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KERNEL DETERMINATION PROBLEM FOR A PARABOLIC INTEGRO–DIFFERENTIAL EQUATION WITH A VARIABLE THERMAL CONDUCTIVITY

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Abstract. The inverse problem of finding a multidimensional memory kernel of a time convolution integral depending on a time variable t and $(n-1)$-dimensional spatial variable $x' = (x_1, ..., x_{n-1})$ in the n-dimensional heat equation with a time-dependent coefficient of thermal conductivity is studied. The direct problem is the Cauchy problem for this equation. The integral term has the time convolution form of kernel and an elliptic operator of direct problem solution. As additional information, the solution of the direct problem on the hyperplane $x_n = 0$ is given. The problem reduces to an auxiliary problem which is more convenient for further consideration. Then the auxiliary problem is replaced by an equivalent system of Volterra-type integral equations with respect to unknown functions. Applying the method of contraction mappings to this system in the Hölder class of functions, it is proved the main result of the paper representing a local existence and uniqueness theorem.

Keywords: Heat equation, memory kernel, Hölder space, convolution integral, contraction mapping.

1. Introduction. Setting up problem

The constitutive relations for a linear non-homogeneous heat propagation and diffusion processes in medium with memory contain a time- and space-dependent kernel in an integral term of time variable convolution type [1]-[5]. Often, in practical

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applications these kernels are unknown functions and it is required to determine them. Kernel determination inverse problems in parabolic integro differential equations were the object of studying since the end of last century. In literature are most often found the linear inverse source and nonlinear inverse coefficient problems with different type of over determination conditions (see, for example [6]-[15] and references there). In these works authors discussed the unique solvability and stability estimates of solution as well a numerical approach for solving such problems.

Among the works devoted to finding the kernel depending on one time variable (one dimensional inverse problem), we note [2], [4], [16]. When kernel, in addition to time variable also depends on all or part of spatial variables, i.e. multidimensional inverse problems are few studied. In this direction we observe [4], [17]-[19]. In the article [17] the problem of determining of the kernel depending on time variable t and $(n-1)-$dimensional spatial variable $x' = (x_1, ..., x_{n-1})$ was investigated. The principal part of integro–differential equation in [17] is $n-$ dimensional heat conduction operator and integral part has a form of time-convolution with regard to unknown functions: the solutions of direct and inverse problems. However in applications, the great interest is the study of kernel determination problems when in convolution type integral the kernel is multiplied by an elliptic operator of solution to the direct problem. The present work considers this kind of parabolic integro–differential equations for which will be studied inverse problem.

Consider the problem of determining functions $u(x,t)$, $k(x',t)$, $x = (x', x_n) = (x_1, ..., x_{n-1}, x_n)$, $t > 0$ from the equations

$$u_t = a(t)\Delta u - \int_0^t k(x', t-\tau)a(\tau)\Delta u(x,\tau)d\tau, \quad (x,t) \in \mathbb{R}^n_T, \tag{1}$$

$$u(x,0) = \varphi(x),\, x \in \mathbb{R}^n, \tag{2}$$

$$u(x',0,t) = f(x',t),\, (x',t) \in \mathbb{R}^{n-1}_T, \;\; f(x',0) = \varphi(x',0), \tag{3}$$

where $\triangle$ is the Laplace operator with respect to spatial variables $x = (x_1, ..., x_n)$, $\mathbb{R}^n_T = \{(x,t) | x = (x', x_n) \in \mathbb{R}^n, \; 0 < t < T\}$ is a strip with thickness T, $T > 0$ is an arbitrary fixed number; $a(t) \in C^2[0,T]$, $0 < a_0 \le a(t) \le a_1 < \infty$, a_0 and a_1 are given numbers.

Our investigations generalize the result of works [17], [18] to the case of the integro-differential heat equation with a variable coefficient of thermal conductivity and special convolution integral.

In the sequel, we will use the Hölder space H with exponent α for functions depending only on spatial variables and for functions depending on both spatial and time variables - Hölder space $H^{\alpha,\alpha/2}$ with exponents α and $\alpha/2$, α is non-negative integer. Throughout this paper we will assume that $\varphi(x) \in H^{l+8}(\mathbb{R}^n)$, $\varphi(x) \ge \varphi_0 = const > 0$, $f(x',t) \in H^{l+6,(l+6)/2}\left(\bar{\mathbb{R}}^{n-1}_T\right)$. Spaces $H^l(Q)$, $H^{l,l/2}(Q_T)$ and norms in them are defined in [20, p. 16-27]. In what follows, for norm of functions in the space $H^{l,l/2}(Q_T)$ (in concrete cases $Q_T = \mathbb{R}^n_T$ or $Q_T = \mathbb{R}^{n-1}_T$) depending on spatial and time variables will be used notation $|\cdot|^{l,l/2}_T$, while for functions depending only on spatial variables we use $|\cdot|^l$ (in this case $Q = \mathbb{R}^n$ or $Q = \mathbb{R}^{n-1}$).

The article is organized as follows. In Section 2, we reduce the problem (1)-(3) into an auxiliary problem where the additional condition contains the unknown k outside integral. In Section 3, we replace auxiliary problem by an equivalent system of integral equations with respect to unknown functions. In Section 4, we prove the main result which states the existence and uniqueness of solution of problem by a fixed point argument.

2. Preliminaries. Auxiliary problem

We give the following assertion which will be used in obtaining the auxiliary problem.

