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OPTIMAL LOCATION AND SHAPE OF A RIGID INCLUSION IN A CONTACT PROBLEM FOR INHOMOGENEOUS TWO-DIMENSIONAL BODY

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ABSTRACT. We analyze a well-known mathematical nonlinear model describing equilibrium of an elastic body with one volume (bulk) rigid inclusion. A possible frictionless contact of the body with a non-deformable obstacle by the Signorini condition on a part of the body boundary is assumed. On the remaining part of the boundary we impose a clamping condition. For a family of corresponding variational problems, we analyze the dependence of their solutions on location and shape of the rigid inclusion. External volume forces depend on the parameters defining location and shape of the inclusion. Continuous dependency of the solutions on location and shape parameters of the inclusion is established. The existence of a solution of the optimal control problem is proven. For this problem, a cost functional is defined by an arbitrary continuous functional on the Sobolev space of sought solutions, while the control is given by three real-valued parameters describing location and shape of the rigid inclusion.

Keywords: variational inequality, optimal shape problem, non-linear boundary conditions, rigid inclusion, location

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1. Introduction

The paper aims at shape control of geometry-dependent nonlinear variational problems, which are motivated by application to mechanics of composite solids. Clear advantages of using of composite parts in industry have increased the need for mathematical tools in order to design and optimize in an efficient way composite structures. One of important issues related to creation of reinforced composites concerns investigation both of the best locations and shapes of incorporated components. Nonlinear model approach using well-known Signorini type boundary conditions can be applied for contact problems. This approach leads to variational problems with an unknown contact zone.

Optimal control problems in the framework of variational inequalities with unilateral constraints of the Signorini type for cases where the control given by a volume or Neumann force were investigated, for example, in [1, 2]. A classification of the different optimality systems of strong stationarity for the case of optimal control of the obstacle problem can be found in [3, 4]. The researches on the shape and topological sensitivity analysis of variational inequalities have been actively elaborating [5, 6, 7, 8, 9, 10]. A shape-topological control problem for nonlinear crack - defect interaction was investigated in [9]. In particular, as one of results of this paper, an asymptotic representation of the strain energy release rate at the tip of the crack with respect to diminishing defects like holes and inclusions of varying stiffness has been obtained in [9]. The paper [11] concerns the control of rigid inclusion shapes in 2D elastic bodies with cracks. In this work, for an optimal shape problem a control of inclusion shapes is given by possible functions belonging to some bounded closed set of the space $H_0^2(0,1)$. Existence of optimal shapes for this problem was proved for a cost functional given by the Griffith formula. An optimal control problem of finding the most safe rigid inclusion shapes in elastic Kirchhoff-Love plates with cracks from the viewpoint of the Griffith rupture criterion was investigated in [12]. An inverse problem of the location of a thin elastic inclusion in an elastic body was investigated in [13], where the existence of a solution to the inverse problem is proved and the first variations of the solution of the direct problem with respect to the shape of the domain and the derivative of the functional with respect to the shape are calculated. Explicit expressions for first-order shape derivatives of the energy functionals for elastic bodies with a rigid inclusion and a crack were obtained in [14, 15]. We refer to [16, 17, 18] for a numerical approaches to equilibrium problems for elastic bodies with rigid inclusions. Problems for different models of heterogeneous bodies with both linear and nonlinear boundary conditions have been under active study; see, for example, [19, 20, 21, 22, 23].

In the present paper, we deal with a shape control problem for a nonlinear mathematical model describing mechanical contact of a composite with rigid obstacle. The framework model is formulated on a 2D Lipschitz domain and describes an equilibrium of the composite body consisting of an elastic matrix and a volume rigid inclusion. A Signorini nonpenetration condition and a clamping condition are imposed on different parts of a domain boundary. Varying parameters describing a location and a shape of rigid inclusion we arrive at a corresponding family of problems. For the shape control problem the cost functional is introduced by an arbitrary continuous functional defined on the solution's space, while the location and shape parameters of the rigid inclusions serve as a control. A set describing admissible locations and shapes of the inclusion is supposed to be compact in

$\mathbb{R}^3$. We prove an existence of a solution of the optimal shape problem and the continuous dependence of the solutions on the location and shape parameters. Sufficient conditions for the solvability of the optimal control problem are found and formulated in the framework of propositions. The novelty of the obtained results compared to the previous investigation [11] related to an optimal shape of a rigid inclusion consists in a weakening the requirements for the regularity of the boundary. Furthermore, the result generalizes a problem considered in [24] which concerns only locations parameters. The present problem also takes into account a dependence of external forces on the parameters defining location and shape of the inclusion, whereas the work [24] was dealt with only one unchanged function of external forces. In fact, we adopt and develop the approach of the work [24] for the more general case. At the same time, we establish sufficient conditions for the solvability of the problem under study.

