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TWO-DIMENSIONAL INTERPOLATION OF FUNCTIONS WITH LARGE GRADIENTS IN BOUNDARY LAYERS

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ABSTRACT. The question of interpolation of a function of two variables with large gradients in the regions of the boundary layer is considered. The interpolated function has a representation as a sum of regular and boundary layer components. Such the representation is valid due to the Shishkin decomposition for the solution of the singularly perturbed problem. The development of interpolation formulas for such functions is relevant, since in the case of a uniform grid the error can be of the order of $O(1)$. In the rectangular domain a Bakhvalov mesh is applied, which condenses in the boundary layers. The initial domain is divided into rectangular cells. In each such cell, two-dimensional interpolation based on the Lagrange polynomial is applied. The interpolation formula contains k interpolation nodes in each direction. For each cell, an error estimate is obtained taking into account uniformity in a small parameter. An estimate of the stability of the interpolation formula is obtained on a two-dimensional grid from the class of Bakhvalov grids. The results of numerical experiments are consistent with the obtained error estimates. The study of the interpolation formula is necessary to continue the solution of the difference scheme from the grid nodes to the entire original domain.

Keywords: function of two variables, exponential boundary layer, Bakhvalov mesh, Lagrange polynomial, error estimate.

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1. Introduction

On the basis of singularly perturbed problems, various convective - diffusion processes with prevailing convection are modeled. The use of classical difference schemes [1] for the numerical solution of such problems leads to significant errors if the perturbing small parameter ε is commensurate with the grid step [2]. The problem of constructing difference schemes in the presence of a boundary layer has been widely studied, starting with the works of A.M. Il'in [2] and N.S. Bakhvalov [3]. In [2] it is constructed the difference scheme on a uniform grid based on fitting to a rapidly growing boundary layer component. In [3], it is proposed to apply the classical difference scheme on a grid condensed in the boundary layer. Grids dense enough in the boundary layer were developed in the works of a number of authors. The grids of Bakhvalov [3] and Shishkin [4] are widely known.

The problem of constructing interpolation formulas for functions with large gradients in the boundary layer is less studied and relevant, since in the case of a uniform grid, the error of the interpolation formula is of the order of $O(1)$ if the grid step is commensurate with a small parameter [5]. In [5], it was proposed to pick out, up to a factor, the component responsible for the growth of the function in the boundary layer and build interpolation formulas on a uniform grid that are exact on this component. In [6] is constructed an interpolation formula on a uniform grid with an arbitrarily specified number of interpolation nodes. A uniform error estimate for this formula with respect to the small parameter ε was obtained in [7]. Note that the class of functions for which we construct and study interpolation formulas corresponds to the solution of a singularly perturbed problem. For functions of this class, the decomposition in the form of a sum of regular and boundary layer components is valid, according to the decomposition of Shishkin [4].

In [8], [9], respectively, in the case of a function of one variable, in the presence of a boundary layer, estimates of the interpolation error by a Lagrange polynomial of an arbitrarily given degree on the Shishkin and Bakhvalov meshes are obtained.

Let us dwell on the results on the construction of interpolation formulas in the two-dimensional case in the presence of boundary layers. The considered class of functions corresponds to the solution of a singularly perturbed elliptic problem. Interpolation formulas, with k nodes in each direction, are built in rectangular cells covering the original domain. In [10] the interpolation formula of fitting to the boundary layer component from [6] is generalized to the two-dimensional case. An error estimate of order $O(h^{k-1})$ is obtained, where h is the step of the uniform grid. In the works under consideration, the interpolation error is estimated through the maximum error over all points of the original domain. The work [11] generalizes the result from [8] to the two-dimensional case. Interpolation by the Lagrange polynomial is carried out on a two-dimensional Shishkin mesh, and an error estimate of order $O((\ln(N)/N)^k)$, uniform in ε is obtained.

The purpose of this work is to estimate the error of interpolation by the Lagrange polynomial on the Bakhvalov mesh in the two-dimensional case, which is a generalization of the result from [9], where the error of the Lagrange polynomial is estimated when interpolating a function of one variable.

By C and C_j we mean positive constants that do not depend on the parameter ε and the number of grid steps N. We will restrict different values to one constant C_j, if it is clear from the text.

