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THE ONE-DIMENSIONAL IMPULSIVE BARENBLATT–ZHELTOV–KOCHINA EQUATION

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Abstract. The initial-boundary value problem for the one-dimensional impulsive pseudoparabolic equation is studied. As a coefficient in the second-order diffusion term, this equation contains the smoothed Dirac delta-function concentrated at some time moment. From a physical viewpoint, such term allows to describe impulsive pressure

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drop phenomena in filtration problems. Existence and uniqueness of solutions for fixed values of the small parameter of smoothing is proved. After this, the limiting passage as the small parameter tends to zero is fulfilled and rigorously justified. As the result, the limit instantaneous impulsive microscopic-macroscopic model is established. This model is well-posed and involves the additional equation on a transition layer posed on a 'very fast' timescale.

Key words: pseudoparabolic equation, impulsive equation, strong solution, Fourier series, transition layer

1. Introduction

The classical Barenblatt–Zheltov–Kochina equation

$$\partial_t u = \chi \partial^3_{xxt} u + \nu \partial^2_{xx} u + f \quad (\chi, \nu > 0) \tag{1}$$

(here, in one-dimensional case) describes nonstationary filtration of a viscous fluid in a cracky-porous ground [4]. In this framework, the sought function u in (1) has a physical sense of the distribution of pressure in the gallery of cracks. Equation (1) also arises in studies of non-equilibrium processes in the heat transfer [7], where u plays the role of one of the temperatures in a two-temperature continuum. As well, some dynamical problems regarding non-Newtonian second-order fluid reduce to boundary value problems for equation (1), and in this context u means some expression of velocity components [28].

If processes under study are rather regular, we can confine ourselves to consideration of rather regular coefficients $\chi = \chi(x,t)$ and $\nu = \nu(x,t)$. The relevant theory of equation (1) and of its multi-dimensional version is vast and deep; its systematic exhibition can be found, for example, in [9]. On the other hand, irregular processes, like very fast pressure drop in the cracky gallery in the problem of filtration, lead to the cases when coefficients become highly non-smooth. More certainly, in this case coefficient ν may behave closely to

$\nu_0 + \alpha\delta_{(t=\tau)}$, where $\nu_0 = \text{const} > 0$, $\alpha = \text{const} \neq 0$, and $\delta_{(t=\tau)}$ is the Dirac delta concentrated at the time moment $t = \tau$ of the (instant) pressure drop. For example, such quick capillary pressure drops were experimentally observed in [20] in a two-phase flow. We can also note that the additional term $\alpha\delta_{(t=\tau)}$ may correspond to pressure fluctuations linked with 'impulsive' fast diffusion in porous media. In this view, we remark that fast diffusion can be simulated also with the help of nonstandard gradient growth [6].

In the present paper, we approximate $\delta_{(t=\tau)}$ on the so-called transition ε-layer. To this end, instead of $\delta_{(t=\tau)}$, we put into equation (1) the sequence $\{K_\varepsilon^\tau\}_{\varepsilon\to 0+}$, which converges weakly* to $\delta_{(t=\tau)}$. We observe that the similar approach was applied in [2, 12], where the Dirac delta-function $\delta_{(t=\tau)}$ was encompassed in the minor (source) term, not in the term containing a derivative.

Now, in addition to the modification of (1) described above regarding the approximation of the Dirac delta, we also set $\nu_0 = 1$, $\chi = 1$ and $f = 0$ for simplicity and formulate the following initial-boundary value problem for the Barenblatt–Zheltov–Kochina equation with the homogeneous boundary conditions, which will be studied further in the article:

$$\partial_t u_\varepsilon = \partial^3_{xxt} u_\varepsilon + (1 + \alpha K_\varepsilon^\tau(t))\, \partial^2_{xx} u_\varepsilon, \quad (x,t) \in \Pi_T, \tag{2a}$$

$$u_\varepsilon(0,t) = u_\varepsilon(1,t) = 0, \quad t \in (0,T), \tag{2b}$$

$$u_\varepsilon(x,0) = g(x), \quad x \in (0,1). \tag{2c}$$

In (2), $\Pi_T = (0,1) \times (0,T)$ is a space-time rectangle, $T = \text{const} > 0$ and $\tau \in (0,T)$ are given fixed time moments, $\alpha \in \mathbb{R}$ is a given fixed value, $u_\varepsilon = u_\varepsilon(x,t)$ is the sought function, $g = g(x)$ is a given function, and K_ε^τ is a given smooth kernel. For a fixed value τ, kernel $K_\varepsilon^\tau = K_\varepsilon^\tau(t)$ is supported on $\left\{\tau - \frac{\varepsilon}{2} \leqslant t \leqslant \tau + \frac{\varepsilon}{2}\right\}$ and is defined by the formula

$$K_\varepsilon^\tau(t) = \frac{1}{\varepsilon} K\left(\frac{t-\tau}{\varepsilon}\right), \quad t \in [0,T],$$

where $K = K(\vartheta)$ is a nonnegative smooth even function supported on segment $\left\{-\frac{1}{2} \leqslant \vartheta \leqslant \frac{1}{2}\right\}$, with the mean value equal to unity, i.e., $\int_{-\frac{1}{2}}^{\frac{1}{2}} K(\vartheta)\, d\vartheta = 1$, and $\varepsilon > 0$ is a small parameter. Thus, K_ε^τ approximates the Dirac delta-function $\delta_{(t=\tau)}$ in weak* sense as $\varepsilon \to 0+$, i.e., the limiting relation $\lim\limits_{\varepsilon\to 0+} \int_{\mathbb{R}} \phi(t) K_\varepsilon^\tau(t)\, dt = \phi(\tau)$ holds for any integrable in a neighborhood of $\{t = \tau\} \subset \mathbb{R}$ function ϕ having the trace at the point $t = \tau$. Note that

$$\varepsilon K_\varepsilon^\tau(\tau + \varepsilon \bar{t}) = K(\bar{t}). \tag{3}$$

By the analogy with the theory of generalized ordinary differential equations [10, 11, 21], we may call (2a) *the generalized Barenblatt–Zheltov–Kochina equation* corresponding to the pressure of a liquid in cracks.