2. Family of equilibrium problems

Let $\Omega \subset \mathbb{R}^2$ be a bounded domain with a boundary $\Gamma \in C^{0,1}$, $\Gamma = \Gamma_0 \cup \Gamma_s$, $\text{meas}(\Gamma_0) > 0$. Suppose that a simply connected domain ω lies strictly inside of Ω, i.e. $\bar{\omega} \subset \Omega$. We consider the following family of continuous mappings $f_s : \omega \to \mathbb{R}^2$, $s \in [0, T]$ such that $f_0(\omega) = \omega$ and the domains $f_s(\omega) = \omega_s$ are simply connected Lipschitz domains with the property $\overline{\omega}_s \subset \Omega$ for all $s \in [0, T]$. In addition, a family of domains $\{\omega_s\}$, $s \in [0, T]$ satisfies the following proposition.

Proposition 1. *We suppose that for every fixed values of* $s \in [0, T]$ *and* $\varepsilon > 0$ *there exists a positive number* $\delta > 0$ *such that*

$$\omega_t \subset \omega_s^\varepsilon \quad \text{for all} \quad |t - s| < \delta, \tag{1}$$

where ω_s^ε *is defined according to the following relation*

$$\mathcal{O}^\varepsilon = \{x \in \mathbb{R}^2 \,|\, dist(x, \mathcal{O}) < \varepsilon\} \tag{2}$$

with an arbitrary subdomain $\mathcal{O} \subset \Omega$ *and any positive number* ε.

Remark 1. *It can be noted that a mapping defining rotations of* ω *by angles* $2\pi s$ *relative to an arbitrary fixed point of the domain* ω *satisfies* (1).

Without loss of generality we suppose that $(0, 0) \in \omega$. For every (s, y), $s \in [0, T]$, $y = (y_1, y_2) \in \Omega$, we can consider induced domains

$$\omega(s, y) = \{x = (x_1, x_2) \,|\, (x_1, x_2) = (y_1, y_2) + (z_1, z_2), \quad \text{where} \quad (z_1, z_2) \in \omega_s\},$$

which describe transformations and translations of initial domain ω. In what follows we will consider some given compact set $\mathcal{A} \subset [0, T] \times \Omega \subset \mathbb{R}^3$ consisting of all (s, y) such that

$$s \in [0, T], \quad y = (y_1, y_2) \in \Omega, \quad \text{and} \quad \overline{\omega(s, y)} \subset \Omega.$$

The set $\mathcal{A}$ defines allowable shapes and locations (translations) of one incorporated rigid inclusion. Note that, for all $(s, y) \in \mathcal{A}$ we have that a domain $\Omega \setminus \overline{\omega(s, y)}$ has a Lipschitz boundary.

As an evident example providing an appropriate set $\mathcal{A}$, we propose following set $\mathcal{A}_{ex} = [0, 1] \times [-l, l] \times [-l, l]$, with $0 < l < 2 - \sqrt{2}$, for square domains $\Omega = [-2, 2] \times [-2, 2]$, $\omega = [-1, 1] \times [-1, 1]$ and ω_s defined by rotations of ω by angles $2\pi s$ relative to the center of the square subdomain ω.

Furthermore, we assume that the following proposition is valid.

Proposition 2. *For an arbitrary strictly inner subdomain $D \subset \omega(s, y)$ there exists a positive sufficiently small number $\delta > 0$ such that $D \subset \omega(\hat{s}, \hat{y}) \cap \omega(s, y)$ for all $(\hat{s}, \hat{y}) \in A$ such that*

$$\|(\hat{s}, \hat{y}) - (s, y)\|_{\mathbb{R}^3} < \delta.$$

Denote by $W = (w_1, w_2)$ the displacement vector. Introduce the Sobolev spaces

$$H^{1,0}(\Omega) = \{v \in H^1(\Omega) \mid v = 0 \quad \text{on} \quad \Gamma_0\}, \quad H(\Omega) = H^{1,0}(\Omega)^2.$$

