \in N(u)$ such that $d_{ve}(w) > d_{ve}(u)$. Then, w, u is a $w - u$ path in G with the property $d_{ve}(w) > d_{ve}(u)$. This implies, $w > u$ in V, which is a contradiction to our assumption. Hence, u is a ve-strong vertex of G.
Similarly, we can prove that u is a ve-weak vertex of G if and only if u is a minimal element of V.
(ii) Let u be a ve-balanced vertex of G. Then there exists $v, w \in N(u)$ such that $d_{ve}(v) > d_{ve}(u) > d_{ve}(w)$. Then v, u is a $v - u$ path and u, w is a $u - w$ path in G such that $d_{ve}(v) > d_{ve}(u) > d_{ve}(w)$. This implies, $v > u > w$ in V. Hence, u is neither a minimal nor a maximal element of V. Conversely, let $u \in V$ such that u is neither a minimal nor a maximal element. Then there exists $v, w \in V$ such that $v > u > w$. Then there is a $v - u$ path in G say $v = v_1, v_2, ..., v_n = u$ such that $d_{ve}(v) > d_{ve}(v_2) > ... > d_{ve}(v_{n-1}) > d_{ve}(u)$ and a $u - w$ path in G say $u = u_1, u_2, ..., u_k = w$ such that $d_{ve}(u) > d_{ve}(u_2) > ... > d_{ve}(w)$. We observe that $v_{n-1}, u_2 \in N(u)$ such that $d_{ve}(v_{n-1}) > d_{ve}(u) > d_{ve}(u_2)$. Hence, u is a ve-balanced vertex of G.
(iii) A vertex u of G is ve-regular if and only if u is both ve-strong and ve-weak vertex of G if and only if u is both maximal and minimal element of V if and only if u is not related to any element of $V - \{u\}$ with respect to $\geq$. □

Theorem 6.2. *Let $G = (V, E)$ be a connected graph. Then the Hasse diagram of the ve-poset $(V, \geq)$ is a connected graph if and only if for any two vertices u and v of G, there exists a $u - v$ path in G having adjacent vertices of distinct ve-degrees.*

Proof. Assume that the Hasse diagram of the ve-poset $(V, \geq)$ is a connected graph. Suppose there exists two vertices u and v such that every $u - v$ path in G contains some adjacent vertices of same ve-degree. Then u and v are not related to each other in $(V, \geq)$. Now, suppose a lowerbound of u and v exists in V. We choose a lowerbound l of u and v such that l is a maximal element of the set of all lowerbounds of u and v. Then $u > l$ and $v > l$ in V. This implies, there exists a $u - l$ path in G say $u = u_1, u_2, ..., u_k = l$ such that $d_{ve}(u) > d_{ve}(u_2) > ... > d_{ve}(l)$ and a $v - l$ path in G say $v = v_1, v_2, ..., v_n = l$ such that $d_{ve}(v) > d_{ve}(v_2) > ... > d_{ve}(l)$. Then $u, u_2, ..., u_{k-1}, l, v_{n-1}, ..., v_2, v$ is a $u - v$ path in G such that adjacent vertices has the distinct ve-degrees, which is a contradiction. Hence, a lowerbound u and v does not exists in V. Similarly, we can prove that a upperbound u and v does not exists in V. This implies, u and v does not lie in the same component of the Hasse diagram graph of ve-poset $(V, \geq)$, which is a contradiction to our assumption. Thus, for any two vertices u and v of G, there exists a $u - v$ path in G having adjacent vertices of distinct ve-degrees. Conversely, assume that for any two vertices u and v of G, there exists a $u - v$ path in G having adjacent vertices of distinct ve-degrees. Let $u, v \in V$. Then by our assumption there exists a $u - v$ path say $u = u_1, u_2, ..., u_n = v$ in G having adjacent vertices of distinct ve-degrees. For any i, $1 \leq i \leq n - 1$, consider u_i and u_{i+1}. Then either $d_{ve}(u_i) > d_{ve}(u_{i+1})$ or $d_{ve}(u_i) < d_{ve}(u_{i+1})$. This implies, either $u_i > u_{i+1}$ or $u_i < u_{i+1}$ in V. Then there is a line between u_i and u_{i+1} in the Hasse diagram of $(V, \geq)$. Therefore, u and v lies in the same component of the Hasse diagram graph of the ve-poset $(V, \geq)$. Hence, the Hasse diagram of the ve-poset $(V, \geq)$ is a connected graph. □

Theorem 6.3. *Let $G = (V, E)$ be a graph of order $p > 1$. If $(V, \geq)$ is a lattice, then $S_{ve}(G) = 1$, $W_{ve}(G) = 1$, $R_{ve}(G) = 0$ and $B_{ve}(G) = p - 2$.*

Proof. Assume that the ve-poset $(V, \geq)$ is a lattice. Then V has unique maximal and minimal elements. By the Theorem 6.1, we have $S_{ve}(G) = 1$ and $W_{ve}(G) = 1$. Also, if G has a ve-regular vertex say u, then by the Theorem 6.1. we observe that the greatest lower bound of u with any element of the set $V - \{u\}$ does not exists in V, which is a contradiction to our assumption. This implies, $R_{ve}(G) = 0$. Hence, by Theorem 5.1., $B_{ve}(G) = p - 2$. □

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