Further, in Section 2, for any fixed rather small ε (say, $\varepsilon \in (0,1]$), we construct a solution to problem (2) in the form of Fourier series (7) and establish that this solution is unique and satisfies the first and the second energy estimates uniform in ε, see Propositions 1. In Section 3, we formulate Theorems 1 and 2, which are the main results of the paper and which relate to the limiting transition in problem (2), as $\varepsilon \to 0+$. Theorem 1 asserts the compactness of the family $\{u_\varepsilon\}_{\varepsilon\in(0,1]}$ as $\varepsilon \to 0+$ and the fact that the limit function $u = \lim\limits_{\varepsilon\to 0+} u_\varepsilon$ satisfies the homogeneous Barenblatt–Zheltov–Kochina equation outside the section $\{t = \tau\}$. The proof of Theorem 1 is given in Section 4. In order to prove compactness of $\{u_\varepsilon\}_{\varepsilon\in(0,1]}$, we apply the Aubin–Lions–Simon lemma [24, Theorem 3], see also [3, 8, 14, 15, 19]. Theorem 2 deals with the study of the family $\{u_\varepsilon\}_{\varepsilon\in(0,1]}$ near the section $\{t = \tau\}$ and is proved in Section 5. More certainly, in Section 5, in order to link the one-sided traces of the limit solution at $t = \tau$, we rescale u_ε on $\left(0, \tau - \frac{\varepsilon}{2}\right)$, $\left(\tau - \frac{\varepsilon}{2}, \tau + \frac{\varepsilon}{2}\right)$, and

$\left(\tau + \dfrac{\varepsilon}{2}, T\right)$ and apply the Aubin–Lions lemma [16, Lemma 2.48, p. 36],[24, Corollary 6] to the rescaled solution. As the result, for the rescaled solution we derive the so-called *equation on the transition layer* between $t = \tau - 0$ and $t = \tau + 0$, see (11b), which incorporates not the 'macroscopic' ('slow') time variable t, but the 'microscopic' ('fast') time variable $\bar{t}$, so that $\bar{t} \in [0, 1]$, $t = \tau - 0$ corresponds to $\bar{t} = 0+$, and $t = \tau + 0$ corresponds to $\bar{t} = 1 - 0$.

The main difference between the aforementioned results from [2, 12] and the results from the present article is that, in the present article, the dissipative term persists on a transition layer $\{0 \leqslant \bar{t} \leqslant 1\}$ after rescaling, since $\varepsilon\partial^2_{xx}\bar{u}_\varepsilon$ does not vanish, as $\varepsilon \to 0+$ in equation (10). This is one of the novelties in the present paper.

Concluding this introduction, let us remark that pseudoparabolic equations are of Sobolev type [1, 9] and are also applied for regularization of forward-backward parabolic equations [5, 17, 18, 25]. The presence of $\partial^3_{xxt}u_\varepsilon$ is essential in our study. The purely parabolic equation of the form (2a), i.e., the equation where the third order derivative $\partial^3_{xxt}u_\varepsilon$ is discarded, requires an additional research and this question lays beyond the present article. As well, an interesting direction of further research may be devoted to inclusion of a non-local term of the Fredholm type into (2a). Specifically, inserting the term

$$K^\tau_\varepsilon(t) \int_0^T K^\tau_\varepsilon(s)\partial^2_{xx}u_\varepsilon(x, s)\, ds$$

on the place of $K^\tau_\varepsilon(t)\partial^2_{xx}u_\varepsilon(x, t)$, we encounter the generalized Fredholm type integro-differential pseudoparabolic equation, which may have a significant place in the theory of non-local in time partial differential equations [26, 27, 30] and in the theory of impulsive integro-differential equations [13].

2. Existence. The first and the second energy estimates

Proposition 1. *Whenever* $g \in W_0^{2,2}(0,1)$, *there is a unique solution* $u_\varepsilon = u_\varepsilon(x,t)$ *to problem* (2). *Moreover, the solution satisfies the uniform in* ε *first and second energy estimates*

$$\|u_\varepsilon\|^2_{L^\infty(0,T;L^2(0,1))} + \|\partial_x u_\varepsilon\|^2_{L^\infty(0,T;L^2(0,1))} \leqslant C\big(\|g\|^2_{L^2(0,1)} + \|\partial_x g\|^2_{L^2(0,1)}\big) \stackrel{\text{def}}{=} M_1, \quad \forall\, \varepsilon \in (0, \varepsilon_0], \tag{4}$$

$$\|\partial_x u_\varepsilon\|^2_{L^\infty(0,T;L^2(0,1))} + \|\partial^2_{xx} u_\varepsilon\|^2_{L^\infty(0,T;L^2(0,1))} \leqslant M_2, \tag{5}$$

and the maximum principle

$$\|\partial_x u_\varepsilon\|_{L^\infty(\Pi_T)} \leqslant M_3, \tag{6}$$

where $M_1, M_2, M_3 = \text{const} > 0$ *do not depend on* ε.