Introduce the tensors describing the deformation of an elastic part of the inhomogeneous body

$$\varepsilon_{11}(W) = \frac{\partial w_1}{\partial x_1}, \quad \varepsilon_{12}(W) = \varepsilon_{21}(W) = \frac{1}{2}\left(\frac{\partial w_1}{\partial x_2} + \frac{\partial w_2}{\partial x_1}\right), \quad \varepsilon_{22}(W) = \frac{\partial w_2}{\partial x_2}.$$

$$\sigma_{ij}(W) = c_{ijkl}\varepsilon_{kl}(W), \ i, j = 1, 2,$$

where c_{ijkl} is the given elasticity tensor, assumed to be symmetric and positive definite:

$$c_{ijkl} = c_{klij} = c_{jikl}, \quad i, j, k, l = 1, 2, \quad c_{ijkl} = const,$$

$$c_{ijkl}\xi_{ij}\xi_{kl} \ge c_0|\xi|^2, \quad \forall\xi, \quad \xi_{ij} = \xi_{ji}, \quad i, j = 1, 2, \quad c_0 = const, \quad c_0 > 0.$$

By the assumption concerning the domain Ω and the Korn's inequality [27, 28], the following inequality holds

$$\int_\Omega \sigma_{ij}(W)\varepsilon_{ij}(W)d\Omega \ge c\|W\|^2_{H(\Omega)}, \quad \forall W \in H(\Omega), \tag{3}$$

with a constant $c > 0$ independent of W.

Remark 2. *The inequality (3) yields the equivalence of the standard norm in $H(\Omega)$ and the semi-norm determined by the left-hand side of (3).*

To formulate the mathematical model for a composite body with a rigid inclusion $\omega(s, y)$, we will use the notion of a rigid inclusion which in general can occupy an arbitrary subdomain $\mathcal{O} \subset \Omega$. In this case the displacements on the domain $\mathcal{O}$ should have a special structure $W|_{\mathcal{O}} = \rho$, where $\rho \in R(\mathcal{O})$ and $R(\mathcal{O})$ is the space of infinitesimal rigid displacements on $\mathcal{O}$

$$R(\mathcal{O}) = \{\rho = (\rho_1, \rho_2) \,|\, \rho(x_1, x_2) = b(x_2, -x_1) + (c_1, c_2); \\ b, c_1, c_2 \in \mathbb{R}, \ (x_1, x_2) \in \mathcal{O}\},$$

see, [29, 20].

Next, we fix the element $(s, y) \in \mathcal{A}$ and suppose that the domain $\omega(s, y)$ fits a volume (bulk) rigid inclusion, while the domain

$$\Omega \setminus \overline{\omega(s, y)},$$

corresponds to the elastic part of the body. The Signorini condition of a possible mechanical contact with a non-deformable obstacle is written as

$$W\nu \le 0 \quad \text{on} \quad \Gamma_s,$$

where $\nu = (\nu_1, \nu_2)$ is an outward normal to Γ. For further studies of Signorini's contact problem we refer to [30]. The homogeneous Dirichlet boundary condition is imposed on the external boundary Γ_0. We introduce the energy functional

$$\Pi(W, s, y) = \frac{1}{2}\int_\Omega \sigma_{ij}(W)\varepsilon_{ij}(W)d\Omega - \int_\Omega F(s, y)W d\Omega, \tag{4}$$

where $F(s,y) = \Big(f_1(s,y), f_2(s,y)\Big) \in C([0,T]\times\Omega; L_2(\Omega))^2$ is a given vector of exterior forces.

An equilibrium problem of the composite body can be formulated as the following minimization problem.

$$\text{Find } U = U(s,y) \in K(s,y),$$

$$\text{such that} \quad \Pi(U,s,y) = \inf_{W\in K(s,y)} \Pi(W,s,y), \tag{5}$$

where the set of admissible displacements is defined as follows

$$K(s,y) = \{W \in H(\Omega) \mid W\nu \le 0 \quad \text{on} \quad \Gamma_s,$$
$$W|_{\omega(s,y)} = \rho, \text{ where } \rho \in R(\omega(s,y))\}.$$

The problem (5) is known to have a unique solution $U(s,y) \in K(s,y)$, which satisfies the variational inequality [31, 32]

$$\int\limits_\Omega \sigma_{ij}(U(s,y))\varepsilon_{ij}(W - U(s,y))d\Omega \ge \int\limits_\Omega F(s,y)(W - U(s,y))d\Omega, \tag{6}$$

for all $W \in K(s,y)$.

3. Optimal control problem

Let's define a cost functional $J : \mathcal{A} \to \mathbb{R}$ of an optimal control problem with the use of the equality $J_G(s,y) = G(U(s,y))$, where $U(s,y)$ is the solution of the problem (5) and $G : H(\Omega) \to \mathbb{R}$ is an arbitrary continuous functional.