Lemma 1. *If $\{k(t), r(t)\} \in C[0,T]$ for a fixed $T>0$ and $k(t), r(t)$ are connected by the integral equation*

$$r(t) = k(t) + \int_0^t k(t-\tau) r(\tau) d\tau, \quad t \in [0,T],$$

then the solution of the integral equation

$$\varphi(t) = \int_0^t k(t-\tau)\varphi(\tau) d\tau + f(t), \quad f(t) \in C[0,T]$$

is expressed by formula

$$\varphi(t) = \int_0^t r(t-\tau) f(\tau) d\tau + f(t).$$

Proof. The proof of Lemma follows from the general theory of Volterra -type integral equations (see, for example [21]).

Rewriting the equation (1) in the form of Volterra integral equation with respect to $a(t)\triangle u$

$$a(t)\Delta u = \int_0^t k(x', t-\tau) a(\tau) \Delta u(x,\tau) d\tau + u_t \tag{4}$$

and at any fixed $x \in \mathbb{R}^n$ applying Lemma to (4), we have

$$u_t - a(t)\Delta u = -\int_0^t r(x', t-\tau) u_\tau(x,\tau) d\tau. \tag{5}$$

In (5) $r(x',t)$ is the resolvent of the kernel $k(x',t)$ and it satisfies the integral equation

$$r(x',t) = k(x',t) + \int_0^t k(x', t-\tau) r(x',t) d\tau, \quad (x,t) \in \mathbb{R}^n_T. \tag{6}$$

In the sequel we investigate the problem of determining the functions $u(x,t)$, $r(x',t)$ satisfying the equations (5), (2), (3). Then after solving this problem, $k(x',t)$ can be found from (6).

We introduce new function $\vartheta^{(1)}(x,t)$ by formula $\vartheta^{(1)}(x,t) = u_{x_n x_n}(x,t)$. Then the straightaway differentiation of equations (5), (2) with respect to x_n twice leads us to the following relations for $\vartheta^{(1)}(x,t)$:

$$\vartheta_t^{(1)} - a(t)\Delta\vartheta^{(1)} = -\int_0^t r(x', t-\tau)\vartheta_\tau^{(1)}(x,\tau)d\tau, \tag{7}$$

$$\vartheta^{(1)}(x,0) = \varphi_{x_n x_n}(x). \tag{8}$$

The overdetermination condition can be transformed as follows. We allocate the term $a(t)u_{x_n x_n}$ in the expression $a(t)\Delta u$ of (5) and set $x_n = 0$. Then in view of $a(t)u_{x_n x_n} = a(t)\vartheta^{(1)}$ and using (2), we obtain

$$\vartheta^{(1)}(x', 0, t) = \frac{1}{a(t)} f_t(x', t) - \sum_{i=1}^{n-1} f_{x_i x_i}(x', t) + \frac{1}{a(t)}\int_0^t r(x', t-\tau) f_\tau(x', \tau)d\tau. \tag{9}$$

Requiring the continuity of the function $\vartheta^{(1)}(x,t)$ for $x_n = t = 0$, $x \in \mathbb{R}^{n-1}$, from (8) and (9) it follows the matching condition

$$\varphi_{x_n x_n}(x', 0) = \frac{1}{a(0)} f_t(x', 0) - \sum_{i=1}^{n-1} f_{x_i x_i}(x', 0). \tag{10}$$

Hereinafter, the values of functions $a(t)$, $f(x', t)$ and their derivatives at $t = 0$ we will understand as the limit for $t \to +0$.

We carry out the next converting of problem. Denoting for this purpose the derivative of $\vartheta^{(1)}(x,t)$ with respect to t by $\vartheta^{(2)}(x,t)$, i.e. $\vartheta^{(2)}(x,t) := \vartheta_t^{(1)}(x,t)$ and $h(x', t) := r_t(x', t)$, from (7)-(9) we get

$$\vartheta_t^{(2)} - a(t)\Delta\vartheta^{(2)} = a'(t)\Delta\vartheta^{(1)} - r(x', 0)\vartheta^{(2)} - \int_0^t h(x', t-\tau)\vartheta^{(2)}(x,\tau)d\tau, \tag{11}$$

$$\vartheta^{(2)}(x, 0) = a(0)\Delta\varphi_{x_n x_n}(x), \tag{12}$$

$$\vartheta^{(2)}(x', 0, t) = \frac{a'(t)}{a^2(t)} f_t(x', t) + \frac{1}{a(t)} f_{tt}(x', t) - \sum_{i=1}^{n-1} f_{t x_i x_i}(x', t) - \tag{13}$$

$$-\frac{a'(t)}{a^2(t)}\int_0^t r(x', t-\tau) f_\tau(x', \tau)d\tau + \frac{1}{a(t)}\int_0^t h(x', \tau) f_\tau(x', t-\tau)d\tau + \frac{1}{a(t)} r(x', 0) f_t(x', t).$$

Here the initial condition (11) is derived from (7) by setting $t = 0$ and using (8). The equations (11) and (13) also include the unknown function $r(x', 0)$. We can determine this function by following approach. Same as when getting the equality (10), we require the continuity of the function $\vartheta^{(2)}(x,t)$ for $x_n = t = 0$, $x \in \mathbb{R}^{n-1}$. Then, from (12) and (13) we have some equation the solving of which with respect to $r(x', 0)$ yields

$$r(x', 0) = \frac{1}{f_t(x', 0)}\Bigg[a^2(0)\Delta\varphi_{x_n x_n}(x', 0) - \frac{a'(0)}{a(0)} f_t(x', 0) - f_{tt}(x', 0) + \tag{14}$$

$$+ a(0)\sum_{i=1}^{n-1} f_{t x_i x_i}(x', 0)\Bigg]$$

Further the function $r(x',0)$ is assumed as known one.