The proof of this proposition can be divided into four stages and is given in Sections 2.1–2.4 further.

2.1. **Existence and uniqueness.** Since we deal with the linear case, we apply the Fourier series method. Taking into account homogeneous conditions, a solution is sought as a Fourier series:

$$u_\varepsilon(x,t) = \sum_{n=1}^{\infty} c_{\varepsilon,n}(t)\varphi_n(x), \tag{7}$$

where $\varphi_n(x) = \sin(\lambda_n x)$, $\lambda_n = n\pi$. Therefore, the coefficients satisfy the set of Cauchy problems ($n \in \mathbb{N}$)

$$c'_{\varepsilon,n}(t) = \frac{-\lambda_n^2}{1+\lambda_n^2}(1 + \alpha K_\varepsilon^\tau(t))c_{\varepsilon,n}(t), \quad t \in (0,T), \tag{8a}$$

$$c_{\varepsilon,n}(0) = g_n, \tag{8b}$$

where, in turn, g_n are the Fourier coefficients of initial data $g \in W_0^{2,2}(0,1)$.

Now, let us apply the results from [22, 23] to (8). Namely, for every $t \in [0, +\infty)$ and $\varepsilon > 0$ we introduce the linear operator $E_\varepsilon(t)$: $L^2(0,1) \mapsto L^2(0,1)$ by the rule:

$$E_\varepsilon(t)v = \sum_{n=1}^{\infty} \exp\left(-\frac{\lambda_n^2}{1+\lambda_n^2}\left(t + \alpha\int_0^t K_\varepsilon^\tau(s)\,ds\right)\right) v_n\varphi_n(x), \quad t > 0,$$

for an arbitrary function v with $\|v\|^2_{L^2(0,1)} = \sum\limits_{n=1}^{\infty} v_n^2 < \infty$.

Operator $E_\varepsilon(t)$ depends on t as on a parameter and has the following properties.

(i) $E_\varepsilon(0) = I$,

(ii) $\|E_\varepsilon\| \leqslant \exp(|\alpha|)$.

Proof of assertion (ii). By Parseval's inequality, we have

$$\|E_\varepsilon(t)v\|^2_{L^2(0,1)} = \sum_{n=1}^{\infty} v_n^2 \exp\left(-\frac{2\lambda_n^2}{1+\lambda_n^2}\left(t + \alpha\int_0^t K_\varepsilon^\tau(s)\,ds\right)\right) \leqslant \exp(2|\alpha|)\sum_{n=1}^{\infty} v_n^2 = \exp(2|\alpha|)\|v\|^2_{L^2(0,1)},$$

and the estimate follows.

(iii) $E_\varepsilon(t)v \in C([0,+\infty); L^2(0,1))$ for any $v \in L^2(0,1)$.

Proof of assertion (iii). We remark that inclusion $g \in W_0^{2,2}(0,1)$ is equivalent to

$$\sum_{n=1}^{\infty} \lambda_n^4 g_n^2 < +\infty.$$

Getting ahead of the proof of Proposition 1, we can take $v = \partial_{xx} g$ in Section 2.3. With this note, we have

$$\int_0^1 (E_\varepsilon(t_2)v - E_\varepsilon(t_1)v)^2\,dx =$$
$$\int_0^1 \left(\sum_{n=1}^{\infty}\left(\exp\left(-\frac{\lambda_n^2}{1+\lambda_n^2}\left(t_2+\alpha\int_0^{t_2} K_\varepsilon^\tau(s)\,ds\right)\right)-\right.\right.$$
$$\left.\left.\exp\left(-\frac{\lambda_n^2}{1+\lambda_n^2}\left(t_1+\alpha\int_0^{t_1} K_\varepsilon^\tau(s)\,ds\right)\right) v_n \sin(\lambda_n x)\right)^2 dx \leqslant$$
$$\sum_{n=1}^{\infty}\left(\exp\left(-\frac{\lambda_n^2}{1+\lambda_n^2}\left(t_2+\alpha\int_0^{t_2} K_\varepsilon^\tau(s)\,ds\right)\right)-\right.$$
$$\left.\exp\left(-\frac{\lambda_n^2}{1+\lambda_n^2}\left(t_1+\alpha\int_0^{t_1} K_\varepsilon^\tau(s)\,ds\right)\right)\right)^2 v_n^2 \int_0^1 \sin^2(\lambda_n x)\,dx =$$
$$\frac{1}{2}\sum_{n=1}^{\infty}\left(\exp\left(-\frac{\lambda_n^2}{1+\lambda_n^2}\left(t_2+\alpha\int_0^{t_2} K_\varepsilon^\tau(s)\,ds\right)\right)-\right.$$
$$\left.\exp\left(-\frac{\lambda_n^2}{1+\lambda_n^2}\left(t_1+\alpha\int_0^{t_1} K_\varepsilon^\tau(s)\,ds\right)\right)\right)^2 v_n^2 \to 0 \text{ as } t_2 - t_1 \to 0,$$

and the assertion (iii) follows.

2.2. **The first energy estimate uniform in $\boldsymbol{\varepsilon}$.** We start by establishing the first energy estimate for the solution of problem (2).