As examples of such functionals having physical sense, we can provide the functional $G_1(W) = \|W - W_0\|_{H(\Omega)}$ characterizing the deviation of the displacement vector from a given function W_0. Consider the optimal control problem:

$$\text{Find} \quad (s^*,y^*) \in \mathcal{A} \quad \text{such that} \quad J_G(s^*,y^*) = \sup_{(s,y)\in\mathcal{A}} J_G(s,y). \tag{7}$$

This means that we want to find the best location and shape of inclusion which provide the maximal value for the cost functional. The following is the main result of the paper.

Theorem 1. *There exists a solution of the optimal control problem (7).*

Proof. Let $\{(s_n,y_n)\}$ be a maximizing sequence. By the compactness of the set $\mathcal{A} \subset \mathbb{R}^{2m}$, we can extract a convergent subsequence $\{(s_{n_k},y_{n_k})\} \subset \{(s_n,y_n)\}$ such that

$$(s_{n_k},y_{n_k}) \to (s^*,y^*) \quad \text{as} \quad k\to\infty, \quad (s^*,y^*) \in \mathcal{A}.$$

Without loss of generality we can assume that $(s_{n_k},y_{n_k}) \ne (s^*,y^*)$ for sufficiently large k.

Otherwise there would exist a sequence $\{(s_{n_k},y_{n_k})\}$ such that $(s_{n_k},y_{n_k}) \equiv (s^*,y^*)$, which implies that $J_G(s^*,y^*)$ is a solution of (7).

Thanks to Lemma 2 proved below, it follows that the solutions $U(s_{n_k},y_{n_k})$ of (5) corresponding to the parameters (s_{n_k},y_{n_k}) converge to the solution $U(s^*,y^*)$ strongly in $H(\Omega)$ as $k\to\infty$. This allows us to obtain the convergence

$$J_G(s_{n_k},y_{n_k}) \to J_G(s^*,y^*),$$

indicating that

$$J_G(s^*,y^*) = \sup_{(s,y)\in\mathcal{A}} J_G(s,y).$$

The theorem is proved.

It is important to note that we have proved this result without introducing of curves that describe possible shifting of inclusions as it was done in [25, 26].

4. Auxiliary lemmas

Now we have to provide a formulation and a proof of Lemma 2 that have been used in the proof of Theorem 1. However, the assertion of this Lemma 2 is based on the following lemma.

Lemma 1. *Let* $(s^*, y^*) \in \mathcal{A}$ *be a fixed vector and let* $\{(s_n, y_n)\} \subset \mathcal{A}$ *be a sequence of vectors converging to* (s^*, y^*) *in* $\mathbb{R}^3$ *as* $n \to \infty$. *Then for an arbitrary function* $W \in K(s, y)$ *there exist a subsequence* $\{s_k, y_k\} \subset \{s_n, y_n\}$ *and a sequence of functions* $\{W_k\}$ *such that* $W_k \in K(s_k, y_k)$, $k \in \mathbb{N}$ *and* $W_k \to W$ *strongly in* $H(\Omega)$ *as* $k \to \infty$.

Proof. We denote by

$$\rho^* = (b^* x_2 + c_1^*, -b^* x_1 + c_2^*),$$

is the function describing the structure of W in the domain $\omega(s^*, y^*)$. We consider an arbitrary decreasing sequence of positive numbers $\{\epsilon_k\}$ satisfying $\epsilon_k \to 0$ as $k \to \infty$. According to (2) we construct a family of domains $\omega^{\epsilon_k}(s^*, y^*)$, $k = 1, 2, \dots$.

$$\omega^{\epsilon_1}(s^*, y^*) \supset \dots \supset \omega^{\epsilon_k}(s^*, y^*) \supset \dots$$

Without loss of generality we suppose that $\overline{\omega^{\epsilon_1}(s^*, y^*)} \subset \Omega$ and each domain $\Omega \setminus \overline{\omega^{\epsilon_k}(s^*, y^*)}$, $k = 1, 2, \dots$ is a Lipschitz domain. It follows from the fact that there exists a positive number ϵ_0 small enough such that for all $0 < \epsilon < \epsilon_0$ the domains

$$\omega^{\epsilon}(s^*, y^*),$$

would be Lipschitz domains [33].