Introducing also function $\vartheta(x,t)$ as $\vartheta(x,t) := \vartheta_t^{(2)}(x,t)$, in this way, we obtain the final problem of determining $\vartheta(x,t)$ and $h(x',t)$ satisfying the equations

$$\vartheta_t - a(t)\Delta\vartheta = 2a'(t)\Delta\vartheta^{(2)} + a''(t)\Delta\vartheta^{(1)} - r(x',0)\vartheta - h(x',t)a(0)\Delta\varphi_{x_n x_n}(x) - \int_0^t h(x',\tau)\vartheta(x,t-\tau)d\tau, \tag{15}$$

$$\vartheta(x,0) = \Psi(x), \tag{16}$$

$$\vartheta(x',0,t) = F(x',t) + \left(2\frac{(a'(t))^2}{a^3(t)} - \frac{a''(t)}{a^2(t)}\right)\int_0^t r(x',t-\tau)f_\tau(x',\tau)d\tau - \tag{17}$$
$$-2\frac{a'(t)}{a^2(t)}\int_0^t h(x',\tau)f_\tau(x',t-\tau)d\tau - \frac{1}{a(t)}\int_0^t h(x',\tau)f_{tt}(x',t-\tau)d\tau + \frac{1}{a(t)}h(x',t)f_t(x',0),$$

where

$$\Psi(x) = a^2(0)\Delta^2\varphi_{x_n x_n}(x) + a'(0)\Delta\varphi_{x_n x_n}(x) - r(x',0)a(0)\Delta\varphi_{x_n x_n}(x),$$

and through $F(x',t)$ in (17) is denoted the known function:

$$F(x',t) = \left(\frac{a''(t)}{a^2(t)} - \frac{(a'(t))^2}{a^3(t)}\right)f_t(x',t) + \frac{1}{a(t)}f_{ttt}(x',t) - \sum_{i=1}^{n-1} f_{ttx_ix_i}(x',t) -$$
$$-2\frac{a'(t)}{a^2(t)}r(x',0)f_t(x',t) + \frac{1}{a(t)}r(x',0)f_{tt}(x',t).$$

The equation (15) includes the expression $2a'(t)\Delta\vartheta^{(2)} + a''(t)\Delta\vartheta^{(1)}$ on the right side. Taking into account $\vartheta_t^{(1)} = \vartheta^{(2)}$ and using (7) we represent it through $\vartheta^{(2)}$:

$$a''(t)\Delta\vartheta^{(1)} = \frac{a''(t)}{a(t)}\vartheta^{(2)} + \frac{a''(t)}{a(t)}\int_0^t r(x',t-\tau)\vartheta^{(2)}(x,\tau)d\tau. \tag{18}$$

In similar way from (7) and (11), we obtain

$$2a'(t)\Delta\vartheta^{(2)} = 2(\ln a(t))'\left[\vartheta - (\ln a(t))'\left(\vartheta^{(2)} + \int_0^t r(x',t-\tau)\vartheta^{(2)}(x,\tau)d\tau\right) - \right. \tag{19}$$
$$\left. -r(x',0)\vartheta^{(2)} - \int_0^t h(x',t-\tau)\vartheta^{(2)}(x,\tau)d\tau\right].$$

In the future, we assume that in equation (15) the expression $2a'(t)\Delta\vartheta^{(2)} + a''(t)\Delta\vartheta^{(1)}$ is excluded with the help of (18) and (19).

At fulfilling the matching condition (10) and relation (14) it is not difficult carrying out the inverse transforms to derive from (11), (12) and (15)-(17) the equations (1)-(3) [17]. Thus the inverse problem (1)-(3) is equivalent to problem (11), (12) and (15)-(17) of determining the functions $\vartheta^{(2)}(x,t), \vartheta(x,t), h(x',t),\ r(x,'t)$. $\square$

3. Reduction of the Auxiliary problem

Here we prove the main result of this Section, which is the following assertion:

Lemma 2. *The auxiliary problems (11)-(12), (15)-(16) and the equality* $h(x',t) := r_t(x',t)$, *is equivalent to the problem of determination functions* $\vartheta^{(2)}(x,t), \vartheta(x,t)$, $h(x',t)$, $r(x,'t)$ *from the following system of integral equations:*

Proof.

$$\vartheta^{(2)}(x,t) = \int\limits_{\mathbb{R}^n} a(0)\Delta\varphi_{\xi_n\xi_n}(\xi) G\left(x-\xi,\theta(t)\right) d\xi + \int\limits_0^{\theta(t)} \frac{d\tau}{a(\theta^{-1}(\tau))} \times \tag{20}$$

$$\times \int\limits_{\mathbb{R}^n} \Bigg[\left(\ln a(\theta^{-1}(\tau))\right)' \Big(\vartheta^{(2)}\left(\xi,\theta^{-1}(\tau)\right) + \int\limits_0^{\theta^{-1}(\tau)} r(\xi',\theta^{-1}(\tau)-\alpha)\vartheta^{(2)}(\xi,\alpha)d\alpha\Big) -$$

$$-r(\xi',0)\vartheta^{(2)}\left(\xi,\theta^{-1}(\tau)\right) - \int\limits_0^{\theta^{-1}(\tau)} h(\xi',\theta^{-1}(\tau)-\alpha)\vartheta^{(2)}(\xi,\alpha)d\alpha \Bigg] G\left(x-\xi,\theta(t)-\tau\right) d\xi,$$

$$\vartheta(x,t) = \int\limits_{\mathbb{R}^n} \Psi(\xi) G(x-\xi,\theta(t))d\xi + \int\limits_0^{\theta(t)} \frac{d\tau}{a(\theta^{-1}(\tau))} \int\limits_{\mathbb{R}^n} \Bigg[\Big(\frac{a''(\theta^{-1}(\tau))}{a(\theta^{-1}(\tau))} - \tag{21}$$

