S©MR ISSN 1813-3304

СИБИРСКИЕ ЭЛЕКТРОННЫЕ
МАТЕМАТИЧЕСКИЕ ИЗВЕСТИЯ

Siberian Electronic Mathematical Reports
http://semr.math.nsc.ru

Том 16, стр. 144–144 (2019) УДК 512.57
DOI 10.33048/semi.2019.16.xxx MSC 08C15

ON DIRECTED AND FINITELY PARTITIONABLE BASES FOR QUASI-IDENTITIES

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Abstract. We prove that, under certain conditions on a quasivariety, there exists continuum many subquasivarieties of this quasivariety with both finitely partitionable (independent) and directed bases for quasi-identities. We also notice that such a situation is impossible for bases for anti-identities.

Keywords: quasivariety, basis for quasi-identities.

1. Introduction

In [3], the notion is introduced of a B-class. Existence of such a class with respect to a quasivariety $\mathbf{K}$ witnesses a very complicated structure of $\mathbf{K}$. A series of examples is listed in [4]. They include quasivarieties of well-known structures with no independent basis for their quasi-identities. As is shown in [5], the intersection of a family of such subquasivarieties admits a finitely partitionable (independent) basis for its quasi-identities.

In [1], the notion is introduced of a directed basis for sentences of a class of structures; moreover, conditions are indicated that are equivalent to existence of such a basis for anti-identities. The notions of directed and independent bases are quite opposite; namely, a basis is both directed and independent if and only if it is a singleton. However, existence of a B-class with respect to a quasivariety $\mathbf{K}$ implies existence of continuum many subquasivarieties of $\mathbf{K}$ admitting a finitely partitionable basis and (another) directed basis for its quasi-identities. We also notice that such a situation is impossible for anti-identities.

Kravchenko, A. V., On directed and finitely partitionable bases for quasi-identities.

The work is supported by ???.

Received May 3, 2022, published ???.

2. Preliminaries

Throughout the article, we consider structures and classes of structures of a fixed finite similarity type σ.

Recall that a *quasi-identity* is a universal Horn sentence of the form

$$\forall \overline{x} \left(\big(\varphi_1(\overline{x}) \& \ldots \& \varphi_n(\overline{x})\big) \to \varphi_0(\overline{x}) \right)$$

and an *anti-identity* is a (negative) universal Horn sentence of the form

$$\forall \overline{x} \left(\neg\varphi_1(\overline{x}) \vee \cdots \vee \neg\varphi_n(\overline{x}) \right),$$

where each φ_i is an atomic formula with free variables among the entries of $\overline{x}$. A *quasivariety* (*antvariety*) is a class of structures defined by a set of quasi-identities (anti-identities).

Let $\mathbf{K} \subseteq \mathbf{M}$, where $\mathbf{K}$ and $\mathbf{M}$ are universal Horn classes.

Definition 1. A set Φ of quasi-identities (anti-identities) is a *finitely paritionable basis* for quasi-identities (anti-identities) of $\mathbf{K}$ in $\mathbf{M}$ if there is a countable set J and a partition $\Phi = \bigcup_{n\in J} \Phi_n$ such that Φ_n is finite for every $n \in J$, we have $\mathbf{K} = \mathbf{M} \cap \mathrm{Mod}(\Phi)$, and the condition $\mathbf{K} \neq \mathbf{M} \cap \mathrm{Mod}\big(\Phi \backslash \Phi_n\big)$ holds for every $n \in J$.

Notice that no block Φ_n can be empty. If Φ is a finitely partitionable basis with $|\Phi_n| = 1$ for every $n \in J$ then Φ is said to be *independent*.

Notice that our definition slightly differs from the generally accepted one (see, for example, [7]). Namely, infinite bases are usually considered and the set of natural numbers is taken for J. Consideration of both infinite and finite bases allows us to establish a more convenient connection between properties of a basis (see Proposition 1).

The following notion was introduced in [1].

Definition 2. A set $\Phi = \{\varphi_i : i \in I\}$ of quasi-identities (anti-identities) is *directed in* $\mathbf{M}$ if, for all $i, j \in I$, there exists $k \in I$ such that φ_i and φ_j are consequences of φ_k in the class $\mathbf{M}$. If Φ is a directed in $\mathbf{M}$ set of quasi-identities (anti-identities) and $\mathbf{K} = \mathbf{M} \cap \mathrm{Mod}(\Phi)$ then we say that Φ is a *directed basis* for quasi-identities (anti-identities) of $\mathbf{K}$ in $\mathbf{M}$.