2. Formulation of the problem

We assume that the interpolated function $u(x,y)$ is a solution of a singularly perturbed elliptic problem, so we use the following decomposition [4, 12]:

$$u(x,y) = p(x,y) + E_1(x,y) + E_2(x,y) + E_{1,2}(x,y), \tag{1}$$

where $(x,y) \in \bar{\Omega},\ \bar{\Omega} = [0,1]^2$. We assume that in the representation (1) $p(x,y)$ is a regular component with bounded derivatives up to some order, $E_1, E_2, E_{1,2}$ are boundary-layer components with large gradients in boundary layers. All components in (1) are not explicitly given, but estimates of derivatives are known for them, namely, for some constant C we have

$$\Big|\frac{\partial^n p}{\partial x^n}(x,y)\Big| \le C,\ \Big|\frac{\partial^n p}{\partial y^n}(x,y)\Big| \le C,\ 0 \le n \le k, \tag{2}$$

$$\Big|\frac{\partial^n E_1}{\partial x^n}(x,y)\Big| \le \frac{C}{\varepsilon^n}e^{-\alpha x/\varepsilon},\ \Big|\frac{\partial^n E_1}{\partial y^n}(x,y)\Big| \le C,\ 0 \le n \le k, \tag{3}$$

$$\Big|\frac{\partial^n E_2}{\partial x^n}(x,y)\Big| \le C,\ \Big|\frac{\partial^n E_2}{\partial y^n}(x,y)\Big| \le \frac{C}{\varepsilon^n}e^{-\beta y/\varepsilon},\ \ 0 \le n \le k, \tag{4}$$

$$\Big|\frac{\partial^n E_{1,2}}{\partial x^n}(x,y)\Big| \le \frac{C}{\varepsilon^n}e^{-\alpha x/\varepsilon},\ \Big|\frac{\partial^n E_{1,2}}{\partial y^n}(x,y)\Big| \le \frac{C}{\varepsilon^n}e^{-\beta y/\varepsilon},\ \ 0 \le n \le k, \tag{5}$$

where $\alpha > 0, \beta > 0,\ \varepsilon \in (0,1]$. The constant k will be given as the number of interpolation nodes in x and y. According to (3)–(5) the derivatives of the functions $E_1, E_2, E_{1,2}$ can grow indefinitely as $\varepsilon \to 0$.

According to [4, 12], for a given k, one can perform a decomposition of the form (1) for the solution of an elliptic problem with regular boundary layers:

$$\varepsilon u_{xx} + \varepsilon u_{yy} + a_1(x)u_x + a_2(y)u_y - c(x,y)u = f(x,y),\ (x,y) \in \Omega;$$

$$a_1(x) \ge \alpha > 0,\ a_2(y) \ge \beta > 0,\ \ u(x,y) = g(x,y),\ (x,y) \in \Gamma, \tag{6}$$

where Γ is the boundary of $\bar{\Omega}$.

Let us estimate the error of polynomial interpolation formulas on the Bakhvalov mesh in the case of a function with large gradients of the form (1).

3. Setting of the mesh

First, let's define the grid in general form:

$$\Omega^h = \{(x_i, y_j),\ i = 0,1,\dots,N,\ j = 0,1,\dots,N\},$$

$$h_i = x_i - x_{i-1},\ \tau_j = y_j - y_{j-1},\ x_0 = 0, x_N = 1,\ y_0 = 0, y_N = 1.$$

Next, we define a grid related to Bakhvalov-type grids, in accordance with [13], where the case of a singularly perturbed problem for a second-order ordinary differential equation is considered. So let

$$\sigma_1 = \min\Big\{\frac{1}{2}, -\frac{k\varepsilon}{\alpha}\ln\varepsilon\Big\},\ \varepsilon \le e^{-1}. \tag{7}$$

For $\varepsilon > e^{-1}$ we set $\sigma_1 = 1/2$.

For $\sigma_1 = 1/2$ we define a grid Ω^h uniform in x, $h_i = 1/N$.

For $\sigma_1 < 1/2$ we set

$$x_i = -\frac{k\varepsilon}{\alpha}\ln\Big[1 - 2(1-\varepsilon)i/N\Big],\ i = 0,1,\dots,\frac{N}{2}, \tag{8}$$

$$x_i = \sigma_1 + (2i/N - 1)(1 - \sigma_1), \quad N/2 \le i \le N. \tag{9}$$

According to the y variable, we set the grid nodes in the same way:

$$\sigma_2 = \min\left\{\frac{1}{2}, -\frac{k\varepsilon}{\beta}\ln\varepsilon\right\}, \ \varepsilon \le e^{-1}. \tag{10}$$

For $\varepsilon > e^{-1}$ we set $\sigma_2 = 1/2$.