$$-2((\ln a(\theta^{-1}(\tau)))')^2 \Big)\vartheta^{(2)}(\xi,\theta^{-1}(\tau)) + \Big(2(\ln a(\theta^{-1}(\tau)))' - r(\xi',0)\Big)\vartheta(\xi,\theta^{-1}(\tau)) -$$

$$- \int\limits_0^{\theta^{-1}(\tau)} h(\xi',\alpha)\vartheta(\xi,\theta^{-1}(\tau)-\alpha)d\alpha + \Big(\frac{a''(\theta^{-1}(\tau))}{a(\theta^{-1}(\tau))} + 2(\ln a(\theta^{-1}(\tau)))' -$$

$$-2((\ln a(\theta^{-1}(\tau)))')^2 \Big) \int\limits_0^{\theta^{-1}(\tau)} r(\xi',\theta^{-1}(\tau)-\alpha)\vartheta^{(2)}(\xi,\alpha)d\alpha + +2(\ln a(\theta^{-1}(\tau)))' \times$$

$$\times \int\limits_0^{\theta^{-1}(\tau)} h(\xi',\theta^{-1}(\tau)-\alpha)\vartheta^{(2)}(\xi,\alpha)d\alpha - h(\xi'\theta^{-1}(\tau))a(0)\Delta\varphi_{\xi_n\xi_n}(\xi) \Bigg] G(x-\xi,\theta(t)-\tau)d\xi,$$

$$h(x',t) = \frac{a(t)}{f_t(x',0)} \Bigg[\int\limits_{\mathbb{R}^n} \Psi(\xi) G(x'-\xi',\xi_n,\theta(t))d\xi - F(x',t) \Bigg] + \tag{22}$$

$$+ \frac{a(t)}{f_t(x',0)} \Bigg[\int\limits_0^{\theta(t)} \frac{d\tau}{a(\theta^{-1}(\tau))} \int\limits_{\mathbb{R}^n} \Bigg(\Big[\frac{a''(\theta^{-1}(\tau))}{a(\theta^{-1}(\tau))} - 2((\ln a(\theta^{-1}(\tau)))')^2 \Big] \vartheta^{(2)}(\xi,\theta^{-1}(\tau)) +$$

$$+ \Big[2(\ln a(\theta^{-1}(\tau)))' - r(\xi',0) \Big] \vartheta(\xi,\theta^{-1}(\tau)) - \int\limits_0^{\theta^{-1}(\tau)} h(\xi',\alpha)\vartheta(\xi,\theta^{-1}(\tau)-\alpha)d\alpha +$$

$$+ \Big[\frac{a''(\theta^{-1}(\tau))}{a(\theta^{-1}(\tau))} + 2(\ln a(\theta^{-1}(\tau)))' - 2((\ln a(\theta^{-1}(\tau)))')^2 \Big] \int\limits_0^{\theta^{-1}(\tau)} r(\xi',\tau-\alpha)\vartheta^{(2)}(\xi,\alpha)d\alpha +$$

$$+2(\ln a(\theta^{-1}(\tau)))' \int\limits_0^{\theta^{-1}(\tau)} h(\xi',\theta^{-1}(\tau)-\alpha)\vartheta^{(2)}(\xi,\alpha)d\alpha - h(\xi',\theta^{-1}(\tau))a(0)\Delta\varphi_{\xi_n\xi_n}(\xi)\Bigg)\times$$

$$\times G(x'-\xi',\xi_n,\theta(t)-\tau)d\xi\Bigg] - f_t(x',0)(2((\ln(a(t))')^2 - \frac{a''(t)}{a(t)}) \int\limits_0^t r(x',t-\tau)f_\tau(x',\tau)d\tau +$$

$$+2f_t(x',0)(\ln(a(t))' \int\limits_0^t h(x',\tau)f_\tau(x',t-\tau)d\tau + f_t(x',0)\int\limits_0^t h(x',\tau)f_{tt}(x',t-\tau)d\tau,$$

$$r(x,'t) = r(x',0) + \int\limits_0^t h(x',\tau)d\tau. \tag{23}$$

For **proof** Lemma we use the formula

$$p(x,t) = \int\limits_{\mathbb{R}^n} \varphi(\xi)G(x-\xi;\theta(t))d\xi + \int\limits_0^{\theta(t)} \frac{d\tau}{a(\theta^{-1}(\tau))}\times \tag{24}$$

$$\times \int\limits_{\mathbb{R}^n} F(\xi,\theta^{-1}(\tau))G(x-\xi;\theta(t)-\tau)d\xi,$$

which provides the solution of the following Cauchy problem for the heat equation with time-variable coefficient of thermal conductivity:

$$p_t - a(t)\Delta p = F(x,t),\ x\in\mathbb{R}^n, t>0,$$

$$p(x,0) = \varphi(x),\ x\in\mathbb{R}.$$

In (3.5) $\theta(t) = \int\limits_0^t a(\tau)d\tau$ and $\theta^{-1}(t)$ is the inverse function to $\theta(t)$; $G(x-\xi;\theta(t)-\tau) = \frac{1}{(2\sqrt{\pi(\theta(t)-\tau)})^n}e^{\frac{-|x-\xi|^2}{4(\theta(t)-\tau)}}$, $\xi = (\xi_1,...,\xi_n)$, $\xi' = (\xi_1,...,\xi_{n-1})$, $d\xi = d\xi_1...d\xi_n$, $|x|^2 = x_1^2+...+x_n^2$.