We multiply (2a) by the solution u_ε, integrate over $(0,1)\times(0,t)$, where $t \in (0,T)$, integrate by parts in x, taking into account (2c), and then use (2b) to get the first energy identity

$$\frac{1}{2}\int_0^1 \left(|u_\varepsilon(x,t)|^2 + |\partial_x u_\varepsilon(x,t)|^2\right)\,dx - \frac{1}{2}\int_0^1 \left(|g(x)|^2 + |\partial_x g(x)|^2\right) dx =$$

$$= -\int_0^t \int_0^1 (1+\alpha K_\varepsilon^\tau(t'))\left|\partial_x u_\varepsilon(x,t')\right|^2 \, dxdt' \leqslant$$
$$|\alpha| \int_0^t \int_0^1 K_\varepsilon^\tau(t')\left|\partial_x u_\varepsilon(x,t')\right|^2 \, dxdt', \quad \forall\, t \in [0,T].$$

Applying the Grönwall–Bellman lemma to this equality, we immediately establish estimate (4).

2.3. **The second energy estimate uniform in $\boldsymbol{\varepsilon}$.** Since $u_\varepsilon = E_\varepsilon(t)g$, $\partial_{xx}^2 g \in L^2(0,1)$ and $\partial_{xx}^2 u_\varepsilon(\cdot,t) \in L^2(0,1)$, we are eligible to multiply (2a) by $-\partial_{xx}^2 u_\varepsilon$, integrate over $(0,1)\times(0,t)$, where $t \in (0,T)$, integrate by parts in x in the first two terms, taking into account (2c), and then use (2b) to get *the second energy identity*

$$\frac{1}{2}\int_0^1 \left(|\partial_x u_\varepsilon(x,t)|^2 + |\partial_{xx}^2 u_\varepsilon(x,t)|^2\right) dx$$
$$- \frac{1}{2}\int_0^1 \left(|\partial_x g(x)|^2 + |\partial_{xx}^2 g(x)|^2\right) dx =$$
$$= -\int_0^t \int_0^1 (1+\alpha K_\varepsilon^\tau(t'))\,|\partial_{xx}^2 u_\varepsilon(x,t')|^2 \, dxdt' \leqslant$$
$$\int_0^t \int_0^1 |\alpha| K_\varepsilon^\tau(t') |\partial_{xx}^2 u_\varepsilon(x,t')|^2 \, dxdt', \quad \forall\, t \in [0,T].$$

Applying the Grönwall–Bellman lemma to this inequality, we arrive at the second energy estimate (5).

2.4. **The maximum principle uniform in $\boldsymbol{\varepsilon}$.** The maximum principle (6) directly follows from the two energy estimates and the Newton–Leibnitz formula.

3. The main results

The main results of the article are the following Theorems 1 and 2. In Theorem 1 we deal with the limit $u(x,t) = \lim\limits_{\varepsilon\to 0+} u_\varepsilon$ apart from the section $\{t=\tau\}$. In Theorem 2 we formulate the initial-boundary value problem on the 'transition layer' between $t=\tau-0$ and $t=\tau+0$. Solution of this problem links $u(x,\tau-0)$ with $u(x,\tau+0)$.

Theorem 1. *The family $\{u_\varepsilon\}_{\varepsilon\in(0,\varepsilon_0]}$ is relatively compact in $L^2(0,T;W_0^{1,2}(0,1))$ and relatively weakly* compact in $L^\infty(0,T;W_0^{2,2}(0,1))$, as $\varepsilon\to 0+$. In other terms, there exist a subsequence from $\{u_\varepsilon\}_{\varepsilon>0}$ (still labeled by ε) and a limit function $u\in L^\infty(0,T;W_0^{2,2}(0,1))$ such that*

$$u_\varepsilon \underset{\varepsilon\to 0+}{\longrightarrow} u \quad \textit{strongly in } L^2(0,T;W_0^{1,2}(0,1)) \quad \textit{and weakly}^\star \textit{ in } L^\infty(0,T;W_0^{2,2}(0,1)), \tag{9a}$$

$$\partial_t u = \partial^3_{xxt}u + \partial^2_{xx}u \quad \textit{for } (x,t)\in \Pi_T\setminus\{t=\tau\}, \tag{9b}$$

$$u(0,t) = u(1,t) = 0 \quad \textit{for } x\in(0,1),\ t\in(0,\tau)\cup(\tau,T), \tag{9c}$$

$$u(x,0) = g(x) \quad \textit{for } x\in(0,1). \tag{9d}$$

In Sections 4 and 5, the identities for the one-sided limits are proved:

$$u(x,\tau-0) = \lim_{\varepsilon\to 0+} u_\varepsilon(x,\tau-0) = \lim_{\varepsilon\to 0+} u_\varepsilon\Big(x,\tau-\frac{\varepsilon}{2}\Big),$$
$$u(x,\tau+0) = \lim_{\varepsilon\to 0+} u_\varepsilon(x,\tau+0) = \lim_{\varepsilon\to 0+} u_\varepsilon\Big(x,\tau+\frac{\varepsilon}{2}\Big).$$