For every fixed $k \in \mathbb{N}$, we formulate the following auxiliary problem

$$\text{Find an element} \quad Q_k \in K_k \quad \text{such that} \quad p(Q_k) = \inf_{\chi \in K_k} p(\chi), \tag{8}$$

where $p(\chi) = \int\limits_{\Omega} \sigma_{ij}(\chi - W) \varepsilon_{ij}(\chi - W) d\Omega$,

$$K_k = \{\chi \in H(\Omega) \mid \quad \chi = W \quad \text{on} \quad \Gamma_s, \ \chi|_{\omega^{\epsilon_k}(s^*, y^*)} = \rho^*\}.$$

It is easy to see that the functional $p(\chi)$ is coercive and weakly lower semicontinuous on the space $H(\Omega)$. Besides, one can verify that the set K_k is closed and convex in the reflexive space $H(\Omega)$. This properties provide the existence of a solution Q_k of the problem (8) for each $k \in \mathbb{N}$ [27]. Since the functional $p(\chi)$ is convex and Gateaux differentiable, this solution is characterized equivalently by a variational inequality

$$Q_k \in K_k, \quad \int\limits_{\Omega} \sigma_{ij}(Q_k - W) \varepsilon_{ij}(\chi - Q_k) d\Omega \geq 0 \quad \forall \chi \in K_k. \tag{9}$$

Inequality (3) guarantees that solution Q_k is unique.

Note that applying a lifting operator for the Lipschitz domain $\Omega \setminus \overline{\omega^{\epsilon_1}(s^*, y^*)}$, we can construct a function $\hat{\chi} \in H(\Omega)$ such that

$$\hat{\chi} = \rho^* \quad \text{on} \quad \omega^{\epsilon_1}(s^*, y^*), \quad \hat{\chi} = W \quad \text{on} \quad \Gamma.$$

Since $\hat{\chi} \in K_k$ for all $k \in \mathbb{N}$, we can substitute $\hat{\chi}$ in (9) as the test functions which yields inequalities

$$\int\limits_{\Omega} \sigma_{ij}(Q_k - W)\varepsilon_{ij}(\hat{\chi})d\Omega + \int\limits_{\Omega} \sigma_{ij}(W)\varepsilon_{ij}(Q_k)d\Omega \geq \int\limits_{\Omega} \sigma_{ij}(Q_k)\varepsilon_{ij}(Q_k)d\Omega \quad \forall k \in \mathbb{N}.$$

From this relation using the Korn inequality we obtain the following uniform upper bound:

$$\|Q_k\|_{H(\Omega)} \leq c \quad \forall k \in \mathbb{N}.$$

Therefore, we can extract from the sequence $\{Q_k\}$ a subsequence $\{Q_{k_l}\}$, which we denote by

$$(10) \qquad Q_l = Q_{k_l}, \quad l \in \mathbb{N} \quad \text{and} \quad Q_l \to \widetilde{W} \quad \text{weakly in} \quad H(\Omega),$$

where $\widetilde{W}$ is some function in $H(\Omega)$. It is now necessary to show that $\widetilde{W} = W$.

By construction

$$(Q_l - W) \in H_0^1\left(\Omega \backslash \overline{\omega(s^*, y^*)}\right)^2.$$

Consequently, bearing in mind the weak closedness of $H_0^1\left(\Omega \backslash \overline{\omega(s^*, y^*)}\right)^2$ we have $(\widetilde{W} - W) \in H_0^1\left(\Omega \backslash \overline{\omega(s^*, y^*)}\right)^2$. We consider now the functions of the form $\chi_l^{\pm} = Q_l \pm \alpha$, where α is the function defined by zero extension of an arbitrary function $\tilde{\alpha} \in C_0^{\infty}\left(\Omega \backslash \overline{\omega(s^*, y^*)}\right)^2$ into Ω. It is observed that for sufficiently large l we have $\chi_l^{\pm} \in K_{k_l}$. Substituting the elements of these sequences, χ_l^+ and χ_l^-, as test functions into the inequalities (9), we obtain that

$$(11) \qquad \int\limits_{\Omega} \sigma_{ij}(Q_l - W)\varepsilon_{ij}(\alpha)d\Omega = 0.$$