The equations (20) and (21) follow from the Cauchy problems (11), (12) and (15), (16) on bases of (24), respectively. In (21) we set $x_n = 0$ and use the additional condition (17). Then we obtain the equation (22). The equality (23) is obvious. □

We join to the equations (20)-(23) the integral equation with respect to the solution of the direct problem (1), (2), i.e. $u(x,t)$. It can be obtained from equalities (5) and (2). At first, we integrate by parts in the integral on the right-hand side of (5), then use the formula (24). As a result, one gets

$$u(x,t) = \int\limits_{\mathbb{R}^n}\varphi(\xi)G(x-\xi;\theta(t))d\xi + \int\limits_0^{\theta(t)}\frac{d\tau}{a(\theta^{-1}(\tau))}\int\limits_{\mathbb{R}^n}\Big[r(\xi',\theta^{-1}(\tau))\varphi(\xi)- \tag{25}$$

$$-r(\xi',0)u(\xi,\theta^{-1}(\tau)) - \int\limits_0^{\theta^{-1}(\tau)} h(\xi',\theta^{-1}(\tau)-\alpha)u(\xi,\alpha)d\alpha\Big]G(x-\xi;\theta(t)-\tau)d\xi.$$

4. Existence and uniqueness

In this section we prove the existence and uniqueness theorem for the system of the integral equations (20)-(23), (25). Then, from this result will be followed the existence unique solution of inverse problem (1)-(3). Here we use the contraction mapping principle [22, pp. 87-97]. The idea is to write the integral equations for unknown functions $\vartheta^{(2)}(x,t)$, $\vartheta(x,t)$, $h(x',t)$, $r(x',t)$ as a system with a nonlinear operator, and prove that this operator is a contraction mapping operator. The existence and uniqueness then follows immediately.

We recall that some operator F is a contraction mapping operator in a closed set Ω which is a subset of a Banach space if it satisfies the following two properties:

1) if $y \in \Omega$, then $Fx \in \Omega$ (i.e. F maps Ω into itself);

2) if $y, z \in \Omega$, then $\|Fy - Fz\| \le \rho \|y - z\|$ with $\rho < 1$ (ρ - is a constant independent of y and z).

Now we bring the main result of this work:

Theorem 1. *Suppose that all conditions of Section 1 with respect to the given functions $a(t)$, $\varphi(x)$, $f(x',t)$ and the matching conditions of (3) and (10) are satisfied. Besides $|f_t(x',0)| > f_0 = const > 0$, f_0 is a given number.*

Then there exists sufficiently small number $T > 0$ that the unique solution to the inverse problem (1)-(3) exists in the class of functions $u(x,t) \in H^{l+2,(l+2)/2}\left(\bar{\mathbb{R}}^n_T\right)$, $k(x',t) \in H^{l,l/2}\left(\bar{\mathbb{R}}^{n-1}_T\right)$.

Proof. The system of equations (20)–(23), (25) are closed system for the unknown functions $\vartheta^{(2)}(x,t), \vartheta(x,t), h(x',t),\ r(x,'t),\ u(x,t)$ in the domain $\mathbb{R}^n_T$. It can be rewritten in a nonlinear operator equation

$$\psi = A\psi, \tag{26}$$

where $\psi = (\psi_1,\ \psi_2,\ \psi_3, \psi_4)^* = \left(\vartheta^{(2)}(x,t), \vartheta(x,t), h(x',t),\ r(x,'t)\right)^*$, $*$ is the symbol of transposition, and according to the equations (20), (21), (22) are operator $A\psi = [(A\psi)_1,\ (A\psi)_2,\ (A\psi)_3, (A\psi)_4]$ has form

$$\begin{aligned}(A\psi)_1 = \psi_{01}(x,t) + \int\limits_0^{\theta(t)} \frac{d\tau}{a(\theta^{-1}(\tau))} \int\limits_{\mathbb{R}^n} \Bigg[(\ln a(\theta^{-1}(\tau)))' \Big(\psi_1(\xi, \theta^{-1}(\tau)) + \\ + \int\limits_0^{\theta^{-1}(\tau)} \psi_4(\xi', \theta^{-1}(\tau) - \alpha)\psi_1(\xi,\alpha)d\alpha \Big) - r(\xi',0)\psi_1(\xi,\alpha) - \\ - \int\limits_0^{\theta^{-1}(\tau)} \psi_3(\xi', \theta^{-1}(\tau) - \alpha)\psi_1(\xi,\alpha)d\alpha \Bigg] G(x-\xi, \theta(t)-\tau)d\xi,\end{aligned} \tag{27}$$

$$\begin{aligned}(A\psi)_2 = \psi_{02}(x,t) + \int\limits_0^{\theta(t)} \frac{d\tau}{a(\theta^{-1}(\tau))} \int\limits_{\mathbb{R}^n} \Bigg(\Big[\frac{a''(\theta^{-1}(\tau))}{a(\theta^{-1}(\tau))} - 2((\ln a(\theta^{-1}(\tau)))')^2 \Big] \times \\ \times \psi_1(\xi, \theta^{-1}(\tau)) + \Big[2(\ln a(\theta^{-1}(\tau)))' - r(\xi',0) \Big] \psi_2(\xi, \theta^{-1}(\tau)) - \int\limits_0^{\theta^{-1}(\tau)} \psi_3(\xi',\alpha) \times\end{aligned} \tag{28}$$

$$\times\psi_2(\xi,\theta^{-1}(\tau)-\alpha)d\alpha+\Big[\frac{a''(\theta^{-1}(\tau))}{a(\theta^{-1}(\tau))}+2(\ln a(\theta^{-1}(\tau)))'-2((\ln a(\theta^{-1}(\tau)))')^2\Big]\times$$