Therefore, following the idea presented in [11, 29, 12], along with the family $\{u_\varepsilon\}_{\varepsilon>0}$, in order to link $u(x,\tau-0)$ and $u(x,\tau+0)$, we use rescaling on the transition layer $\Big[\tau-\frac{\varepsilon}{2},\tau+\frac{\varepsilon}{2}\Big]$ and deal with rescaled solutions $\bar{u}_\varepsilon(x,\bar{t}) = u_\varepsilon(x,\tau+\varepsilon\bar{t})$, where $\bar{u}_\varepsilon$: $(0,1)\times\Big(-\frac{1}{2},\frac{1}{2}\Big)\mapsto\mathbb{R}$. For every fixed $\varepsilon>0$, with the help of a new variable $\bar{t} = \dfrac{t-\tau}{\varepsilon} \in \left[-\dfrac{1}{2}, \dfrac{1}{2}\right]$ and multiplication by ε, we deduce from (2a) the rescaled equation

$$\partial_{\bar{t}} \bar{u}_\varepsilon = \partial^3_{xx\bar{t}} \bar{u}_\varepsilon + (\varepsilon + \alpha K(\bar{t}))\, \partial^2_{xx} \bar{u}_\varepsilon, \quad (x, \bar{t}) \in \Pi, \tag{10}$$

where $\Pi = (0,1) \times \left(-\dfrac{1}{2}, \dfrac{1}{2}\right)$.

Theorem 2. *The family of rescaled solutions* $\{\bar{u}_\varepsilon\}_{\varepsilon>0}$ *is relatively compact in* $L^2\Big((0,1)\times\left(-\dfrac{1}{2}, \dfrac{1}{2}\right)\Big)$ *and relatively weakly** *compact in* $L^\infty\Big(-\dfrac{1}{2}, \dfrac{1}{2}; W_0^{2,2}(0,1)\Big)$. *In other terms, there exist a subsequence from* $\{\bar{u}_\varepsilon\}_{\varepsilon>0}$*, still labeled by* ε*, and a limit function* $\bar{u} \in L^\infty\Big(-\dfrac{1}{2}, \dfrac{1}{2}; W_0^{2,2}(0,1)\Big)$ *such that*

$$\bar{u}_\varepsilon \underset{\varepsilon\to 0+}{\longrightarrow} \bar{u} \quad \text{strongly in } L^2\Big((0,1)\times\left(-\frac{1}{2}, \frac{1}{2}\right)\Big) \\ \text{and weakly}^\star \text{ in } L^\infty\Big(-\frac{1}{2}, \frac{1}{2}; W_0^{2,2}(0,1)\Big). \tag{11a}$$

Moreover, on the transition layer we write out the pseudo-parabolic equation supplemented with the homogeneous boundary conditions and the initial condition:

$$\partial_{\bar{t}} \bar{u} = \partial^3_{xx\bar{t}} \bar{u} + \alpha K(\bar{t}) \partial^2_{xx} \bar{u} \text{ for } (x, \bar{t}) \in \Pi, \tag{11b}$$

$$\bar{u}(0, \bar{t}) = \bar{u}(1, \bar{t}) = 0 \text{ for } \bar{t} \in \left(-\frac{1}{2}, \frac{1}{2}\right), \tag{11c}$$

$$\bar{u}\left(x, -\frac{1}{2} + 0\right) = u(x, \tau - 0) \text{ for } x \in (0,1). \tag{11d}$$

4. Proof of Theorem 1

4.1. **Application of the Aubin–Lions–Simon lemma.**

Lemma 1. *For all $\varepsilon \in (0, \varepsilon_0]$, the family of strong solutions of problem* (2) *satisfies the following demands:*

- *uniform boundedness in $L^2(0,T;W_0^{2,2}(0,1))$;*
- *integral equi-continuity*

$$\lim_{h\to 0+} \|\tau_h u_\varepsilon - u_\varepsilon\|_{L^2(0,T-h;W_0^{1,2}(0,1))} = 0, \text{ where } \tau_h u_\varepsilon(x,t) = u_\varepsilon(x,t+h).$$

Proof. The uniform boundedness follows from (5). In order to prove the integral equi-continuity, we rewrite

$$\|\tau_h u_\varepsilon - u_\varepsilon\|^2_{L^2(0,T-h;W_0^{1,2}(0,1))} = \int_0^{T-h} \big(\|u_\varepsilon(\cdot,t+h) - u_\varepsilon(\cdot,t)\|^2_{L^2(0,1)} + \|\partial_x u_\varepsilon(\cdot,t+h) - \partial_x u_\varepsilon(\cdot,t)\|^2_{L^2(0,1)}\big)\, dt.$$

Therefore,

$$\begin{aligned}
&\|u_\varepsilon(\cdot,t+h) - u_\varepsilon(\cdot,t)\|^2_{L^2(0,1)} + \|\partial_x u_\varepsilon(\cdot,t+h) - \partial_x u_\varepsilon(\cdot,t)\|^2_{L^2(0,1)} = \\
&= \sum_{n=1}^\infty (1+\lambda_n^2)\left(c_{\varepsilon,n}(t+h) - c_{\varepsilon,n}(t)\right)^2 = \\
&= \sum_{n=1}^\infty (1+\lambda_n^2) \int_t^{t+h} c'_{\varepsilon,n}(s)\, ds\, (c_{\varepsilon,n}(t+h) - c_{\varepsilon,n}(t)) = \\
&= \sum_{n=1}^\infty \lambda_n^2 \int_t^{t+h} c_{\varepsilon,n}(s)(1+\alpha K_\varepsilon^\tau(s))\, ds\, (c_{\varepsilon,n}(t+h) - c_{\varepsilon,n}(t)) \leqslant \\
&2\int_t^{t+h} (1+|\alpha| K_\varepsilon^\tau(s))\, ds \sum_{n=1}^\infty \lambda_n^2 \max_{t\in[0,T]} |c_{\varepsilon,n}(t)|^2 \leqslant \\
&2C \int_t^{t+h} (1+|\alpha| K_\varepsilon^\tau(s))\, ds \sum_{n=1}^\infty \lambda_n^2 g_n^2.
\end{aligned}$$