For $\sigma_2 = 1/2$, we set the grid Ω^h uniform in y with steps $\tau_j = 1/N$.

For $\sigma_2 < 1/2$ we set

$$y_j = -\frac{k\varepsilon}{\alpha}\ln\Big[1 - 2(1-\varepsilon)j/N\Big], \ j = 0, 1, \dots, \frac{N}{2}, \tag{11}$$

$$y_j = \sigma_2 + (2j/N - 1)(1 - \sigma_2), \quad N/2 \le j \le N. \tag{12}$$

So, the grid Ω^h is built.

4. Two-dimantional polynomial interpolation

We will interpolate the function $u(x, y)$ in rectangular cells $K_{i,j} = [x_i, x_{i+k-1}] \times [y_j, y_{j+k-1}]$, forming the cover of the original domain $\bar{\Omega}$. For interpolation, we will use the Lagrange polynomial with k interpolation nodes in x and y. To interpolate over x for a given value of y, we use the Lagrange polynomial:

$$L_x(u, x, y) = \sum_{m=i}^{i+k-1} u(x_m, y) \prod_{l=i, l\neq m}^{i+k-1} \frac{x - x_l}{x_m - x_l}. \tag{13}$$

Similarly, for a given x, we interpolate with respect to y

$$L_y(u, x, y) = \sum_{m=j}^{j+k-1} u(x, y_m) \prod_{l=j, l\neq m}^{j+k-1} \frac{y - y_l}{y_m - y_l}. \tag{14}$$

Based on (13), (14) we build a two-dimensional interpolation formula:

$$L_{x,y}(u, x, y) = L_y(L_x(u, x, y), x, y), \ (x, y) \in K_{i,j}. \tag{15}$$

5. Preliminary results

In [9] the error of interpolation by the Lagrange polynomial on the Bakhvalov mesh of a function of one variable with a boundary layer component is estimated. It is assumed that the decomposition is valid for the function $v(x)$:

$$v(x) = p(x) + \Phi(x), \ x \in [0, 1], \tag{16}$$

where for some constant C_1

$$|p^{(n)}(x)| \le C_1, \quad |\Phi^{(n)}(x)| \le \frac{C_1}{\varepsilon^n} e^{-\alpha x/\varepsilon}, \ 0 \le n \le k. \tag{17}$$

The functions $p(x)$ and $\Phi(x)$ do not explicitly given, $\alpha > 0, \varepsilon \in (0, 1]$.

The decomposition (16) is valid for the solution of singular perturbed problem in the case of a second-order ordinary differential equation [4].

Let $L_k(v, x)$ be the Lagrange interpolation polynomial for the function $v(x)$, corresponding to (13) when interpolated over x on the interval $[x_i, x_{i+k-1}]$. According to [9], the following theorem is valid.

Theorem 1. *Let the function $v(x)$ have the representation (16), N is a multiple of $2(k-1)$ and the nodes of the one-dimensional grid correspond to the nodes of the constructed grid Ω^h in the variable x. Then for some constant C for all $x \in [x_i, x_{i+k-1}]$ and $i = 0, k-1, \dots, N-k+1$ depending on the value of i, the following error estimates are valid:*

$$|L_k(v,x) - v(x)| \le \frac{C}{N^k}, \quad i+k-1 \ne \frac{N}{2}, \tag{18}$$

$$|L_k(v,x) - v(x)| \le \frac{C}{N^k}\Big[\ln^{k-1}\Big(1+\frac{1}{N\varepsilon}\Big)+1\Big], \quad i+k-1 = \frac{N}{2}. \tag{19}$$

In [9], stability estimates for the Lagrange polynomial on the Bakhvalov mesh are obtained. Let $w(x)$ be some bounded function on the interval $[0,1]$, $\tilde{w}(x)$ is some perturbation of this function. Then for some constant C at $x \in [x_i, x_{i+k-1}]$, depending on the value of i, stability estimates are obtained [9]:

$$|L_k(w,x) - L_k(\tilde{w},x)| \le C \max_{i \le n \le i+k-1} |w_n - \tilde{w}_n|, \; w_n = w(x_n), \; i+k-1 \ne N/2, \tag{20}$$

$$|L_k(w,x) - L_k(\tilde{w},x)| \le C \max_{i \le n \le i+k-1} |w_n - \tilde{w}_n| \Big[1 + \ln^{k-2}\Big(1+\frac{1}{N\varepsilon}\Big)\Big],$$

$$i+k-1 = N/2. \tag{21}$$

6. The error of a two-dimensional interpolation formula

Let us estimate the error of the two-dimensional interpolation formula (15) on the given grid Ω^h.