The function α is now fixed and by passing to the limit in (11) it is established that

$$\int\limits_{\Omega} \sigma_{ij}(\widetilde{W} - W)\varepsilon_{ij}(\alpha)d\Omega = \int\limits_{\Omega \backslash \overline{\omega(s^*, y^*)}} \sigma_{ij}(\widetilde{W} - W)\varepsilon_{ij}(\alpha)d\Omega = 0,$$

for all $\alpha \in C_0^{\infty}(\Omega \backslash \overline{\omega(s^*, y^*)})^2$. Hence, by consideration of the density of $C_0^{\infty}\left(\Omega \backslash \overline{\omega(s^*, y^*)}\right)^2$ in $H_0^1\left(\Omega \backslash \overline{\omega(s^*, y^*)}\right)^2$, we conclude that $\widetilde{W} - W = 0$ in $H_0^1\left(\Omega \backslash \overline{\omega(s^*, y^*)}\right)^2$. Finally, by construction, the equality $\widetilde{W} = W$ is satisfied in the domains $\omega(s^*, y^*)$ and on the external boundary Γ. Therefore, $\widetilde{W} = W$ in $H(\Omega)$. This means that there is a sequence $\{Q_l\}$ such that $Q_l \in K_{k_l}$, $l \in \mathbb{N}$ and $Q_l \to W$ weakly in $H(\Omega)$ as $l \to \infty$. Now we are in a position to prove the strong convergence. By the Mazur theorem there exist a function $T : \mathbb{N} \to \mathbb{N}$ and a sequence of sets of real numbers $\{\alpha(l)_i \,|\, i = l, \ldots, T(l)\}$ satisfying $\alpha(l)_i \geq 0$ and $\sum\limits_{i=l}^{T(l)} \alpha(l)_i = 1$ such that the sequence $\{\hat{Q}_l\}$ defined by the convex combination

$$\hat{Q}_l = \sum_{i=l}^{T(l)} \alpha(l)_i Q_i$$

converges to W strongly in $H(\Omega)$. According to this construction, we have subsequence $\{k_l\}$ of natural numbers which corresponds to the subsequence $\{Q_l\}$ from (10), therefore to the sequence $\{T(l)\}$ we will have the corresponding subsequence of natural numbers $\{k_{T(l)}\}$, which we denote by $\{m(l)\}$, i.e. $m(l) = k_{T(l)}$, for all $l \in \mathbb{N}$. Note that we have the following inclusion

$$\hat{Q}_l \in R(\omega^{\epsilon_{m(l)}}(s^*, y^*)),$$

which we will use in the sequel.

As the next step, we should determine a subsequence $\{(s_{n_i}, y_{n_i})\} \subset \{(s_n, y_n)\}$ providing the assertion of the theorem. At first we prove that for every k there exists a natural number $N(k)$ such that for every $n \geq N(k)$

$$\omega(s_n, y_n) \subset \omega^{\epsilon_k}(s^*, y^*) \tag{12}$$

Indeed, for ϵ_k there exists some $\delta >$ such that for $|s_n - s^*| < \delta$ we have

$$\omega_{s_n} \subset \omega_{s^*}^{\frac{\epsilon_k}{2}} \quad \text{or} \quad \omega(s_n, y_l) \subset \omega^{\frac{\epsilon_k}{2}}(s^*, y_l),$$

for all $l \in \mathbb{N}$. By the convergence of $\{s_n\}$ there exists some natural number $\hat{N}(k)$ such that (12) holds for $n \geq \hat{N}(k)$.

It is obvious that parallel translation of the set $\omega^{\frac{\epsilon_k}{2}}(s^*, y_l)$ by the vector $y^* - y_l$ with length $\|y_l - y^*\|_{\mathbb{R}^2} < \frac{\epsilon_k}{2}$ gives us that

$$\omega^{\frac{\epsilon_k}{2}}(s^*, y_l) \subset \omega^{\epsilon_k}(s^*, y^*).$$

Therefore, to obtain (12), we can choose the number $N(k)$ as the maximum of two natural numbers $\hat{N}(k)$ and $\tilde{N}(k)$, i.e. $N(k) = max\{\hat{N}(k), \tilde{N}(k)\}$, where the number $\tilde{N}(k)$ provided by the condition $\|y_l - y^*\|_{\mathbb{R}^2} < \frac{\epsilon_k}{2}$ for all $l \geq \tilde{N}(k)$.

Now, we determine a subsequence $\{(s_{n_i}, y_{n_i})\} \subset \{(s_n, y_n)\}$ by the following procedure, for every $i \in \mathbb{N}$ we set $n_i = N(m(i))$. In this case we have $\omega(s_{n_1}, y_{n_1}) \subset \omega^{\epsilon_{m(1)}}(s^*, y^*)$, whereas the function $\hat{Q}_1$ belongs $K(s_{n_1}, y_{n_1})$, and analogously

$$\omega(s_{n_i}, y_{n_i}) \subset \omega^{\epsilon_{m(i)}}(s^*, y^*), \quad \hat{Q}_i \in K(s_{n_i}, y_{n_i}).$$

As a consequence, we set

$$W_k = \hat{Q}_k, \quad k = 1, 2, ...$$

This completes the proof.