This provides

$$\|\tau_h u_\varepsilon - u_\varepsilon\|^2_{L^2(0,T-h;W_0^{1,2}(0,1))} \leqslant$$
$$2C\int_0^{T-h}\left(\int_t^{t+h}(1+|\alpha|K_\varepsilon^\tau(s))\,ds\right)dt\sum_{n=1}^{\infty}\lambda_n^2 g_n^2 \leqslant Ch,$$

since

$$\int_0^{T-h}\left(\int_t^{t+h}K_\varepsilon^\tau(s)\,ds\right)\,dt = \int_0^h K_\varepsilon^\tau(s)s\,ds + \int_h^{T-h}K_\varepsilon^\tau(s)h\,ds$$
$$+\int_{T-h}^{T}K_\varepsilon^\tau(s)(T-s)\,ds \leqslant h\int_0^T K_\varepsilon^\tau(s)\,ds = h,$$

for $0<\tau<T$, h, $\varepsilon \ll 1$. □

4.2. Relative compactness of $\{u_\varepsilon\}_{\varepsilon\in(0,\varepsilon_0]}$ in $L^2(0,T;W_0^{1,2}(0,1))$.

Lemma 1 enables to apply the Aubin–Lions–Simon lemma, see [24, Theorem 3], where $X = W_0^{2,2}(0,1)$ and $B = W_0^{1,2}(0,1)$, such that $X \hookrightarrow\hookrightarrow B$. Under this result and Lemma 1, sequence $\{u_\varepsilon\}_{\varepsilon\to 0+}$ is relatively compact in $L^2(0,T;W_0^{1,2}(0,1))$. Bound (5) implies $\{u_\varepsilon\}_{\varepsilon\to 0+}$ is relatively weakly* compact in $L^\infty(0,T;W_0^{2,2}(0,1))$. Due to these properties, there exist a subsequence from $\{u_\varepsilon\}_{\varepsilon\to 0+}$ and a limit function $u \in L^\infty(0,T;W_0^{2,2}(0,1))$ satisfying the limiting relation (9a) and the initial-boundary value problem (9b)–(9d). This completes the proof of Theorem 1.

5. Proof of Theorem 2

In the present section, we rescale the solution u_ε in the time variable, correspondingly, in three domains:

- $(0,1)\times(0,\tau-\frac{\varepsilon}{2})$ is mapped into $(0,1)\times(0,\tau)$ and a rescaled solution $\widehat{u}_\varepsilon$, defined by Fourier coefficients (12), satisfies the problem (13), stated further;

- $(0,1)\times(\tau-\frac{\varepsilon}{2},\tau+\frac{\varepsilon}{2})$ is mapped into $(0,1)\times\left(-\dfrac{1}{2},\dfrac{1}{2}\right)$ and a rescaled solution $\overline{u}_\varepsilon$, defined by the Fourier coefficients (14), satisfies problem (10), (15a), (15b);
- $(0,1)\times(\tau+\frac{\varepsilon}{2},T)$ is mapped into $(0,1)\times(\tau,T)$ and a rescaled solution $\widetilde{u}_\varepsilon$, defined by the Fourier coefficients (16), satisfies problem (17).

Therefore, with the the help of rescaling, we can link $\lim\limits_{\varepsilon\to 0+} u_\varepsilon(x,\tau-0)$ with $\lim\limits_{\varepsilon\to 0+} u_\varepsilon(x,\tau+0)$ and, correspondingly, $\lim\limits_{\varepsilon\to 0+}\partial_x u_\varepsilon(x,\tau-0)$ with $\lim\limits_{\varepsilon\to 0+}\partial_x u_\varepsilon(x,\tau+0)$. Moreover, in each domain we apply the original version of the Aubin–Lions lemma.

5.1. **Rescaling on $\boldsymbol{t\in\left[0,\tau-\dfrac{\varepsilon}{2}\right]}$.** Here we are going to rescale system (8). On $\left\{0\leqslant t\leqslant\tau-\frac{\varepsilon}{2}\right\}$ we take $\widehat{t}:=\frac{t\tau}{(\tau-\frac{\varepsilon}{2})}\in[0,\tau]$ and

$$\widehat{c}_{\varepsilon,n}(\widehat{t}):=c_{\varepsilon,n}\left(\frac{(\tau-\frac{\varepsilon}{2})}{\tau}\widehat{t}\right).$$