$$\times\int\limits_0^{\theta^{-1}(\tau)}\psi_4(\xi',\theta^{-1}(\tau)-\alpha)\psi_1(\xi,\alpha)d\alpha+2(\ln a(\theta^{-1}(\tau)))'\int\limits_0^{\theta^{-1}(\tau)}\psi_3(\xi',\theta^{-1}(\tau)-\alpha)\times$$

$$\times\psi_1(\xi,\alpha)d\alpha-\psi_3(\xi',\theta^{-1}(\tau))a(0)\Delta\varphi_{\xi_n\xi_n}(\xi)\Bigg)G(x-\xi,\theta(t)-\tau)d\xi,$$

$$(A\psi)_3=\psi_{03}(x',t)+\frac{a(t)}{f_t(x',0)}\Bigg[\int\limits_0^{\theta(t)}\frac{d\tau}{a(\theta^{-1}(\tau))}\int\limits_{\mathbb{R}^n}\Bigg(\Big[\frac{a''(\theta^{-1}(\tau))}{a(\theta^{-1}(\tau))}- \tag{29}$$

$$-2((\ln a(\theta^{-1}(\tau)))')^2\Big]\psi_1(\xi,\theta^{-1}(\tau))+\Big[2(\ln a(\theta^{-1}(\tau)))'-r(\xi',0)\Big]\psi_2(\xi,\theta^{-1}(\tau))-$$

$$-\int\limits_0^{\theta^{-1}(\tau)}\psi_3(\xi',\alpha)\psi_2(\xi,\theta^{-1}(\tau)-\alpha)d\alpha+\Big[\frac{a''(\theta^{-1}(\tau))}{a(\theta^{-1}(\tau))}+2(\ln a(\theta^{-1}(\tau)))'-$$

$$-2((\ln a(\theta^{-1}(\tau)))')^2\Big]\int\limits_0^{\theta^{-1}(\tau)}\psi_1(\xi',\theta^{-1}(\tau)-\alpha)\psi_1(\xi,\alpha)d\alpha+2(\ln a(\theta^{-1}(\tau)))'\times$$

$$\times\int\limits_0^{\theta^{-1}(\tau)}\psi_3(\xi',\theta^{-1}(\tau)-\alpha)\psi_1(\xi,\alpha)d\alpha-\psi_3(\xi',\theta^{-1}(\tau))a(0)\Delta\varphi_{\xi_n\xi_n}(\xi)\Bigg)\times$$

$$\times G(x'-\xi',\xi_n,\theta(t)-\tau)d\xi\Bigg]-f_t(x',0)(2((\ln(a(t))')^2-\frac{a''(t)}{a(t)})\int\limits_0^t\psi_4(x',t-\tau)f_\tau(x',\tau)d\tau+$$

$$+2f_t(x',0)(\ln(a(t))'\int\limits_0^t\psi_3(x',\tau)f_\tau(x',t-\tau)d\tau+f_t(x',0)\int\limits_0^t\psi_3(x',\tau)f_{tt}(x',t-\tau)d\tau,$$

$$(A\psi)_4=\psi_{04}(x',t)+\int\limits_0^t\psi_3(x',\tau)d\tau, \tag{30}$$

$$(A\psi)_5=\psi_{05}(x,t)+\int\limits_0^{\theta(t)}\frac{d\tau}{a(\theta^{-1}(\tau))}\int\limits_{\mathbb{R}^n}\Big[\psi_4(\xi',\theta^{-1}(\tau))\varphi(\xi)- \tag{31}$$

$$-r(\xi',0)\psi_5(\xi,\theta^{-1}(\tau))-\int\limits_0^{\theta^{-1}(\tau)}\psi_3(\xi',\theta^{-1}(\tau)-\alpha)\psi_5(\xi,\alpha)d\alpha\Big]G(x-\xi;\theta(t)-\tau)d\xi.$$

In (27)–(31) we introduced notations:

$$\psi_{01}(x,t)=\int\limits_{\mathbb{R}^n}a(0)\Delta\varphi_{\xi_n\xi_n}(\xi)G(x-\xi,\theta(t))d\xi,$$

$$\psi_{02}(x,t)=\int\limits_{\mathbb{R}^n}\Psi(\xi)G(x-\xi,\theta(t))d\xi,$$

$$\psi_{03}(x',t)=\frac{a(t)}{f_t(x',0)}\Big[\int\limits_{\mathbb{R}^n}\Psi(\xi)G(x'-\xi',\xi_n,\theta(t))d\xi-F(x',t)\Big],$$

$$\psi_{04}(x',t)=r(x',0),\quad \psi_{05}(x,t)=\int\limits_{\mathbb{R}^n}\varphi(\xi)G(x-\xi;\theta(t))d\xi.$$

Denote $|\psi|^l_T=\max\left(|\psi_1|^l_{T_0},\ |\psi_2|^l_{T_0},\ |\psi_3|^l_{T_0},|\psi_4|^l_{T_0},\ |\psi_5|^l_{T_0}\right),T<T_0$ and consider in the space $H^{l,l/2}\left(\mathbb{R}^n_T\right)$ the set $S(T)$ of functions $\psi(x,t)$, which obey the inequality

$$|\psi-\psi_0|^l_T\le|\psi_0|^l_{T_0}\,, \tag{32}$$

where $\psi_0=(\psi_{01},\ \psi_{02},\ \psi_{03},\ \psi_{04},\ \psi_{05})$ and

$$|\psi_0|^l_{T_0}=\max\left(|\psi_{01}|^l_{T_0},\ |\psi_{02}|^l_{T_0},\ |\psi_{03}|^l_{T_0}\ \ |\psi_{04}|^l_{T_0},\ |\psi_{05}|^l_{T_0}\right).$$

It can be demonstrated that sufficiently small T the operator A is contraction mapping operator in $S(T)$. The theorem of existence and uniqueness then follows immediately from the contraction mapping principle.