Theorem 2. *Let the function $u(x,y)$ have the representation (1) and nodes of grid Ω^h correspond to relations (7)– (12), N is a multiple of $2(k-1)$. Let the cells $K_{i,j} = [x_i, x_{i+k-1}] \times [y_j, y_{j+k-1}]$ do not intersect and form the coverage of $\bar{\Omega}$. Then there is a constant C such that for all $(x,y) \in K_{i,j}$, depending on the values of i,j, the following error estimate is valid:*

$$|L_{x,y}(u,x,y) - u(x,y)| \le \frac{C}{N^k}, \quad i+k-1 \ne \frac{N}{2}, \; j+k-1 \ne \frac{N}{2}, \tag{22}$$

$$|L_{x,y}(u,x,y) - u(x,y)| \le \frac{C}{N^k}\Big[\ln^{k-1}\Big(1+\frac{1}{N\varepsilon}\Big)+1\Big],$$

$$i+k-1 = \frac{N}{2}, j+k-1 \ne \frac{N}{2} \;\; \textit{or} \;\; j+k-1 = \frac{N}{2}, i+k-1 \ne \frac{N}{2}, \tag{23}$$

$$|L_{x,y}(u,x,y) - u(x,y)| \le \frac{C}{N^k}\Big[\ln^{2k-3}\Big(1+\frac{1}{N\varepsilon}\Big)+1\Big],$$

$$i+k-1 = \frac{N}{2}, j+k-1 = \frac{N}{2}. \tag{24}$$

Proof. From (15), we get

$$\Big|L_{x,y}(u,x,y) - u(x,y)\Big| \le$$

$$\le \Big|L_y(L_x(u,x,y) - u(x,y), x, y)\Big| + \Big|L_y(u,x,y) - u(x,y)\Big|. \tag{25}$$

Taking into account the relations (1)–(5), we obtain that, for a fixed value of y, the function $u(x,y)$ can be represented in a form similar to (16) :

$$u(x,y) = p(x,y) + \Phi(x,y),\ x,y \in [0,1], \tag{26}$$

where for some constant C

$$\Big|\frac{\partial^n p}{\partial x^n}(x,y)\Big| \le C,\ \ \Big|\frac{\partial^n \Phi}{\partial x^n}(x,y)\Big| \le \frac{C}{\varepsilon^n} e^{-\alpha x/\varepsilon}, 0 \le n \le k. \tag{27}$$

Similarly, for a fixed value of x we have

$$u(x,y) = q(x,y) + \Theta(x,y),\ x,y \in [0,1], \tag{28}$$

where for some constant C

$$\Big|\frac{\partial^n q}{\partial y^n}(x,y)\Big| \le C,\ \ \Big|\frac{\partial^n \Theta}{\partial y^n}(x,y)\Big| \le \frac{C}{\varepsilon^n} e^{-\beta x/\varepsilon}, 0 \le n \le k. \tag{29}$$

Let us prove the estimate (22). In this case $i+k-1 \neq N/2,\ j+k-1 \neq N/2$. Let $w(x,y)$ be some bounded function in the domain $\bar{\Omega},\ \tilde{w}(x,y)$ is some perturbation of this function. Considering that when interpolating over y for a given x the same grid is used as in the one-dimensional case, by analogy with (20) we obtain the stability estimate:

$$|L_y(w-\tilde{w},x,y)| \le C \max_{j \le n \le j+k-1} |w(x,y_n) - \tilde{w}(x,y_n)|, j+k-1 \neq \frac{N}{2}. \tag{30}$$

Considering (30) in (25), for some constant C_1 we get

$$\Big|L_{x,y}(u,x,y) - u(x,y)\Big| \le C_1 \max_{j \le n \le j+k-1} |L_x(u,x,y_n) - u(x,y_n)| +$$

$$+\Big|L_y(u,x,y) - u(x,y)\Big|. \tag{31}$$

According to (26)–(27), when interpolated over x for a given y, the function $u(x,y)$ have a decomposition similar to that of (16)– (17) in the case of a function of one variable, and the same grid is used. The decomposition (28)–(29) is used similarly to estimate the interpolation error over y. Thus, an estimate of the form (18) is preserved under interpolation with respect to x and y, and we get the estimates

$$\Big|L_x(u,x,y) - u(x,y)\Big| \le C/N^k,\ \ \Big|L_y(u,x,y) - u(x,y)\Big| \le C/N^k.$$

Taking into account these estimates in (31), we obtain the required estimate (22).