With $dt=\dfrac{(\tau-\frac{\varepsilon}{2})}{\tau}d\widehat{t}$, system (8) is rewritten in the form:

$$\widehat{c}\,'_{\varepsilon,n}(\widehat{t})=-\frac{(\tau-\frac{\varepsilon}{2})}{\tau}\frac{\lambda_n^2}{1+\lambda_n^2}\widehat{c}_{\varepsilon,n}(\widehat{t}),\quad \widehat{t}\in(0,\tau),\quad n\in\mathbb{N}, \tag{12a}$$

$$\widehat{c}_{\varepsilon,n}(0)=g_n. \tag{12b}$$

We introduce a function

$$\widehat{u}_\varepsilon(x,\widehat{t})=\sum_{n=1}^{\infty}\widehat{c}_{\varepsilon,n}(\widehat{t})\varphi_n(x)$$

as a solution of the rescaled equation

$$\partial_{\widehat{t}}\widehat{u}_\varepsilon=\frac{(\tau-\frac{\varepsilon}{2})}{\tau}\partial^2_{xx}\widehat{u}_\varepsilon+\partial^3_{xx\widehat{t}}\widehat{u}_\varepsilon \tag{13a}$$

satisfying the respective initial and homogeneous boundary conditions

$$\widehat{u}_\varepsilon(x,0) = g(x), \tag{13b}$$

$$\widehat{u}_\varepsilon(0,\widehat{t}) = \widehat{u}_\varepsilon(1,\widehat{t}) = 0. \tag{13c}$$

Lemma 2. *The following energy estimate holds true:*

$$\sup_{\widehat{t}\in(0,\tau)} \|\widehat{u}_\varepsilon(\cdot,\widehat{t})\|^2_{W_0^{2,2}(0,1)} + \sup_{\widehat{t}\in(0,\tau)} \|\partial_{\widehat{t}}\widehat{u}_\varepsilon(\cdot,\widehat{t})\|^2_{W_0^{2,2}(0,1)} \leqslant C\|g\|^2_{W_0^{2,2}(0,1)}.$$

Proof. The proof of this lemma is similar to the deduction of the first and the second energy estimates, see Sections 2.2 and 2.3. □

The family of the functions $\widehat{u}_\varepsilon$ is compact in $C([0,\tau];W_0^{1,2}(0,1))$. This is guaranteed by Proposition 1, Lemma 2 and the original version of the Aubin–Lions lemma, since the space

$$\{v \in L^2(0,\tau;W_0^{2,2}(0,1)) : \partial_t v \in L^\infty(0,\tau;W_0^{1,2}(0,1))\}$$

is compactly embedded in $C([0,\tau];W_0^{1,2}(0,1))$.

5.2. **Rescaling on $\boldsymbol{t \in \left[\tau - \frac{\varepsilon}{2}, \tau + \frac{\varepsilon}{2}\right]}$.** Let $\overline{t} := \dfrac{t-\tau}{\varepsilon}$ for $t \in \left[\tau - \dfrac{\varepsilon}{2}, \tau + \dfrac{\varepsilon}{2}\right]$. Note that $\overline{t} \in \left[-\dfrac{1}{2}, \dfrac{1}{2}\right]$, $dt = \varepsilon d\overline{t}$, and $t = \tau + \varepsilon\overline{t}$. Applying (3), for

$$\overline{c}_{\varepsilon,n}(\overline{t}) := c_{\varepsilon,n}(\tau + \varepsilon\overline{t})$$

system (8) transforms into

$$\overline{c}\,'_{\varepsilon,n}(\overline{t}) = -\frac{\lambda_n^2}{1+\lambda_n^2}(\varepsilon + \alpha K(\overline{t}))\overline{c}_{\varepsilon,n}(\overline{t}), \quad \overline{t} \in \left(-\frac{1}{2}, \frac{1}{2}\right), \quad n \in \mathbb{N}, \tag{14a}$$

$$\overline{c}_{\varepsilon,n}\left(-\frac{1}{2}\right) = \widehat{c}_{\varepsilon,n}(\tau). \tag{14b}$$

As in Section 5.1, it follows from (14) that

$$\overline{u}_\varepsilon(x,\overline{t}) = \sum_{n=1}^{\infty} \overline{c}_{\varepsilon,n}(\overline{t})\varphi_n(x).$$

satisfies equation (10) with initial and boundary conditions

$$\overline{u}_\varepsilon\Big(x,-\frac{1}{2}\Big) = \widehat{u}_\varepsilon(x,\tau), \tag{15a}$$

$$\overline{u}_\varepsilon(0,\widehat{t}) = \overline{u}_\varepsilon(1,\widehat{t}) = 0. \tag{15b}$$

Lemma 3. *The following energy estimate holds true:*

$$\sup_{\overline{t}\in(-\frac{1}{2},\frac{1}{2})} \|\overline{u}_\varepsilon(\cdot,\overline{t})\|^2_{W^{2,2}_0(0,1)} + \sup_{\overline{t}\in(-\frac{1}{2},\frac{1}{2})} \|\partial_{\overline{t}}\overline{u}_\varepsilon(\cdot,\overline{t})\|^2_{W^{2,2}_0(0,1)} \leqslant \overline{C}_\alpha \|g\|^2_{W^{2,2}_0(0,1)}.$$

Proof. The proof is based on Lemma 2 and is similar to the proof of the first and the second energy inequalities, see Secs. 2.2 and 2.3. □

The family of functions $\overline{u}_\varepsilon$ is compact in $C\Big(\Big[-\frac{1}{2},\frac{1}{2}\Big];W^{1,2}_0(0,1)\Big)$. This is guaranteed by Lemma 3 and the original version of the Aubin–Lions lemma, since the space

$$\Big\{v\in L^2\Big(-\frac{1}{2},\frac{1}{2};W^{2,2}_0(0,1)\Big) :\ \partial_t v\in L^\infty\Big(-\frac{1}{2},\frac{1}{2};W^{1,2}_0(0,1)\Big)\Big\}$$

is compactly embedded in $C\Big(\Big[-\frac{1}{2},\frac{1}{2}\Big];W^{1,2}_0(0,1)\Big)$.