The following fact is immediate from Definitions 1 and 2.

Proposition 1. *If a basis Φ is simultaneously directed and finitely partitionable then it is finite. In particular, if Φ is simultaneously directed and independent then it is a singleton.*

Proof. Consider two representations of Φ. Since it is finitely partitionable, we have $\Phi = \bigcup_{n\in J} \Phi_n$, where each Φ_n is finite. Since it is directed, for every $n \in J$, there exists $i_n \in I$ such that every sentence in Φ_n is a consequence of φ_{i_n} in $\mathbf{M}$. If $\varphi_{i_n} \notin \Phi_n$ then $\mathbf{M} \cap \mathrm{Mod}(\Phi) = \mathbf{M} \cap \mathrm{Mod}\big(\Phi \backslash \Phi_n\big)$, which is a contradiction. If $|J| > 1$ then we consider $n, m \in J$ with $n \neq m$ and find $k \in J$ such that φ_{i_n} and φ_{i_m} are consequences of φ_k in $\mathbf{M}$. Since $\varphi_k \notin \Phi_n \cap \Phi_m = \varnothing$, we again arrive at a contradiction. We conclude that $J = 1$, i.e., Φ is a finite set (which is a singleton if Φ is an independent basis). $\square$

Kartashov [2] found a quasivariety of monounary algebras that admits an infinite directed basis and another infinite independent basis for its quasi-identities. We show that, under certain conditions, a quasivariety possesses 2^ω subquasivarieties with a similar property; however, such a situation is impossible for anti-identities.

The main tool is the notion of a B-class introduced in [3]. Let $\mathbf{M}$ be a quasivariety. We denote by $\mathcal{P}_{\mathrm{fin}}(\omega)$ the set of finite subsets of the set ω of natural numbers. We denote by $\mathbf{Q}(\mathbf{K})$ the least quasivariety extending a class $\mathbf{K}$ and by $\mathbf{H}(\mathbf{K})$ the class of homomorphic images of structures in $\mathbf{K}$. A class $\mathbf{A} = \{\mathcal{A}_F : F \in \mathcal{P}_{\mathrm{fin}}(\omega)\} \subseteq \mathbf{M}$ is called a B-*class with respect to* $\mathbf{M}$ if $\mathbf{A}$ satisfies the following conditions:

(B_0) for every nonempty $F \in \mathcal{P}_{\mathrm{fin}}(\omega)$, the structure $\mathcal{A}_F$ is finitely presented in $\mathbf{M}$; the structure $\mathcal{A}_\varnothing$ is trivial;

(B_1) if $F = G \cup H$ in $\mathcal{P}_{\mathrm{fin}}(\omega)$ then $\mathcal{A}_F \in \mathbf{Q}(\mathcal{A}_G, \mathcal{A}_H)$;

(B_2) if $F, G \in \mathcal{P}_{\mathrm{fin}}(\omega)$, $F \neq \varnothing$, and $\mathcal{A}_F \in \mathbf{Q}(\mathcal{A}_G)$ then $F = G$;

(B_3) if $F \in \mathcal{P}_{\mathrm{fin}}(\omega)$, $i \in \omega$, and f is a homomorphism from $\mathcal{A}_F$ to $\mathcal{A}_{\{i\}}$ then either $f(\mathcal{A}_F) \cong \mathcal{A}_\varnothing$ or $i \in F$;

(B_4) if $F \in \mathcal{P}_{\mathrm{fin}}(\omega)$ then $\mathbf{H}(\mathcal{A}_F) \cap \mathbf{M} \subseteq \mathbf{A}$.

As is shown in [3, 4, 5, 6], existence of a B-class with respect to a quasivariety $\mathbf{M}$ witnesses complexity of $\mathbf{M}$ from many points of view. Namely, such a quasivariety is $\mathcal{Q}$-universal, there exists a subclass $\mathbf{K} \subseteq \mathbf{M}$ such that the set of (isomorphism types) of finite sublattices of the lattice $\mathrm{L_q}(\mathbf{K})$ of its relative subquasivarieties is undecidable, and there exist 2^ω elements of $\mathrm{L_q}(\mathbf{M})$ with no covers, 2^ω subquasivarieties of $\mathbf{M}$ with no finitely partitionable bases for their quasi-identities, and 2^ω nonstandard subquasivarieties of $\mathbf{M}$; moreover, a series of natural algorithmic problems is undecidable for 2^ω subquasivarieties of $\mathbf{M}$. In the present article, we indicate one more point of view from which a quasivariety with a B-class is complicated.