Let us dwell on the estimate (23). Let be

$$i+k-1 \neq N/2, j+k-1 = N/2.$$

Taking this case into account in the estimates (18)–(21), by analogy with the proof of the estimate (22), we obtain the estimate (23).

In the case of the conditions $j+k-1 = N/2, i+k-1 \neq N/2$, the estimate (23) can be proved similarly. To obtain the estimate (24), we take into account (19), (21). The theorem is proven. □

Stability of two-dimentional interpolation. Let us dwell on the estimation of the stability of the two-dimensional interpolation formula on the Bakhvalov mesh. Let $w(x,y)$ be a function bounded in the domain $\bar{\Omega}$ and $\tilde{w}(x,y)$ be some perturbation for this function. When estimating the stability of the interpolant $L_{x,y}(w,x,y)$, we use the stability estimates (20), (21) and the relation (15) for the two-dimensional interpolation formula.

Let us dwell on the case of the cell $K_{i,j}$, when $i+k-1 \neq N/2,\ j+k-1 \neq N/2$. Then according to (15), (30) we have

$$|L_{x,y}(w,x,y) - L_{x,y}(\tilde{w},x,y)| = |L_y(L_x(w-\tilde{w},x,y),x,y)| \leq$$
$$\leq C_1 \max_{y_n} |L_x(w-\tilde{w},x,y_n)| \leq C_1^2 \max_{m,n} |w_{m,n} - \tilde{w}_{m,n}|,$$
$$i \leq m \leq i+k-1,\ j \leq n \leq j+k-1.$$

Thus, for some constant C we obtain the estimate

$$|L_{x,y}(w,x,y) - L_{x,y}(\tilde{w},x,y)| \leq C \max_{m,n} |w_{m,n} - \tilde{w}_{m,n}|, \tag{32}$$

where $w_{m,n} = w(x_m,y_n), \tilde{w}_{m,n} = \tilde{w}(x_m,y_n), i \leq m \leq i+k-1,\ j \leq n \leq j+k-1$.

Let us dwell on the case $i+k-1 \neq N/2,\ j+k-1 = N/2$. By analogy with the considered case, applying the stability estimates (20), (21), for some constant C we get

$$|L_{x,y}(w,x,y) - L_{x,y}(\tilde{w},x,y)| \leq C \max_{m,n} |w_{m,n} - \tilde{w}_{m,n}| \Big[1 + \ln^{k-2}\Big(1+\frac{1}{N\varepsilon}\Big)\Big]. \tag{33}$$

In the case $i+k-1 = N/2,\ j+k-1 \neq N/2$, similarly the estimate (33) is valid.

Let $i+k-1 = N/2,\ j+k-1 = N/2$. Applying the estimate (21), we obtain

$$|L_{x,y}(w,x,y) - L_{x,y}(\tilde{w},x,y)| \leq C \max_{m,n} |w_{m,n} - \tilde{w}_{m,n}| \Big[1 + \ln^{2k-4}\Big(1+\frac{1}{N\varepsilon}\Big)\Big]. \tag{34}$$

So, in the two-dimensional case, stability estimates (32)– (34) of polynomial interpolation in the cell $K_{i,j}$ of the Bakhvalov mesh are obtained depending on the values of i, j.

Note that in the case of the Shishkin mesh, the grid steps in each cell $K_{i,j}$ are the same in x and y, so the stability estimate (32) is valid for all i, j.

7. Results of numerical experiments

Consider a function that can be represented in the form (1)

$$u(x,y) = \Big(1-e^{-x/\varepsilon}\Big)\Big(1-e^{-2y/\varepsilon}\Big)(1-x)(1-y) + \cos\frac{\pi x}{2}e^{-y},\ \varepsilon > 0, x,y \in [0,1].$$

Let us define the Shishkin mesh [4]. As in [8], the grid steps in x are given as:

$$h_i = \frac{2\sigma}{N},\ 1 \leq i \leq \frac{N}{2};\quad h_i = \frac{2(1-\sigma)}{N},\ \frac{N}{2} < i \leq N,$$

where

$$\sigma = \min\Big\{\frac{1}{2}, \frac{k\varepsilon}{\alpha}\ln N\Big\}.$$

Similarly, we set the steps in y.