5.3. **Rescaling on $t\in\Big[\tau+\frac{\varepsilon}{2},T\Big)$.** For $t\in[\tau+\frac{\varepsilon}{2},T)$ we set

$$\widetilde{t} := \frac{-\varepsilon T}{2(T-\tau-\frac{\varepsilon}{2})} + \frac{(T-\tau)t}{(T-\tau-\frac{\varepsilon}{2})} \in [\tau,T),$$

and, correspondingly,

$$\widetilde{c}_{\varepsilon,n}(\widetilde{t}) = c_{\varepsilon,n}\left(\frac{(T-\tau-\frac{\varepsilon}{2})}{(T-\tau)}\widetilde{t} + \frac{-\varepsilon T}{2(T-\tau)}\right), \quad dt = \frac{(T-\tau-\frac{\varepsilon}{2})}{(T-\tau)}d\widetilde{t}.$$

Finally, we rewrite (8) as

$$\widetilde{c}_{\varepsilon,n}'(\widetilde{t}) = -\frac{(T-\tau-\frac{\varepsilon}{2})}{(T-\tau)}\cdot\frac{\lambda_n^2}{1+\lambda_n^2}\,\widetilde{c}_{\varepsilon,n}(\widetilde{t}), \quad \widetilde{t}\in(\tau,T),\ n\in\mathbb{N}, \tag{16a}$$

$$\widetilde{c}_{\varepsilon,n}(\tau) = \overline{c}_{\varepsilon,n}(\frac{1}{2}+0). \tag{16b}$$

We introduce a solution

$$\widetilde{u}_\varepsilon(x,\widetilde{t}) = \sum_{n=1}^{\infty}\widetilde{c}_{\varepsilon,n}(\widetilde{t})\varphi_n(x)$$

of the rescaled problem

$$\partial_{\widetilde{t}}\widetilde{u}_\varepsilon = \frac{(T-\tau-\frac{\varepsilon}{2})}{(T-\tau)}\partial_{xx}^2\widetilde{u}_\varepsilon + \partial_{xx\widetilde{t}}^3\widetilde{u}_\varepsilon, \tag{17a}$$

$$\widetilde{u}_\varepsilon(x,\tau) = \overline{u}_\varepsilon(x,\frac{1}{2}+0), \tag{17b}$$

$$\widetilde{u}_\varepsilon(0,\widetilde{t}) = \widetilde{u}_\varepsilon(1,\widetilde{t}) = 0. \tag{17c}$$

Lemma 4. *The following energy estimate holds true:*

$$\sup_{\widetilde{t}\in(\tau,T)}\|\widetilde{u}_\varepsilon(\cdot,\widetilde{t})\|^2_{W_0^{2,2}(0,1)} + \sup_{\widetilde{t}\in(\tau,T)}\|\partial_{\widetilde{t}}\widetilde{u}_\varepsilon(\cdot,\widetilde{t})\|^2_{W_0^{2,2}(0,1)} \leqslant \widetilde{C}_\alpha\|g\|^2_{W_0^{2,2}(0,1)}.$$

The family of the functions $\widetilde{u}_\varepsilon$ is compact in $C([\tau,T];W_0^{1,2}(0,1))$. This is guaranteed by Lemma 4 and the original version of the Aubin–Lions lemma, since the space

$$\left\{v\in L^2(\tau,T;W_0^{2,2}(0,1))\,:\,\partial_t v\in L^\infty(\tau,T;W_0^{1,2}(0,1))\right\}$$

is compactly embedded in $C([\tau,T];W_0^{1,2}(0,1))$.

5.4. **Matching conditions.** Note that the sequences $\{\widehat{u}_\varepsilon\}$, $\{\overline{u}_\varepsilon\}$, and $\{\widetilde{u}_\varepsilon\}$ are compact in the following spaces, respectively:

$$C([0,\tau]; W_0^{1,2}(0,1)), \quad C\Big(\Big[-\frac{1}{2},\frac{1}{2}\Big]; W_0^{1,2}(0,1)\Big), \quad \text{and } C([\tau,T]; W_0^{1,2}(0,1)).$$

Moreover, $\lim\limits_{\varepsilon\to 0+} \widehat{u}_\varepsilon$ and $\lim\limits_{\varepsilon\to 0+} \widetilde{u}_\varepsilon$ satisfy problem (9b)–(9d), and, correspondingly, $\lim\limits_{\varepsilon\to 0+} \overline{u}_\varepsilon$ satisfies the problem (11b)–(11d). From initial conditions (15a) and (17b), the matching conditions

$$\lim_{\varepsilon\to 0+} u_\varepsilon(x,\tau-0) = \lim_{\varepsilon\to 0+} \widehat{u}_\varepsilon(x,\tau-0) = \lim_{\varepsilon\to 0+} \overline{u}_\varepsilon\Big(x,-\frac{1}{2}+0\Big)$$

and

$$\lim_{\varepsilon\to 0+} \overline{u}_\varepsilon\Big(x,\frac{1}{2}-0\Big) = \lim_{\varepsilon\to 0+} \widetilde{u}_\varepsilon(x,\tau+0) = \lim_{\varepsilon\to 0+} u_\varepsilon(x,\tau+0)$$

follow.

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