3. Bases of quasi-identities

By [5, Theorem 8.1], if there exists a B-class with respect to a quasivariety $\mathbf{M}$ then there exist 2^ω subquasivarieties of $\mathbf{M}$ without finitely partitionable bases for their quasi-identities such that the intersection admits a finitely partitionable basis for its quasi-identities. Moreover, in [5, Theorem 8.2], conditions are found for such a basis to be independent.

The proofs of those assertions are "relatively constructive," i.e., they provide us with a description of the intersection of subquasivarieties in terms of the B-class.

We denote by $\mathcal{P}_{\mathrm{inf}}$ the set of infinite subsets $I \subseteq \omega$ such that the complement $\omega \setminus I$ is infinite too. For every $I \in \mathcal{P}_{\mathrm{inf}}$, we consider the class $\mathbf{K}_I$ of all structures $\mathcal{A} \in \mathbf{M}$ with the following property: The structure $\mathcal{A}_F$ is embeddable into $\mathcal{A}$ if and only if $F \subseteq I$. We put $\mathbf{K} = \bigcap_{I \in \mathcal{P}_{\mathrm{inf}}} \mathbf{K}_I$. The following assertion is immediate from [5, Claim 8.3].

Proposition 2. *The class* $\mathbf{K}$ *is a quasivariety and consists of all structures* $\mathcal{A} \in \mathbf{M}$ *with the following property: If* $\mathcal{A}_F$ *is embeddable into* $\mathcal{A}$ *then* $F = \varnothing$.

By [5, Claim 8.5], there exists a finitely partitionable basis for quasi-identities of $\mathbf{K}$ in $\mathbf{M}$. We find another basis for its quasi-identities.

Proposition 3. *There exists a directed basis for quasi-identities of* $\mathbf{K}$ *in* $\mathbf{M}$.

Proof. Consider a nonempty set $F \in \mathcal{P}_{\mathrm{fin}}(\omega)$. According to condition (B_0), there are a finite set X_F of variables and a finite set Δ_F of atomic formulas with free variables belonging to X_F such that $\mathcal{A}_F \cong \mathcal{F}_{\mathbf{M}}(X_F, \Delta_F)$, i.e., the structure $\mathcal{A}_F$ is finitely presented in $\mathbf{M}$ by the generators X_F and relations Δ_F. Let $\gamma_F\colon X_F \to A_F$ be the corresponding interpretation of the variables from X_F. Consider an arbitrary set $G \subseteq F$. By [3, Lemma 2.2(ii)], there is a homomorphism from $\mathcal{A}_F$ onto $\mathcal{A}_G$. Then there is a finite set Δ_G^F of atomic formulas such that $\mathcal{A}_G \cong \mathcal{F}_{\mathbf{M}}(X_F, \Delta_G^F)$ with an interpretation γ_G^F of the variables from X_F in A_G and $\Delta_G^F \models_{\mathbf{M}} \Delta_F$ (the reader is referred to [3, Sec. 1] for more detail).

Let $F \in \mathcal{P}_{\mathrm{fin}}(\omega)$. We consider the following sentence ψ_F which is equivalent to a finite set of quasi-identities:

$$\forall \overline{x} \left(\& \Delta_F(\overline{x}) \to \& \Delta_{\varnothing}^F(\overline{x}) \right),$$

where the tuple $\overline{x}$ corresponds to the set X_F of generators. Let $\Psi_{\mathbf{K}} = \{\psi_F : F \in \mathcal{P}_{\mathrm{fin}}(\omega)\}$.