The interpolation error on the Shishkin mesh satisfies the estimate [11]:

$$|L_{x,y}(u,x,y) - u(x,y)| \leq C\Big(\frac{\ln N}{N}\Big)^k,\ (x,y) \in K_{i,j}, \forall i,j. \tag{35}$$

Let's compute the error at the nodes $(\tilde{x}_i, \tilde{y}_j)$ of the condensed mesh, obtained from the mesh Ω^h by division of each grid interval into ten equal parts.

For $k=2$, the two-dimensional interpolation formula in a cell $K_{i,j}$ has the form:

$$L_{x,y}(u,x,y) = \Big(u_{i+1,j+1} - u_{i,j+1} - u_{i+1,j} + u_{i,j}\Big)\frac{x-x_i}{x_{i+1}-x_i}\frac{y-y_j}{y_{j+1}-y_j} +$$

$$+\left(u_{i,j+1}-u_{i,j}\right)\frac{y-y_j}{y_{j+1}-y_j}+\left(u_{i+1,j}-u_{i,j}\right)\frac{x-x_i}{x_{i+1}-x_i}+u_{i,j}. \tag{36}$$

TABLE 1. The error of the interpolation formula on the uniform grid, $k=2$

ε	N					
	16	32	64	128	256	512
1	$1.34e-3$	$3.37e-4$	$8.47e-5$	$2.12e-5$	$5.31e-6$	$1.33e-6$
10^{-1}	$8.28e-2$	$2.76e-2$	$8.01e-3$	$2.16e-3$	$5.62e-4$	$1.43e-4$
10^{-2}	$6.77e-1$	$5.09e-1$	$2.98e-1$	$1.40e-1$	$4.98e-2$	$1.50e-2$
10^{-3}	$7.19e-1$	$7.35e-1$	$7.42e-1$	$7.26e-1$	$5.95e-1$	$3.66e-1$
10^{-4}	$7.19e-1$	$7.35e-1$	$7.42e-1$	$7.46e-1$	$7.48e-1$	$7.49e-1$
10^{-5}	$7.19e-1$	$7.35e-1$	$7.42e-1$	$7.46e-1$	$7.48e-1$	$7.49e-1$

TABLE 2. The error of the interpolation formula on the Shishkin mesh, $k=2$

ε	N					
	16	32	64	128	256	512
1	$1.34e-3$	$3.37e-4$	$8.47e-5$	$2.12e-5$	$5.31e-6$	$1.33e-6$
	1.99	1.99	2.00	2.00	2.00	
10^{-1}	$3.58e-2$	$1.48e-2$	$5.72e-3$	$2.04e-3$	$5.62e-4$	$1.43e-4$
	1.28	1.37	1.49	1.86	1.97	
10^{-2}	$4.30e-2$	$1.87e-2$	$7.34e-3$	$2.63e-3$	$8.88e-4$	$2.86e-4$
	1.20	1.35	1.48	1.57	1.63	
10^{-3}	$4.52e-2$	$1.95e-2$	$7.57e-3$	$2.70e-3$	$9.05e-4$	$2.91e-4$
	1.21	1.37	1.49	1.58	1.64	
10^{-4}	$4.55e-2$	$1.96e-2$	$7.62e-3$	$2.72e-3$	$9.13e-4$	$2.94e-4$
	1.21	1.37	1.49	1.57	1.64	
10^{-5}	$4.55e-2$	$1.97e-2$	$7.63e-3$	$2.72e-3$	$9.14e-4$	$2.94e-4$
	1.21	1.37	1.49	1.57	1.64	

In table 1, in the case of $k=2$ and a uniform grid, the following error is given

$$\Delta_{\varepsilon,N}=\max_{i,j}\left|L_{x,y}(u,\tilde{x}_i,\tilde{y}_j)-u(\tilde{x}_i,\tilde{y}_j)\right|$$

for different values of ε and N. It follows from table 1 that the use of a uniform grid is unacceptable if $\varepsilon \le h$.