Now, we are in a position to prove an auxiliary statement which was used in the proof of the theorem.

Lemma 2. *Let $(s^*, y^*) \in \mathcal{A}$ and let $\{(s_n, y_n)\} \subset \mathcal{A}$ be a sequence of real numbers converging to (s^*, y^*) in $\mathbb{R}^3$ as $n \to \infty$. Then $U(s_n, y_n) \to U(s^*, y^*)$ strongly in $H(\Omega)$ as $n \to \infty$, where $U(s_n, y_n)$, $U(s^*, y^*)$ are the solutions of (5) corresponding to parameters (s_n, y_n), (s^*, y^*), respectively.*

Proof. We proceed by contradiction. Let us assume that there exist a number $\epsilon_0 > 0$ and a sequence $\{(s_n, y_n)\} \subset \mathcal{A}$ such that $(s_n, y_n) \to (s^*, y^*)$, $\|U(s_n, y_n) - U(s^*, y^*)\|_{H(\Omega)} \geq \epsilon_0$, where $U_n = U(s_n, y_n)$, $U^* = U(s^*, y^*)$ are the corresponding solutions of (5).

Because of $W^0 \equiv (0, 0) \in K(s_n, y_n)$ for all $n \in \mathbb{N}$, we can insert $W = W^0$ in (6) for fixed $n \in \mathbb{N}$. This provides

$$\int_\Omega \sigma_{ij}(U_n)\varepsilon_{ij}(U_n) d\Omega \leq \int_\Omega F(s_n, y_n) U_n d\Omega, \quad \forall n \in \mathbb{N}. \tag{13}$$

Since $F(s,y) \in C([0,T]\times\overline{\Omega}; L_2(\Omega))^2$, there exists some constant $C>0$ such that

$$\|F(s,y)\|_{L_2(\Omega)^2} < C,$$

for all $(s,y) \in [0,T]\times\overline{\Omega}$. Taking into account the last estimate, from (13) we conclude that for all $n \in \mathbb{N}$ the following uniform estimate holds

$$\|U_n\|_{H(\Omega)} \le c$$

with some constant $c>0$ independent of $n \in \mathbb{N}$. Consequently, replacing U_n by its subsequence if necessary, we can assume that U_n converges to some function $\tilde{U}$ weakly in $H(\Omega)$.

Now we show that $\tilde{U} \in K(s^*,y^*)$. Indeed, we have

$$U_n|_{\omega(s_n,y_n)} = \rho_n \in R(\omega(s_n,y_n))$$

for all $n \in \mathbb{N}$. In accordance with the Sobolev embedding theorem [27], we obtain

$$U_n|_{\omega(s^*,y^*)} \to \tilde{U}|_{\omega(s^*,y^*)} \quad \text{strongly in } L_2(\omega(s^*,y^*))^2 \text{ as } n\to\infty, \tag{14}$$

$$U_n|_{\Gamma} \to \tilde{U}|_{\Gamma} \quad \text{strongly in } L_2(\Gamma)^2 \text{ as } n\to\infty. \tag{15}$$

Choosing a subsequence, if necessary, we assume as $n\to\infty$ that $U_n \to \tilde{U}$ a.e. on $\omega(s^*,y^*)$.

In the next step we fix an arbitrary strictly inner subdomain $D \subset \omega(s^*,y^*)$. According to the Proposition 2 there exists a sufficiently large N such that if $n \ge N$, then $D \subset \omega(s^*,y^*)\cap\omega(s_n,y_n)$. Therefore, the sequence $\{\rho_n\}$ converges to $\tilde{U}$ a.e. on D as n goes to infinity. This allows us to conclude that each of the numerical sequences $\{b^n\}$, $\{c_1^n\}$, $\{c_2^n\}$, defining the structure of functions ρ_n, $n=1,2,...$ on D is bounded in $\mathbb{R}$. Thus, we can extract subsequences (retain notation) such that

$$b^n \to b, \quad c_i^n \to c_i, \quad i=1,2, \quad \text{as} \quad n\to\infty.$$

Therefore, we can choose a subsequence $\{(s_{n_k},y_{n_k})\}$ such that

$$U(s_{n_k},y_{n_k}) \to (bx_2+c_1, -bx_1+c_2) \quad \text{a.e.} \quad \text{on} \quad D \quad \text{as} \quad k\to\infty. \tag{16}$$

Consequently, we obtain that

$$\tilde{U} = (bx_2+c_1, -bx_1+c_2) \quad \text{a.e.} \quad \text{on} \quad D.$$

Due to arbitrariness of the domain $D \subset \omega(s^*,y^*)$ we get that

$$\tilde{U} = (bx_2+c_1, -bx_1+c_2) \quad \text{a.e.} \quad \text{on} \quad \omega(s^*,y^*).$$

Whence we can conclude that $\tilde{U}|_{\omega(s^*,y^*)} \in R(\omega(s^*,y^*))$ holds.