We show that $\Psi_{\mathbf{K}}$ is a directed set. For $F, G \in \Psi_{\mathbf{K}}$, we put $H = F \cup G$. Let $\mathcal{A} \in \mathbf{M}$ and let $\mathcal{A}$ satisfy the sentence ψ_H. Assume that $\mathcal{A}$ satisfies the premise $\&\Delta_F$ with an interpretation γ of the variables from X_F in A. Then there exists a homomorphism f from $\mathcal{A}_F$ to $\mathcal{A}$. We represent $\mathcal{A}_F \cong \mathcal{F}_{\mathbf{M}}(X_H, \Delta_F^H)$ with an interpretation γ_F^H of variables from X_H in A_F. Then $\mathcal{A}$ satisfies the premise $\&\Delta_F^H$ with the interpretation $f\gamma_F^H$. Since $\Delta_F^H \models_{\mathbf{M}} \Delta_H$, we conclude that $\mathcal{A}$ satisfies the premise $\&\Delta_H$ with the same interpretation. Since $\mathcal{A} \models \psi_H$, the structure $\mathcal{A}$ satisfies the conclusion of ψ_H with the same interpretation, i.e., the homomorphic image of $\mathcal{A}_H$ under the composition of homomorphisms is a trivial structure. This means that $f(\mathcal{A}_F)$ is a trivial structure too, i.e., the conclusion of ψ_F holds in $\mathcal{A}$ with the interpretation γ.

It remains to show that $\Psi_{\mathbf{K}}$ is a basis for quasi-identities of $\mathbf{K}$ in $\mathbf{M}$. Let $\mathcal{A} \in \mathbf{K}$ and let $\mathcal{A}$ satisfy the premise of ψ_F with an interpretation γ of the variables from X_F in A. Then there exists a homomorphism f from $\mathcal{A}_F$ to $\mathcal{A}$. The homomorphic image $f(\mathcal{A}_F)$ belongs to $\mathbf{H}(\mathcal{A}_F)$ and is a substructure of $\mathcal{A}$. We have $\mathcal{A} \in \mathbf{K} \subseteq \mathbf{M}$; hence, $f(\mathcal{A}_F) \in \mathbf{M}$. By (B_4), we find that $f(\mathcal{A}_F) \cong \mathcal{A}_G$ for some $G \in \mathcal{P}_{\mathrm{fin}}(\omega)$. By Proposition 2, we have $G = \varnothing$. Therefore, the structure $f(\mathcal{A}_F)$ is trivial and, consequently, satisfies the conclusion of ψ_F with the same interpretation γ. Conversely, if $\mathcal{A} \in \mathbf{M}$ and $\mathcal{A} \models \psi_F$ for every $F \in \mathcal{P}_{\mathrm{fin}}(\omega)$ then no nontrivial structure of the form $\mathcal{A}_F$ admits a homomorphism to $\mathcal{A}$. We conclude that $\mathcal{A}_F$ is embeddable into $\mathcal{A}$ if and only if $F = \varnothing$. It remains to apply Proposition 2. □

We summarise the above results as follows.

Proposition 4. *If there exists a* B*-class with respect to a quasivariety* $\mathbf{M}$ *then there exist distinct bases* $\Phi_{\mathbf{K}}$ *and* $\Psi_{\mathbf{K}}$ *for quasi-identities of a suitable subquasivariety* $\mathbf{K} \subseteq \mathbf{M}$ *such that* $\Phi_{\mathbf{K}}$ *is finitely partitionable and* $\Psi_{\mathbf{K}}$ *is directed.*

By [5, Claim 8.1], existence of a B-class with respect to $\mathbf{M}$ implies existence of 2^ω such classes; moreover, the corresponding quasivarieties for distinct B-classes are distinct too. Hence, we take into account Proposition 4 and obtain the following assertion.

Theorem 5. *If there exists a* B*-class with respect to a quasivariety* $\mathbf{M}$ *then there exist* 2^ω *subquasivarieties* $\mathbf{K}$ *of* $\mathbf{M}$ *such that* $\mathbf{K}$ *admits a finitely partitionable basis* $\Phi_{\mathbf{K}}$ *and a directed basis* $\Psi_{\mathbf{K}}$ *in* $\mathbf{M}$.

An additional condition on a B-class guarantees that the finitely partitionable basis for quasi-identities becomes independent, see [5, Theorem 8.2]. Thus, there exist examples of quasivarieties with (distinct) independent and directed infinite bases for their quasi-identities. The proof of the following assertion is immediate from [5, Corollary 9.1] and Theorem 5.