In table 2 in the case of $k=2$ and the Shishkin mesh the error $\Delta_{\varepsilon,N}$ and the calculated order of accuracy

$$M_{\varepsilon,N}=\log_2\frac{\Delta_{\varepsilon,N}}{\Delta_{\varepsilon,2N}}$$

for different values of ε and N are given. The results of the calculations agree with the estimate (35).

Table 3 for $k=2$ shows the error and the order of accuracy in the case of the Bakhvalov mesh given above. The calculation results give an error of the order of $O(1/N^2)$, which agrees with the estimate (22).

TABLE 3. The error of the interpolation formula on the Bakhvalov mesh, $k = 2$

ε	N					
	16	32	64	128	256	512
1	$1.34e-3$	$3.37e-4$	$8.47e-5$	$2.12e-5$	$5.31e-6$	$1.33e-6$
	1.99	1.99	2.00	2.00	2.00	
10^{-1}	$1.30e-2$	$3.63e-3$	$9.85e-4$	$2.59e-4$	$6.67e-5$	$1.69e-5$
	1.84	1.88	1.93	1.96	1.98	
10^{-2}	$1.49e-2$	$3.98e-3$	$1.11e-3$	$3.21e-4$	$9.32e-5$	$2.62e-5$
	1.90	1.84	1.79	1.79	1.83	
10^{-3}	$1.54e-2$	$3.87e-3$	$9.77e-4$	$2.50e-4$	$6.59e-5$	$1.81e-5$
	1.99	1.99	1.97	1.92	1.86	
10^{-4}	$1.55e-2$	$3.90e-3$	$9.75e-4$	$2.44e-4$	$6.12e-5$	$1.54e-5$
	2.00	2.00	2.00	2.00	1.99	
10^{-5}	$1.56e-2$	$3.90e-3$	$9.76e-4$	$2.44e-4$	$6.10e-5$	$1.53e-5$
	2.00	2.00	2.00	2.00	2.00	

TABLE 4. The error of the interpolation formula on the Bakhvalov mesh, $k = 3$

ε	N					
	16	32	64	128	256	512
1	$8.86e-5$	$1.14e-5$	$1.45e-6$	$1.82e-7$	$2.29e-8$	$2.86e-9$
	2.96	2.98	2.99	2.99	3.00	
10^{-1}	$9.43e-3$	$1.54e-3$	$2.19e-4$	$2.93e-5$	$3.78e-6$	$4.81e-7$
	2.62	2.81	2.90	2.95	2.98	
10^{-2}	$1.81e-2$	$1.74e-3$	$1.69e-4$	$1.71e-5$	$1.86e-6$	$2.22e-7$
	3.38	3.36	3.31	3.20	3.07	
10^{-3}	$3.93e-2$	$4.10e-3$	$4.21e-4$	$4.24e-5$	$4.19e-6$	$4.10e-7$
	3.26	3.28	3.31	3.34	3.35	
10^{-4}	$6.22e-2$	$6.83e-3$	$7.50e-4$	$8.10e-5$	$8.60e-6$	$8.93e-7$
	3.19	3.19	3.21	3.24	3.27	
10^{-5}	$8.56e-2$	$9.65e-3$	$1.10e-3$	$1.24e-4$	$1.39e-5$	$1.53e-6$
	3.15	3.14	3.15	3.16	3.18	

Table 4 for $k = 3$ shows the error and the order of accuracy in the case of the Bakhvalov mesh. The calculation results correspond to the estimate (22).

Application of the interpolation formula on the uniform grid leads to errors of the order of $O(1)$. The most accurate results are obtained on the Bakhvalov mesh. The results of numerical experiments are consistent with the obtained error estimates

CONCLUSION

For the first time, the question of polynomial interpolation of a function of two variables on the Bakhvalov mesh in the presence of exponential boundary layers was studied. The function to be interpolated is given in terms of the Shishkin decomposition, which is valid for the solution of a singularly perturbed problem. Lagrange interpolation is carried out in rectangular cells with an arbitrary number of interpolation nodes k in each direction. The estimate of the interpolation error of order $O(1/N^k\ )$ is obtained on the Bakhvalov mesh, as in the regular case when the interpolated function has bounded derivatives. Only in the cells at the end of the boundary layer there is a weak logarithmic dependence on the parameter ε. Estimates of the stability of a two-dimensional interpolation formula on the Bakhvalov mesh are obtained. The usefulness of stability estimates is due to the fact that the function at the grid nodes can be found with some error.

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