First it is shown that A has the first property of a contraction mapping operator. Let $\psi\in S(T),\ T<T_0$. The from the inequality (32), we have

$$|\psi_i|^l_T\le 2\,|\psi_0|^l_{T_0}\,,\ i=1,2,3,4,5.$$

Let us introduce the notations:

$$a_1:=\|a(t)\|_{C^2[0,T]},\quad a_2:=|(\ln a(t))'|_T,\quad a_3:=|a(0)|$$
$$r_1=|r(x',0)|^l,\quad f_1:=|f(x',t-\tau)|^{l+6,(l+6)/2},\quad \varphi_1:=|\varphi(x)|^{l+6}.$$

It is easy to see that

$$|(A\psi)_1-\psi_{01}|^l_T=\Big|\int\limits_0^{\theta(t)}\frac{d\tau}{a(\theta^{-1}(\tau))}\int\limits_{\mathbb{R}^n}\Bigg[(\ln a(\theta^{-1}(\tau)))'\Big(\psi_1(\xi,\theta^{-1}(\tau))+$$

$$+\int\limits_0^{\theta^{-1}(\tau)}\psi_4(\xi',\theta^{-1}(\tau)-\alpha)\psi_1(\xi,\alpha)d\alpha\Big)-r(\xi',0)\psi_1(\xi,\alpha)-\int\limits_0^{\theta^{-1}(\tau)}\psi_3(\xi',\theta^{-1}(\tau)-\alpha)\times$$

$$\times\psi_1(\xi,\alpha)d\alpha\Bigg]G(x-\xi,\theta(t)-\tau)d\xi\Big|^l_T\le\int\limits_0^{\theta(t)}\frac{d\tau}{|a(\theta^{-1}(\tau))|_T}\int\limits_{\mathbb{R}^n}\Bigg[|(\ln a(\theta^{-1}(\tau)))'|_T\times$$

$$\times\Big(\psi_1(\xi,\theta^{-1}(\tau))|^l_T+\int\limits_0^{\theta^{-1}(\tau)}|\psi_4(\xi',\theta^{-1}(\tau)-\alpha)|^l_T|\psi_1(\xi,\alpha)|^l_Td\alpha\Big)+|r(\xi',0)|^l|\psi_1(\xi,\alpha)|^l_T+$$

$$+\int\limits_0^{\theta^{-1}(\tau)}|\psi_3(\xi',\theta^{-1}(\tau)-\alpha)|^l_T|\psi_1(\xi,\alpha)|^l_Td\alpha\Bigg]G(x-\xi,\theta(t)-\tau)d\xi\le$$

$$\le|\psi_0|^l_{T_0}2T^2a_0(a_2+2Ta_2|\psi_0|^l_{T_0}+r_1+2T|\psi_0|^l_{T_0}):=|\psi_0|^l_{T_0}\beta_1,$$

In similar way we obtain

$$|(A\psi)_2-\psi_{02}|_T^l\le|\psi_0|_{T_0}^l[2T^2a_0(a_0a_1+2a_2^2+2a_2+r_1+2T|\psi_0|_{T_0}^l)+$$
$$+2T|\psi_0|_{T_0}^l(a_0a_2+2a_2+2a_2^2)+4Ta_2|\psi_0|_{T_0}^l+a_3\varphi_1]:=|\psi_0|_{T_0}^l\beta_2,$$
$$|(A\psi)_3-\psi_{03}|_T^l\le|\psi_0|_{T_0}^l\Big(2\frac{T^2}{f_1}\Big[a_0a_1+2a_2^2+2a_2+r_1+2T|\psi_0|_{T_0}^l+2T|\psi_0|_{T_0}^l(a_0a_1+2a_2+$$
$$+2a_2^2)+4T|\psi_0|_{T_0}^la_2+a_3\varphi_1\Big]+Tf_1^2(a_0a_1+2a_2^2+2a_2+1)\Big):=|\psi_0|_{T_0}^l\beta_3$$
$$|(A\psi)_4-\psi_{04}|_T^l\le 2T|\psi_0|_{T_0}^l:=|\psi_0|_{T_0}^l\beta_4,$$
$$|(A\psi)_5-\psi_{05}|_T^l\le|\psi_0|_{T_0}^l\cdot 2T^2a_0(\varphi_1+r_1+2T|\psi_0|_{T_0}^l):=|\psi_0|_{T_0}^l\beta_5,$$

where $\beta_i(T)\to 0$ at $T\to 0$, $i=1,2,3,4,5$. Therefore, if we choose T $(T<T_0)$ so that the following inequality should be satisfied

$$\beta:=\max\{\beta_1,\ \beta_2,\ \beta_3,\ \beta_4,\beta_5\}<1,$$

then the operator A has the first property of a contraction mapping operator, i.e., $A\psi\in S(T)$.