We now show that $\tilde{U}$ satisfies the inequality $\tilde{U}\nu \le 0$ on Γ_s. Bearing in mind the convergence (15), if necessary, we can once again extract a subsequence satisfying $U_n|_\Gamma \to \tilde{U}|_\Gamma$ a.e. on Γ. This fact allows us to pass to the limit in the following inequality

$$U_n\nu \le 0 \quad \text{on} \quad \Gamma_s.$$

This leads to $\tilde{U}\nu \le 0$ on Γ_s. Therefore we get the inclusion $\tilde{U} \in K(s^*,y^*)$.

Our next goals are to prove the following equality $\tilde{U} = U^*$ and to establish the existence of a sequence $U_n = U(s_n,y_n)$, $n=1,2...$ of solutions strongly converging in $H(\Omega)$ to $U(s^*,y^*)$. Now, let us prove that $\tilde{U} = U(s^*,y^*)$. For this purpose we will analyze the variational inequality (6) and its limiting case. From Lemma 1, for any $W \in K(s^*,y^*)$ there exist a subsequence $\{(s_{n_k},y_{n_k})\} \subset \{(s_n,y_n)\}$ and a sequence of functions $\{W_k\}$ such that $W_k \in K(s_{n_k}, y_{n_k})$ and $W_k \to W$ strongly in $H(\Omega)$ as $k \to \infty$.

Bearing in mind that $F(s_{n_k}, y_{n_k}) \to F(s^*, y^*)$ in $L_2(\Omega)^2$ for $k \to \infty$, and properties of the convergent sequences $\{W_k\}$ and $\{U_n\}$ allow us to pass to the limit as $k \to \infty$ in following inequalities derived from (6) for $\{(s_{n_k}, y_{n_k})\}$ and with the test functions $W_k \in K(s_{n_k}, y_{n_k})$

$$\int\limits_{\Omega} \sigma_{ij}(U_{n_k})\varepsilon_{ij}(W_{n_k} - U_{n_k})d\Omega \geq \int\limits_{\Omega} F(s_{n_k}, y_{n_k})(W_{n_k} - U_{n_k})d\Omega. \tag{17}$$

As a result, we have

$$\int\limits_{\Omega} \sigma_{ij}(\tilde{U})\varepsilon_{ij}(W - \tilde{U})d\Omega \geq \int\limits_{\Omega} F(s^*, y^*)(W - \tilde{U})d\Omega \quad \forall\, W \in K(s^*, y^*).$$

The unique solvability of this variational inequality ensures that $\tilde{U} = U^*$.

To complete the proof, it is sufficient to establish the strong convergence $U_n \to U^*$. By substituting $W = 2U_n$ and $W = 0$ into the variational inequalities (6) for $n \in \mathbb{N}$, we get

$$\int\limits_{\Omega} \sigma_{ij}(U_n)\varepsilon_{ij}(U_n)d\Omega = \int\limits_{\Omega} F(s_n, y_n)U_n d\Omega \quad \forall\, n \in \mathbb{N}. \tag{18}$$

The equalities (18) together with a strong convergence $F(s_n, y_n) \to F(s^*, y^*)$ in $L^2(\Omega)^2$ and the weak convergence $U_n \to U^*$ in $H(\Omega)$ as $n \to \infty$ imply

$$\lim_{n\to\infty} \int\limits_{\Omega} \sigma_{ij}(U_n)\varepsilon_{ij}(U_n)d\Omega = \lim_{n\to\infty} \int\limits_{\Omega} F(s_n, y_n)U_n d\Omega =$$

$$\int\limits_{\Omega} F(s^*, y^*)U^* d\Omega = \int\limits_{\Omega} \sigma_{ij}(U^*)\varepsilon_{ij}(U^*)d\Omega.$$

Since we have the equivalence of norms (see Remark 2), one can see that $U_n \to U^*$ strongly in $H(\Omega)$ as $n \to \infty$. But this contradicts to the initial assumption. The Lemma is proved.

Remark 3. *As can be seen from the proofs of the present paper, the main result remains true in 3D case, as well as for equilibrium problems related to the two-dimensional solids with classical linear conditions.*

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