Corollary 6. *If $\mathbf{X}$ is one of the classes below then the conclusion of Theorem 5 holds for a suitable subquasivariety $\mathbf{M} \subseteq \mathbf{X}$:*

- *the variety $\mathbf{U}$ of monounary algebras without 1-cycles,*
- *the variety $\mathbf{R}$ of all commutative rings with unit,*
- *the variety $\mathbf{C}_{mn}$ of all Cantor algebras, where $0 < m < n < \omega$,*
- *the quasivariety $\mathbf{G}$ of all directed loopless graphs,*
- *the quasivariety $\mathbf{M}_{0,1}$ of all modular $(0,1)$-lattices,*
- *every variety of $(0,1)$-lattices that contains a finite non-distributive simple $(0,1)$-lattice,*
- *every quasivariety of undirected antireflexive graphs that contains a non-bipartite graph.*

In the first four cases, the bases $\Phi_{\mathbf{K}}$ are independent.

4. Bases of anti-identities

We prove that an analogue of Theorem 5 is no longer valid for anti-identities.

By [1, Theorem 4.8], if $\mathbf{M}$ is a proper universal Horn class (i.e., no structure in $\mathbf{M}$ possesses a trivial substructure) and $\mathbf{K}$ is an $\mathbf{M}$-antivariety then the following conditions are equivalent:

(1) there exists a directed basis for anti-identities of $\mathbf{K}$ in $\mathbf{M}$;
(2) the element $\mathbf{K}$ is meet irreducible in the lattice of $\mathbf{M}$-antivarieties;
(3) we have $\mathcal{A} \times \mathcal{B} \in \mathbf{M} \setminus \mathbf{K}$ for all $\mathcal{A}, \mathcal{B} \in \mathbf{M} \setminus \mathbf{K}$.

We show that each of these equivalent conditions is equivalent to the following condition:

(4) each basis for anti-identities of $\mathbf{K}$ in $\mathbf{M}$ is directed.

Indeed, it is clear that (4) yields (1); it suffices to prove that, say, (3) implies (4). Essentially, we pass to the set of (isomorphism types) of finitely presented structures in $\mathbf{M} \setminus \mathbf{K}$ partially ordered by homomorphisms and take into account the fact that each subset generating a directed set is directed.

Theorem 7. *If an antivariety possesses a directed basis for its anti-identities then each basis for its anti-identities is directed.*

Proof. Let $\mathbf{K}$ be defined in $\mathbf{M}$ by a basis $\Sigma = \{\varphi_i : i \in I\}$ for its anti-identities. For every anti-identity

$$\varphi \leftrightharpoons \forall \overline{x} \left(\neg\psi_1(\overline{x}) \vee \cdots \vee \neg\psi_n(\overline{x}) \right),$$

let $\mathcal{A}(\varphi)$ denote the structure defined in $\mathbf{M}$ by the generators $\overline{x}$ and the relations $\psi_1, \ldots, \psi_n$. It is easy to see that a structure $\mathcal{B} \in \mathbf{M}$ satisfies an anti-identity φ if and only if $\mathcal{A}(\varphi)$ admits no homomorphism to $\mathcal{B}$.

In particular, for every $i \in I$, the anti-identity φ_i is false in $\mathcal{A}(\varphi_i)$, i.e., for all $i, j, \in I$, we have $\mathcal{A}(\varphi_i), \mathcal{A}(\varphi_j) \notin \mathbf{K}$. By (3), we obtain $\mathcal{A}(\varphi_i) \times \mathcal{A}(\varphi_j) \notin \mathbf{K}$. Hence, there exists $k \in I$ such that $\mathcal{A}(\varphi_i) \times \mathcal{A}(\varphi_j)$ does not satisfy φ_k. This means that $\mathcal{A}(\varphi_k)$ admits a homomorphism to $\mathcal{A}(\varphi_i) \times \mathcal{A}(\varphi_j)$. Hence, $\mathcal{A}(\varphi_k)$ admits a homomorphism to both $\mathcal{A}(\varphi_i)$ and $\mathcal{A}(\varphi_j)$.

Let $\mathcal{B} \in \mathbf{M}$ be an arbitrary structure obeying φ_k. Then $\mathcal{A}(\varphi_k)$ admits no homomorphism to $\mathcal{B}$. Hence, none of $\mathcal{A}(\varphi_i)$, $\mathcal{A}(\varphi_j)$ admits a homomorphism to $\mathcal{B}$. This means that $\mathcal{B}$ satisfies both φ_i and φ_j. Since $\mathcal{B} \in \mathbf{M}$ is arbitrary, we conclude that φ_i and φ_j are consequences of φ_k in $\mathbf{M}$. $\square$

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