Consider next the second property of a contraction mapping operator for A. Let $\psi^{(1)}=\left(\psi_1^{(1)},\ \psi_2^{(1)},\ \psi_3^{(1)},\ \psi_4^{(1)},\ \psi_5^{(1)}\right)\in S(T)$, $\psi^{(2)}=\left(\psi_1^{(2)},\ \psi_2^{(2)},\ \psi_3^{(2)},\ \psi_4^{(2)},\ \psi_5^{(2)}\right)\in S(T)$. In view of inequalities

$$\left|\psi_2^{(1)}\psi_1^{(1)}-\psi_2^{(2)}\psi_1^{(2)}\right|_T^l=\left|\left(\psi_2^{(1)}-\psi_2^{(2)}\right)\psi_1^{(1)}+\psi_2^{(2)}\left(\psi_1^{(1)}-\psi_1^{(2)}\right)\right|_T^l\le$$
$$\le 2\left|\psi^{(1)}-\psi^{(2)}\right|_T^l\max\left(\left|\psi_1^{(1)}\right|_T^l,\ \left|\psi_2^{(2)}\right|_T^l\right)\le 4\,|\varphi_0|_T^l\left|\psi^{(1)}-\psi^{(2)}\right|_T^l,$$

we estimate the difference

$$|((A\psi)^{(1)}-A\psi)^{(2)})_1|_T^l\le$$
$$|((A\psi)^{(1)}-A\psi)^{(2)})_1|_T^l\le\int\limits_0^{\theta(t)}\frac{d\tau}{|a(\theta^{-1}(\tau))|_T}\int\limits_{\mathbb{R}^n}\Bigg[|(\ln a(\theta^{-1}(\tau)))'|_T|\Big((\psi_1^{(1)}(\xi,\theta^{-1}(\tau))-$$
$$-\psi_1^{(2)}(\xi,\theta^{-1}(\tau)))|_T^l+\int\limits_0^{\theta^{-1}(\tau)}|\Big[\psi_4^{(1)}(\xi',\theta^{-1}(\tau)-\alpha)\psi_1^{(1)}(\xi,\alpha)-$$
$$-\psi_4^{(2)}(\xi',\theta^{-1}(\tau)-\alpha)\psi_1^{(2)}(\xi,\alpha)\Big]|_T^ld\alpha\Big)+|r(\xi',0)|^l|(\psi_1^{(1)}(\xi,\alpha)-\psi_1^{(2)}(\alpha))|_T^l+$$
$$+\int\limits_0^{\theta^{-1}(\tau)}|\Big[\psi_3^{(1)}(\xi',\theta^{-1}(\tau)-\alpha)\psi_1^{(1)}(\xi,\alpha)-\psi_3^{(2)}(\xi',\theta^{-1}(\tau)-$$
$$-\alpha)\psi_1^{(2)}(\xi,\alpha)\Big]|_T^ld\alpha\Bigg]G(x-\xi,\theta(t)-\tau)d\xi\le$$
$$\le|\psi^{(1)}-\psi^{(2)}|_{T_0}^l2T^2a_0(a_2+4Ta_2|\psi_0|_{T_0}^l+r_1+4T|\psi_0|_{T_0}^l):=|\psi^{(1)}-\psi^{(2)}|_{T_0}^l\mu_1.$$

For other differences by similar way, we obtain

$$|((A\psi)^{(1)}-A\psi)^{(2)})_2|_T^l\le|\psi^{(1)}-\psi^{(2)}|_{T_0}^l\Big(T^2a_0(a_0a_1+2a_2^2+2a_2+r_1+4T|\psi_0|_{T_0}^l)+$$

$$+4T|\psi_0|^l_{T_0}(a_0a_1+2a_2+2a_2^2)+8Ta_2|\psi_0|^l_{T_0}+a_3\varphi_1\Big):=|\psi^{(1)}-\psi^{(2)}|^l_{T_0}\mu_2$$

$$|((A\psi)^{(1)}-A\psi)^{(2)})_3|^l_T\leq|\psi^{(1)}-\psi^{(2)}|^l_{T_0}\Big(\frac{T^2}{f_1}\Big[a_0a_1+2a_2^2+2a_2+r_1+4T|\psi_0|^l_{T_0}+$$

$$+4T|\psi_0|^l_{T_0}(a_0a_1+2a_2+2a_2^2)+8T|\psi_0|^l_{T_0}a_2+a_3\varphi_1\Big]+$$

$$+Tf_1^2(a_0a_1+2a_2^2+2a_2+1)\Big):=|\psi^{(1)}-\psi^{(2)}|^l_{T_0}\mu_3$$

$$|((A\psi)^{(1)}-A\psi)^{(2)})_4|^l_T\leq|\psi^{(1)}-\psi^{(2)}|^l_{T_0}T:=|\psi^{(1)}-\psi^{(2)}|^l_{T_0}\mu_4$$

$$|((A\psi)^{(1)}-A\psi)^{(2)})_5|^l_T\leq|\psi^{(1)}-\psi^{(2)}|^l_{T_0}T^2a_0(\varphi_1+r_1+4T|\psi_0|^l_{T_0}):=|\psi^{(1)}-\psi^{(2)}|^l_{T_0}\mu_5$$

Hence, $\left|\left(A\psi^{(1)}-A\psi^{(2)}\right)\right|^l_T<\mu\left|\psi^{(1)}-\psi^{(2)}\right|^l_T$, if T satisfies the condition

$$\mu:=\max\{\mu_1,\ \mu_2.\ \mu_3,\ \mu_4,\mu_5\}<1.$$

It is not difficult to see if we choose T_0 as $T_0=\min(\beta,\mu)$, then for $T\in(0,T_0)$ the operator A satisfies both the properties of a contraction mapping operator, i.e., A realizes contracted mapping of the set $S(T)$ onto itself. Hence, according to Banach theorem (see, for instance, [22, pp. 87-97]), in the set $S(T)$ there exists only one fixed point of transformations, i.e. there exists only one solution of (26).

□

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